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# Supplement to 'Real-valued-measurable cardinals'

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This note contains further results on real-valued-measurable cardinals, supplementing my paper 'Real-valued-measurable cardinals' (pp. 151-304 in Israel Math. Conference Proc. 6 (1993), ed. H.Judah), with which it should be read. References of the form '7C', 'A2J' are directions to paragraphs in that paper, and unexplained notation is defined there. References of the form 'SA2Ac' are directions to paragraphs below.

Section numbers follow those of 'Real-valued measurable cardinals', except that a new section S6′ ('Topological implications') has been added.

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## S5. Combinatorial implications

**S5A** The arguments of 5E may be refined, as follows.

**Theorem** Let $\kappa$ be an atomlessly-measurable cardinal, and $\omega \le \delta \le \lambda < \kappa$. If $2^\delta < \kappa^{\delta^+}$, then $2^\lambda = 2^\delta$.

**proof** Induce on $\lambda$.

**(a)** For the inductive step to $\lambda$, where $\mathrm{cf}(\lambda) > \delta$, choose an injection $\theta_\xi : \mathcal{P}\xi \to \mathfrak{c}$ for each $\xi < \lambda$; set $g_A(\xi) = \theta_\xi(A \cap \xi)$ for $A \subseteq \lambda$, $\xi < \lambda$. **?** If $2^\lambda > 2^\delta$, let $N \subseteq 2^\delta$ be a set of minimal cardinal such that

$$\mathcal{A}_N = \{A : A \subseteq \lambda,\ g_A^{-1}[N] \text{ is cofinal with } \lambda\}$$

has cardinal greater than $2^\delta$. Then $\mathrm{cf}(\lambda) \le \mathrm{cf}(\#(N)) \le \lambda$. **P** **(i)** If $\gamma < \lambda$ and $N = \bigcup_{\alpha<\gamma} N_\alpha$ then $\mathcal{A}_N = \bigcup_{\alpha<\gamma} \mathcal{A}_{N_\alpha}$. **(ii)** If $\gamma > \lambda$ is regular and $\langle N_\alpha \rangle_{\alpha<\gamma}$ is an increasing family with union $N$, then again $\mathcal{A}_N = \bigcup_{\alpha<\gamma} \mathcal{A}_{N_\alpha}$. **Q**

But there is no cardinal $\gamma$ such that $\kappa \le \gamma < \kappa^{+\delta^+}$ and $\delta < \mathrm{cf}(\gamma) \le \lambda$, so $\#(N) < \kappa$.

Let $h : \kappa \to \mathcal{A}_N$ be injective. For $\alpha < \beta < \kappa$ set $f(\{\alpha, \beta\}) = \min\{\zeta : h(\alpha) \cap \zeta \ne h(\beta) \cap \zeta\}$. By 5Ca, there is a set $C \subseteq \kappa$ such that $\#(C) = \kappa$ and $M = \{f(I) : I \in [C]^2\}$ is countable, therefore bounded above in $\lambda$; set $\zeta = \sup M$, so that $h(\alpha) \cap \zeta \ne h(\beta) \cap \zeta$ for all distinct $\alpha$, $\beta \in C$. Next, there is a $\xi \in \lambda \setminus \zeta$ such that $C_1 = \{\alpha : \alpha \in C,\ g_{h(\alpha)}(\xi) \in N\}$ has cardinal $\kappa$. But now $\alpha \mapsto g_{h(\alpha)}(\xi) = \theta_\xi(h(\alpha \cap \xi))$ is an injective function from $C_1$ to $M$, while $\#(M) < \kappa = \#(C_1)$. **X**

This deals with the inductive step if $\mathrm{cf}(\lambda) > \delta$.

**(b)** If $\mathrm{cf}(\lambda) \le \delta$, then $2^\lambda \le (\sup_{\gamma<\lambda} 2^\gamma)^{\mathrm{cf}(\lambda)} = 2^\delta$, so the induction proceeds.

**S5B Proposition** Suppose there is an atomlessly-measurable cardinal $\kappa$. Then there is a family $\langle A_\alpha \rangle_{\alpha<\omega_1}$ of sets such that

(i) $A_\alpha \subseteq \alpha$ for every $\alpha < \omega_1$;
(ii) if $\alpha \le \beta < \omega_1$, then $A_\alpha \triangle (A_\beta \cap \alpha)$ is finite;
(iii) if $C \subseteq \omega_1$ is uncountable, then

$$\{\alpha : C \cap A_\alpha \text{ is infinite}\}, \quad \{\alpha : (C \cap \alpha) \setminus A_\alpha \text{ is infinite}\}$$

are both uncountable.

**proof (a)** Choose a family $\langle e_\alpha \rangle_{\alpha<\omega_1}$ such that (i) each $e_\alpha$ is an injective function from $\alpha$ to $\mathbb{N}$ (ii) if $\alpha \le \beta$ then $\{\xi : \xi < \alpha,\ e_\alpha(\xi) \ne e_\beta(\xi)\}$ is finite (SA1A). Let $\nu$ be an extension of the usual measure $\mu$ of $\mathcal{P}\mathbb{N}$ (corresponding to the usual measure on $\{0,1\}^{\mathbb{N}}$) to $\mathcal{P}(\mathcal{P}\mathbb{N})$. For each $a \subseteq \mathbb{N}$ set

$$A_{a\alpha} = e_\alpha^{-1}[a] \subseteq \alpha.$$

Then whenever $\alpha \le \beta < \omega_1$,

$$A_{a\alpha} \triangle (\alpha \cap A_{a\beta}) \subseteq \{\xi : \xi < \alpha,\ e_\alpha(\xi) \ne e_\beta(\xi)\}$$

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is finite. Thus every $a \subseteq \mathbb{N}$ provides a family satisfying conditions (i) and (ii).

**(b)** Let $S$ be the set of those $a \subseteq \mathbb{N}$ for which there is some uncountable set $C_a \subseteq \omega_1$ such that $\{\alpha : C_a \cap A_{a\alpha}$ is infinite$\}$ is countable. **?** Suppose, if possible, that $\nu S > 0$. Then there is an $\alpha_0 < \omega_1$ such that

$$\{\alpha : C_a \cap A_{a\alpha} \text{ is infinite}\} \subseteq \alpha_0$$

for $\nu$-almost every $a \in S$; set

$$T = \{a : a \in S,\ C_a \cap A_\alpha \text{ is finite for every } \alpha \geq \alpha_0\}.$$

For each $\xi < \omega_1$, set $T_\xi = \{a : a \in T,\ \xi \in C_a\}$; then

$$0 < \nu S = \nu T \leq \textstyle\sum_{\xi \geq \beta} \nu T_\xi$$

for every $\beta < \omega_1$, so $\{\xi : \nu T_\xi > 0\}$ is uncountable and there is a $\delta > 0$ such that $B$ is uncountable, where

$$B = \{\xi : \nu T_\xi \geq \delta\}.$$

By S6F (I apologize for the forward reference, but you can easily check that there is no circularity) there is an infinite $D \subseteq B$ such that $\mu^*(\bigcap_{\xi \in D} T_\xi) > 0$. Take $\alpha \geq \alpha_0$ such that $D \subseteq \alpha$. If $a \in \bigcap_{\xi \in D} T_\xi$, then $D \subseteq C_a$, while $C_a \cap A_{a\alpha}$ is finite, so $D \cap A_{a\alpha}$ is finite. But

$$E = \{a : D \cap A_{a\alpha} \text{ is finite}\} = \{a : a \cap e_\alpha[D] \text{ is finite}\}$$

is $\mu$-measurable and has zero measure, so $\mu^*(\bigcap_{\xi \in D} T_\xi) = 0$, which is impossible. **X** Thus $\nu S = 0$.

**(c)** Now consider the set $S'$ of those $a \subseteq \mathbb{N}$ for which there is some uncountable set $C_a$ such that $\{\alpha : (C_a \cap \alpha) \setminus A_{a\alpha}$ is infinite$\}$ is countable. **?** Suppose, if possible, that $\nu S' > 0$. Then there is an $\alpha_0 < \omega_1$ such that

$$\{\alpha : C_a \cap \alpha \setminus A_{a\alpha} \text{ is infinite}\} \subseteq \alpha_0$$

for $\nu$-almost every $a \in S'$; set

$$T = \{a : a \in S,\ C_a \cap \alpha \setminus A_\alpha \text{ is finite for every } \alpha \geq \alpha_0\}.$$

For each $\xi < \omega_1$, set $T_\xi = \{a : a \in T,\ \xi \in C_a\}$; then

$$0 < \nu S' = \nu T \leq \textstyle\sum_{\xi \geq \beta} \nu T_\xi$$

for every $\beta < \omega_1$, so $\{\xi : \nu T_\xi > 0\}$ is uncountable and there is a $\delta > 0$ such that $B$ is uncountable, where

$$B = \{\xi : \nu T_\xi \geq \delta\}.$$

By S6F there is an infinite $D \subseteq B$ such that $\mu^*(\bigcap_{\xi \in D} T_\xi) > 0$. Take $\alpha \geq \alpha_0$ such that $D \subseteq \alpha$. If $a \in \bigcap_{\xi \in D} T_\xi$, then $D \subseteq C_a$, while $C_a \cap \alpha \setminus A_{a\alpha}$ is finite, so $D \setminus A_{a\alpha}$ is finite. But

$$E = \{a : D \setminus A_{a\alpha} \text{ is finite}\} = \{a : e_\alpha[D] \setminus a \text{ is finite}\}$$

is $\mu$-measurable and has zero measure, so $\mu^*(\bigcap_{\xi \in D} T_\xi) = 0$, which is impossible. **X** Thus $\nu S' = 0$.

**(d)** There must therefore be an $a \subseteq \mathbb{N}$ not belonging to either $S$ or $S'$, and the corresponding family $\langle A_{a\alpha} \rangle_{\alpha < \omega_1}$ has the required property.

**Remark** This comes from TODORČEVIĆ 93, §7.

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## S6. Measure-theoretic implications

**S6A Proposition** Let $\kappa$ be an atomlessly-measurable cardinal and $\lambda < \kappa$ a cardinal. Then there is a cardinal $\theta < \kappa$ such that whenever $X$ is a $\mathrm{T}_0$ topological space of cardinal at least $\theta$ and weight at most $\lambda$ then there is an atomless $\tau$-additive Borel probability on $X$.

**proof ?** Suppose, if possible, otherwise.

**(a)** We are supposing that for each $\zeta < \kappa$ there is a $\mathrm{T}_0$ space $X_\zeta$, of cardinal at least $\#(\zeta)$ and of weight at most $\lambda$, with no atomless $\tau$-additive Borel probability. Note that any $\tau$-additive Borel probability $\tilde{\mu}$ on any subspace $D$ of $X_\zeta$ gives rise to a $\tau$-additive Borel probability $\tilde\mu_1$ on $X_\zeta$, writing $\tilde\mu_1 E = \tilde\mu(D\cap E)$ for every Borel set $E\subseteq X_\zeta$, which would be atomless if $\tilde\mu$ were atomless; so we may take it that $\#(X_\zeta)$ is precisely $\#(\zeta)$. Let $\langle U_{\zeta\xi}\rangle_{\xi<\lambda}$ run over a base for the topology of $X_\zeta$. Define $f_\zeta : X_\zeta \to X = \{0,1\}^\lambda$ by writing

$$f_\zeta(x)(\xi) = 1 \text{ if } x\in U_{\zeta\xi},\ 0 \text{ if } x\in X_\zeta\setminus U_{\zeta\xi}.$$

Note that $f_\zeta$ is injective, because $X_\zeta$ is a $\mathrm{T}_0$ space, so we may enumerate $Y_\zeta = f_\zeta[X_\zeta]$ as $\langle y_{\zeta\xi}\rangle_{\xi<\zeta}$.

**(b)** Let $\nu$ be a normal witnessing probability on $\kappa$. Define $\mu : \mathcal{P}(X\times\kappa)\to[0,1]$ by setting

$$\mu E = \int \nu\{\zeta : (y_{\zeta\xi},\zeta)\in E\}\nu(d\xi)$$

for $E\subseteq X\times\kappa$. Then $\mu$ is a $\kappa$-additive probability on $X\times\kappa$. By SA2B below there is a family $\langle\mu_\zeta\rangle_{\zeta<\kappa}$ of Radon probabilities on $X$ such that $\mu(H\times A) = \int_A \mu_\zeta H\nu(d\zeta)$ for every set $H$ in the Baire $\sigma$-algebra of $X$ and every $A\subseteq\kappa$.

Let $\mathcal{V}$ be the algebra of open-and-closed subsets of $X$.

**(c)** For any $C\subseteq X\times\kappa$, $\mu C\le\int\mu^*_\zeta C^{-1}[\{\zeta\}]\nu(d\zeta)$. **P** The set $\mathcal{W}$ of sets expressible as a finite disjoint union of sets $V\times A$, where $V\in\mathcal{V}$ and $A\subseteq\kappa$, is a subalgebra of $\mathcal{P}(X\times\kappa)$, and for every $W\in\mathcal{W}$ we have

$$\mu W = \int\mu_\zeta W^{-1}[\{\zeta\}]\nu(d\zeta).$$

Now take $\epsilon>0$. For each $\zeta<\kappa$ choose an open set $G_\zeta\supseteq C^{-1}[\{\zeta\}]$ such that $\mu_\zeta G_\zeta\le\mu^*_\zeta C^{-1}[\{\zeta\}]+\epsilon$. Set

$$E = \{(y,\zeta):\zeta<\kappa,\ y\in G_\zeta\} = \bigcup_{V\in\mathcal{V}}\{(y,\zeta): y\in V\subseteq G_\zeta\}.$$

Then $C\subseteq E$ so $\mu C\le\mu E$. Because $\mu$ is $\kappa$-additive and $\#(\mathcal{V})\le\max(\lambda,\omega)<\kappa$, there is a finite $\mathcal{V}_0\subseteq\mathcal{V}$ such that $\mu W\ge\mu E-\epsilon$, where

$$W = \bigcup_{V\in\mathcal{V}_0}\{(y,\zeta): y\in V\subseteq G_\zeta\}.$$

But $W\in\mathcal{W}$, so

$$\mu W = \int\mu_\zeta W^{-1}[\{\zeta\}]\nu(d\zeta)\le\int\mu_\zeta G_\zeta\nu(d\zeta).$$

Putting these together,

$$\mu C\le\int\mu_\zeta G_\zeta\nu(d\zeta)+\epsilon\le\int\mu^*_\zeta C^{-1}[\{\zeta\}]\nu(d\zeta)+2\epsilon.$$

As $\epsilon$ is arbitrary, we have the result. **Q**

Taking complements, we see now that

$$\mu C\ge\int(\mu_\zeta)_* C^{-1}[\{\zeta\}]\nu(d\zeta)$$

for every $C\subseteq X\times\kappa$, writing $(\mu_\zeta)_* D = \sup\{\mu_\zeta H : H\subseteq D \text{ is } \mu_\zeta\text{-measurable}\}$ for every $D\subseteq X$, $\zeta<\kappa$.

**(d)** $\mu^*_\zeta Y_\zeta = 1$ for $\nu$-almost every $\zeta$. **P** Set $C = \{(y,\zeta):\zeta<\kappa,\ y\in Y_\zeta\}\subseteq X\times\kappa$. Then $(y_{\zeta\xi},\zeta)\in C$ whenever $\xi<\zeta<\kappa$, so $\mu C = 1$. By (c),

$$1 = \mu C\le\int\mu^*_\zeta C^{-1}[\{\zeta\}]\nu(d\zeta) = \int\mu^*_\zeta Y_\zeta\nu(d\zeta).$$

Thus $\mu^*_\zeta Y_\zeta = 1$ for almost all $\zeta$. **Q**

Set $A_0 = \{\zeta:\mu^*_\zeta Y_\zeta = 1\}$.

**(e)** $\mu_\zeta$ is atomless for almost all $\zeta\in A_0$. **P** Set

$$B = \{\zeta:\zeta\in A_0,\ \mu_\zeta \text{ has an atom}\},$$
$$C = \{(y,\zeta):\zeta<\kappa,\ y\in Y_\zeta,\ \mu_\zeta\{y\}>0\}.$$

If $\zeta\in B$, then (because $\mu_\zeta$ is a Radon measure) there is a $y\in X$ such that $\mu_\zeta\{y\}>0$; but as $\mu^*_\zeta Y_\zeta = 1$, $y\in Y_\zeta$ and $(y,\zeta)\in C$. For each $\zeta<\kappa$, $C^{-1}[\{\zeta\}]$ is countable, so is expressible as $\{y_{\zeta\xi}:\xi\in I_\zeta\}$, where $I_\zeta\subseteq\zeta$ is countable. Let $I\subseteq\kappa$ be a countable set such that $I_\zeta\subseteq I$ for almost all $\zeta$ (5Ab). Now $\nu\{\zeta : y_{\zeta\xi}\in C\} = 0$ for every $\xi\in\kappa\setminus I$, so $\mu C = 0$. Consequently, by the last remark of (c) above,

$$\int(\mu_\zeta)_* C^{-1}[\{\zeta\}]\nu(d\zeta) = 0.$$

But of course $(\mu_\zeta)_* C^{-1}[\{\zeta\}]>0$ for every $\zeta\in B$, so $\nu B = 0$, as claimed. **Q**

**(f)** There must therefore be some $\zeta<\kappa$ such that $\mu_\zeta$ is atomless and $\mu^*_\zeta Y_\zeta = 1$.

Write $\mu'_\zeta$ for the quasi-Radon subspace probability $\mu_\zeta\restriction Y_\zeta$ (A2Ja). $f_\zeta^{-1}: Y_\zeta\to X_\zeta$ is continuous, so we have a $\tau$-additive Borel probability $\tilde\mu$ on $X_\zeta$ defined by saying that $\tilde\mu E = \mu'_\zeta(f_\zeta[E])$ for every Borel set $E\subseteq X_\zeta$. Now take $\epsilon>0$. Because $\mu_\zeta$ is atomless, every point of $X$ is contained in an open set of $\mu_\zeta$-measure at most $\epsilon$. Because $X$ is compact, we have a finite cover of $X$ by basic cylinder sets of the form $U_w = \{y : y\restriction I = w\}$, where $I\subseteq\lambda$ is finite and $w\in\{0,1\}^I$, all of $\mu_\zeta$-measure at most $\epsilon$. Now $f_\zeta^{-1}[U_w]$ is a Borel set in $X_\zeta$ for every $w$ (examine the original definition of $f_\zeta$ in (a) above), and $\tilde\mu f_\zeta^{-1}[U_w] = \mu'_\zeta(Y_\zeta\cap U_w) = \mu_\zeta U_w$, so $X_\zeta$ has a partition into finitely many Borel sets of $\tilde\mu$-measure at most $\epsilon$. As $\epsilon$ is arbitrary, $\tilde\mu$ is atomless.

But this contradicts the choice of $X_\zeta$. **X**

This contradiction completes the proof.

**S6B Corollary** Let $\kappa$ be an atomlessly-measurable cardinal and $\lambda < \kappa$ a cardinal. Then there is a cardinal $\theta < \kappa$ such that

(a) whenever $X$ is a regular $T_0$ topological space of cardinal at least $\theta$ and weight at most $\lambda$ then there is an atomless quasi-Radon probability on $X$;

(b) whenever $Z$ is a compact Hausdorff space of weight at most $\lambda$ and $X \subseteq Z$ has cardinal at least $\theta$, then there is an atomless Radon probability $\mu$ on $Z$ such that $\mu^* X = 1$.

**proof** Take $\theta$ from S6A.

**(a)** There is an atomless $\tau$-additive Borel probability on $X$, which by SA2A extends to a quasi-Radon probability on $X$.

**(b)** There is an atomless $\tau$-additive Borel probability $\mu_0$ on $X$; setting $\mu E = \mu_0(X \cap E)$ for Borel sets $E \subseteq Z$, we get a $\tau$-additive Borel probability on $Z$ for which $\mu^* X = 1$; by SA2A the completion $\hat{\mu}$ of $\mu$ is quasi-Radon, and of course $\hat{\mu}$ is atomless and $\hat{\mu}^* X = 1$. Because $Z$ is compact, $\hat{\mu}$ is inner regular for the compact sets and is therefore a Radon measure.

**S6C Corollary** Let $\kappa$ be an atomlessly-measurable cardinal. Then there is a $\theta < \kappa$ such that no subset of $\mathbb{R}$ of cardinal $\theta$ or more can be universally negligible.

**proof** Apply S6Bb with $\lambda = \omega$, $X = [a, b]$, where $a \leq b$ in $\mathbb{R}$.

**S6D Proposition** If $\kappa$ is an atomlessly-measurable cardinal and $(X, \mu)$ is an atomless Radon probability space of Maharam type at most $\kappa$, then $\mathrm{cf}(\mathcal{N}_\mu) = \mathrm{cf}(\mathcal{N})$, where $\mathcal{N}$ is the Lebesgue null ideal.

**proof** Use S7B below and 6.14 of Fremlin 89.

**S6E Lemma** Let $\kappa$ be an atomlessly-measurable cardinal, and $\lambda \leq \kappa$ a cardinal of uncountable cofinality. Let $(X, \mu)$ be a Radon probability space and $\langle E_\xi \rangle_{\xi < \lambda}$ a family of non-negligible measurable subsets of $X$. Then there is an $x \in X$ such that $\#(\{\xi : x \in E_\xi\}) = \lambda$.

**proof** By 6C, $\lambda$ is a precaliber of the measure algebra $\mathfrak{A}$ of $(X, \mu)$. For each $\xi < \kappa$, choose a compact non-negligible $F_\xi \subseteq E_\xi$. Then there is a set $D \in [\lambda]^\lambda$ such that $\{F_\xi^\bullet : \xi \in D\}$ is centered. But this means that $\{F_\xi : \xi \in D\}$ has the finite intersection property; because the $F_\xi$ are compact, there is a point in $\bigcap_{\xi \in D} F_\xi$, and this will serve for $x$.

**S6F Lemma** Let $\kappa$ be an atomlessly-measurable cardinal. Let $(X, \Sigma, \mu)$ be a probability space, and $\Sigma_0$ a $\sigma$-subalgebra of $\Sigma$; set $\mu_0 = \mu \restriction \Sigma_0$ and let $\mu_0^*$ be the outer measure on $X$ defined from $\mu_0$. Suppose that the Maharam type $\tau_0$ of $\mu_0$ is less than $\kappa$. Let $\langle E_\xi \rangle_{\xi < \lambda}$ be a family in $\Sigma$ with $\inf_{\xi < \lambda} \mu E_\xi = \gamma > 0$, where $\lambda > \max(\omega, \tau_0)$. Then for any $\gamma' < \gamma$ there is an infinite $W \subseteq \lambda$ such that $\mu_0^*(\bigcap_{\xi \in W} E_\xi) \geq \gamma'$.

**Remark** Compare A2S.

**proof** It is enough to consider the case $\lambda = \max(\omega_1, \tau_0^+)$, so that $\mathrm{cf}(\lambda) > \omega$ and $\lambda \leq \kappa$.

**(a)** Consider first the special case in which $X = \{0,1\}^I$ for some set $I$, with $\mu$ the usual measure on $X$, and there is a $J \subseteq I$ of cardinal $\tau_0$ such that $\Sigma_0 = \{\pi_J^{-1}[F] : F \in \Sigma_J\}$, writing $\Sigma_J$ for the domain of the usual measure $\mu_J$ on $\{0,1\}^J$, and $\pi_J : X \to \{0,1\}^J$ for the canonical map.

We may regard $X$ as the product $\{0,1\}^J \times \{0,1\}^{I \setminus J}$. For $\xi < \lambda$, $z \in \{0,1\}^{I \setminus J}$ set

$$E_{\xi z} = \{y : y \in \{0,1\}^J, (y, z) \in E_\xi\}.$$

Then

$$\int \mu_J E_{\xi z} \mu_{I \setminus J}(dz) = \mu E_\xi \geq \gamma$$

by Fubini's theorem (A2Ge). Set

$$G_\xi = \{z : z \in \{0,1\}^{I \setminus J}, \mu_J E_{\xi z} \text{ exists } \geq \gamma\};$$

then $\mu_{I \setminus J} G_\xi$ exists and is greater than 0. By S6E, there is a $z \in \{0,1\}^{I \setminus J}$ such that $U = \{\xi : z \in G_\xi\}$ has cardinal $\lambda$.

Let $(\mathfrak{A}_0, \bar{\mu}_J)$ be the measure algebra of $\mu_J$. Then the topological density of $\mathfrak{A}_0$ is at most $\max(\omega, \tau_0) < \lambda$ (Fremlin 89, 6.3b). So there is a $b \in \mathfrak{A}_0$ such that $\{\xi : \xi \in U, \bar{\mu}_J(b \triangle E_{\xi z}^\bullet) \leq \delta\}$ is infinite for every

$\delta > 0$. Of course $\bar\mu_J b \ge \inf_{\xi\in U} \bar\mu_J E^\bullet_{\xi z} \ge \gamma$. Let $\langle \xi_n\rangle_{n\in\mathbb{N}}$ be a sequence of distinct elements of $U$ such that $\sum_{n\in\mathbb{N}} \bar\mu_J(b\triangle E^\bullet_{\xi_n z}) \le \gamma - \gamma'$. Then

$$\mu_J(\bigcap_{n\in\mathbb{N}} E_{\xi_n z}) = \bar\mu_J(\inf_{n\in\mathbb{N}} E^\bullet_{\xi_n z}) \ge \bar\mu_J b - \sum_{n\in\mathbb{N}} \bar\mu_J(b \setminus E^\bullet_{\xi_n z}) \ge \gamma'.$$

Set $W = \{\xi_n : n \in \mathbb{N}\}$. If $F' \in \Sigma_J$ and $\pi_J^{-1}[F'] \supseteq \bigcap_{\xi\in W} E_\xi$, we must have $F' \supseteq \bigcap_{n\in\mathbb{N}} E_{\xi_n z}$, so

$$\mu(\pi_J^{-1}[F']) = \mu_J F' \ge \gamma'.$$

As $F'$ is arbitrary, $\mu_0^*(\bigcap_{\xi\in W} E_\xi) \ge \gamma'$, as required.

**(b)** It follows that if $(\mathfrak{A},\bar\mu)$ is a probability algebra, $\mathfrak{A}_0$ is a order-closed subalgebra of $\mathfrak{A}$ with $\tau(\mathfrak{A}_0) = \tau_0 < \kappa$, $\lambda$ is a cardinal greater than $\max(\omega,\tau(\mathfrak{A}_0))$, $\langle a_\xi\rangle_{\xi<\lambda}$ is a family in $\mathfrak{A}$ with $\inf_{\xi<\lambda} \bar\mu a_\xi = \gamma > 0$, and $\gamma' < \gamma$, then there is an infinite $W \subseteq \lambda$ such that

$$\min\{\bar\mu b : b \in \mathfrak{A}_0,\, b \supseteq \inf_{\xi\in W} a_\xi\} \ge \gamma'.$$

**P** We can embed $\mathfrak{A}$ as a subalgebra of the measure algebra $\mathfrak{A}_I$ of $\{0,1\}^I$ for some set $I$ (A2Ib). If we take a set $B \subseteq \mathfrak{A}_0$ of cardinal $\tau_0$ which completely generates $\mathfrak{A}_0$, then for each $b \in B$ we can find a set $G_b \subseteq X = \{0,1\}^I$, belonging to the Baire $\sigma$-algebra of $X$, such that $b = G_b^\bullet$ in $\mathfrak{A}$; now there is a set $J \subseteq I$, of cardinal at most $\max(\omega,\tau_0)$, such that every $G_b$ belongs to $\Sigma_0$, if we define $\Sigma_0$ from $J$ as in part (a) above.

Set $\mathfrak{A}_1 = \{G^\bullet : G \in \Sigma_0\}$, so that $\mathfrak{A}_1$ is an order-closed subalgebra of $\mathfrak{A}$ and $\mathfrak{A}_0 \subseteq \mathfrak{A}_1$. Let $\Sigma_J$ be the algebra of measurable subsets of $\{0,1\}^J$, and $\mathfrak{A}_J$ the measure algebra of $\{0,1\}^J$. Then the inverse-measure-preserving map $\pi_J : X \to \{0,1\}^J$ induces an isomorphism $\phi$ between $\mathfrak{A}_1$ and $\mathfrak{A}_J$, taking $\phi(\pi_J^{-1}[G]^\bullet) = G^\bullet$ for every $G \in \Sigma_J$. By the lifting theorem (A2Qb) there is a lifting $\theta_J : \mathfrak{A}_J \to \Sigma_J$. So we have a corresponding Boolean homomorphism $\theta_1 : \mathfrak{A}_1 \to \Sigma_0$ given by setting $\theta_1(a) = \pi_J^{-1}[\theta_J(\phi(a))]$ for each $a \in A_1$. As remarked in A2Qc, there is an extension $\theta$ of $\theta_1$ to a lifting from $\mathfrak{A}$ to $\Sigma$.

Set $E_\xi = \theta(a_\xi)$ for each $\xi < \lambda$. By part (a) above, there is an infinite set $W \subseteq \lambda$ such that $\mu_0^*(\bigcap_{\xi\in W} E_\xi) \ge \gamma'$. Now suppose that $b \in \mathfrak{A}_0$ and $b \supseteq \inf_{\xi\in W} a_\xi$. Then $\theta(b) \supseteq \bigcap_{\xi\in L} E_\xi$ and $\theta(b) \in \Sigma_0$, so

$$\bar\mu b = \mu(\theta(b)) \ge \mu_0^*(\bigcap_{\xi\in W} E_\xi) \ge \gamma'.$$

Thus we have the set $W$ we need. **Q**

**(c)** We are now ready for the general case of the lemma. Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of $(X,\Sigma,\mu)$ and set $\mathfrak{A}_0 = \{G^\bullet : G \in \Sigma_0\}$, $a_\xi = E_\xi^\bullet$ for each $\xi < \lambda$. By (b), there is an infinite $W \subseteq \lambda$ such that $\bar\mu b \ge \gamma'$ whenever $b \in \mathfrak{A}_0$ and $b \supseteq \inf_{\xi\in W} a_\xi$. Now if $G \in \Sigma_0$ and $G \supseteq \bigcap_{\xi\in W} E_\xi$, $b = G^\bullet \in \mathfrak{A}_0$ and $b \supseteq \inf_{\xi\in W} a_\xi$, so that $\mu_0 G = \mu G = \bar\mu G^\bullet \ge \gamma'$. As $G$ is arbitrary, $\mu_0^*(\bigcap_{\xi\in W} E_\xi) \ge \gamma'$, and we have a suitable set $W$.

**S6G Proposition** Let $\kappa$ be an atomlessly-measurable cardinal.

(a) Let $X$ be a Hausdorff space such that $\chi(X) < \kappa$. Then any Radon probability measure on $X$ has Maharam type at most $\max(\omega,\chi(X))$.

(b) Let $X$ be a Hausdorff space such that $\chi(x,X) < \kappa$ for every $x \in X$. Then any Radon probability measure on $X$ has Maharam type less than $\kappa$.

**proof ?** Suppose, if possible, otherwise. By SA2D, $X$ has a Maharam homogeneous Radon probability measure $\mu$ of Maharam type $\lambda$, where $\lambda = \max(\omega_1,\chi(X)^+)$ in (a) and $\lambda = \kappa$ in (b). Let $\mathfrak{A}$ be the measure algebra of $\mu$; then $\mathfrak{A}$ can be expressed as the union of a strictly increasing family $\langle\mathfrak{A}_\xi\rangle_{\xi<\lambda}$ of closed subalgebras. For each $x \in X$ choose a base $\mathcal{U}_x$ of open neighbourhoods of $x$ with $\#(\mathcal{U}_x) = \chi(x,X)$. Set

$$X_\xi = \{x : x \in X,\, U^\bullet \in \mathfrak{A}_\xi \text{ for every } U \in \mathcal{U}_x\}.$$

Because $\mathrm{cf}(\lambda) = \lambda > \chi(x,X)$ for every $x$, $\bigcup_{\xi<\lambda} X_\xi = X$.

By 6C/S6E, $\sup_{\xi<\lambda} \mu^* X_\xi = 1$; because $\mathrm{cf}(\lambda) > \omega$, there is a $\xi < \lambda$ such that $\mu^* X_\xi = 1$. Now let $G$ be any open subset of $X$. Set

$$\mathcal{H} = \{H : H \subseteq G \text{ is open},\, H^\bullet \in \mathfrak{A}_\xi\}, \quad H_0 = \bigcup\mathcal{H}.$$

Then $H_0 \supseteq G \cap X_\xi$. So $G \setminus H_0$ must be negligible, and

$$G^\bullet = H_0^\bullet = \sup_{H\in\mathcal{H}} H^\bullet \in \mathfrak{A}_\xi.$$

As $G$ is arbitrary, $\mathfrak{A} = \mathfrak{A}_\xi$, which is absurd. **X**

**Remark** This is derived from PLEBANEK 95.

Version of 22.5.00

**S6′. Topological implications**

**S6′A Proposition** Suppose there is an atomlessly-measurable cardinal. Then there is a locally compact locally countable Hausdorff topology $\mathfrak{T}$ on $\omega_1$ such that every closed set is either countable or cocountable.

**proof (a)** Let $\kappa$ be an atomlessly-measurable cardinal, and $\nu_0$ a Maharam homogeneous atomless probability measure with domain $\mathcal{P}\kappa$; let $\lambda$ be the Maharam type of $\nu$. By 3F, $\lambda > \kappa > \omega_1$. Let $g : \kappa \to \{0,1\}^\lambda$ be a function representing an isomorphism from the measure algebra of the usual measure on $X = \{0,1\}^\lambda = X$ and the measure algebra of $\nu_0$ (A2Gd). Let $\nu$ be the image measure $\nu_0 g^{-1}$, so that $\nu$ extends the usual measure on $X$, and setting $E_\xi = \{x : x \in X, x(\xi) = 1\}$ for $\xi < \lambda$, $\langle E_\xi \rangle_{\xi<\lambda}$ is a stochastically independent family of sets of measure $\frac{1}{2}$ such that $\{E_\xi^\bullet : \xi < \lambda\}$ $\tau$-generates the measure algebra of $\nu$. For $\Gamma \subseteq \lambda$ write $\Sigma_\Gamma$ for the $\sigma$-algebra generated by $\{E_\xi : \xi \in \Gamma\} \cup \{E : \nu E = 0\}$, so that

$\mathcal{P}X = \bigcup\{\Sigma_\Gamma : \Gamma \in [\lambda]^{\le\omega}\}$. Write $\Omega$ for the set of non-zero countable limit ordinals; for $\alpha \in \Omega$, enumerate $\alpha$ as $\langle e_\alpha(n) \rangle_{n\in\mathbb{N}}$. Fix a bijection $\phi : \{0,1\}^\mathbb{N} \to [0,1]^\mathbb{N}$ which is an automorphism for the usual measures (FREMLIN P00*, 344I). For $\alpha \in \Omega$, $n \in \mathbb{N}$ define $\phi_{\alpha n} : X \to [0,1]$ by writing $\langle \phi_{\alpha n}(x) \rangle_{n\in\mathbb{N}} = \phi(\langle x(\alpha+n) \rangle_{n\in\mathbb{N}})$ for every $x \in X$.

**(b)** For $x \in X$, $\xi < \omega_1$ choose $B_x(\xi)$ as follows. If $\xi < \omega$ then $B_x(\xi) = \{\xi\}$. Suppose that $B_x(\xi)$ has been defined for every $\xi < \alpha$, where $\alpha \in \Omega$. Set $C_x(\alpha, n) = B_x(e_\alpha(n)) \setminus \bigcup_{i<n} B_x(e_\alpha(i))$ for each $n$. Set

$$B_x(\alpha + n) = \{\alpha + n\} \cup \bigcup\{C_x(\alpha, i) : i \in \mathbb{N},\ 2^{-n-1} \le \phi_{\alpha i}(x) < 2^{-n}\}$$

for each $n$.

Let $\mathfrak{T}_x$ be the topology on $\omega_1$ generated by $\{B_x(\xi) : \xi < \omega_1\} \cup \{\omega_1 \setminus B_x(\xi) : \xi < \omega_1\}$. Then $\mathfrak{T}_x$ is Hausdorff and locally countable. **P** It is Hausdorff because if $\xi < \eta$ then $\xi \in B_x(\xi)$ and $\eta \notin B_x(\xi)$. (The point is that if $\alpha$ is a limit ordinal and $n \in \mathbb{N}$, then

$$\alpha + n \in B_x(\alpha + n) \subseteq \{\alpha + n\} \cup \alpha.)$$

It is locally countable because every $B_x(\xi)$ is countable. **Q**

**(c)** For $\alpha \in \Omega$, let $\mathfrak{T}_{x\alpha}$ be the topology on $\alpha$ generated by $\{B_x(\xi) : \xi < \alpha\} \cup \{\alpha \setminus B_x(\xi) : \xi < \alpha\}$. Then an easy induction shows that $C_x(\beta, n) \cap \alpha$ and $B_x(\eta) \cap \alpha$ are open-and-closed in $\mathfrak{T}_{x\alpha}$ for every $\beta \in \Omega$, $n \in \mathbb{N}$ and $\eta \in \omega_1$, so that $\mathfrak{T}_{x\alpha}$ is the subspace topology on $\alpha$ induced by $\mathfrak{T}_x$. Now $B_x(\xi)$ is $\mathfrak{T}_{x\alpha}$-compact whenever $\alpha \in \Omega$, $n \in \mathbb{N}$ and $\xi < \alpha$. **P** Induce on $\alpha$. For $\alpha = \omega$, every $B_x(\xi)$ is a singleton, so the induction starts. If $\alpha = \beta + \omega$, where $\beta \in \Omega$, and $\xi < \alpha$, either $\xi < \beta$ so $B_x(\xi) \subseteq \beta$ is $\mathfrak{T}_{x\beta}$-compact, therefore $\mathfrak{T}_{x\alpha}$-compact, or $\xi = \alpha + n$ for some $n$. In this case, $B_x(e_\beta(n))$ is always $\mathfrak{T}_{x\beta}$-compact and $\mathfrak{T}_{x\beta}$-open, so $\langle C_x(\beta, i) \rangle_{i\in\mathbb{N}}$ is a partition of $\beta$ into $\mathfrak{T}_{x\beta}$-compact and $\mathfrak{T}_{x\beta}$-open sets, and $B_x(\beta + n)$ must be $\mathfrak{T}_{x,\beta+\omega}$-compact. If $\alpha$ is a limit member of $\Omega$, then for every $\xi < \alpha$ there is a $\beta \in \Omega$ such that $\xi < \beta < \alpha$, and the inductive hypothesis tells us that $B_x(\xi)$ is $\mathfrak{T}_{x\beta}$-compact, therefore $\mathfrak{T}_{x\alpha}$-compact. Thus the induction proceeds. **Q**

It follows that every $B_x(\xi)$ is $\mathfrak{T}_x$-compact, so that $\mathfrak{T}_x$ is locally compact, for every $x \in X$.

**(d)** The essential fact to note about this construction is that $\{x : \xi \in B_x(\eta)\}$, $\{x : \xi \in C_{\beta,n}\}$ belong to $\Sigma_\alpha$ whenever $\alpha \in \Omega$, $\xi < \omega_1$, $\eta < \alpha$, $\beta \in \alpha \cap \Omega$ and $n \in \mathbb{N}$.

**(e) ?** Suppose, if possible, that for every $x \in X$ there is a set $D_x \subseteq \omega_1$ which is $\mathfrak{T}_x$-closed but neither countable nor cocountable. For $\xi < \omega_1$ set

$H_\xi = \{x : \xi \in D_x\} \subseteq X$. Then there is a countable set $\Gamma_\xi \subseteq \lambda$ such that $H_\xi \in \Sigma_{\Gamma_\xi}$. Note also that, for any $\xi < \omega_1$, $\bigcup_{\eta>\xi} H_\eta = X$, so there is a countable $\Delta_\xi \subseteq \omega_1 \setminus \xi$ such that $\bigcup_{\eta\in\Delta_\xi} H_\eta$ is conegligible. Let $A$ be

$$\{\alpha : \alpha \in \Omega,\ (\Gamma_\xi \cap \omega_1) \cup \Delta_\xi \subseteq \alpha \text{ for every } \xi < \alpha\};$$

then $A$ is a club set in $\omega_1$. Take any $\alpha \in A$. Set $\Gamma = \bigcup_{\xi<\alpha} \Gamma_\xi$, so that $\Gamma \cap \omega_1 \subseteq \alpha$.

Let $Y$ be $\{y : y \in X, D_y \cap \alpha$ is cofinal with $\alpha\}$, so that $Y \supseteq \bigcap_{\xi<\alpha} \bigcup_{\eta\in\Delta_\xi} H_\eta$ is conegligible. For every $x \in X$, $D_x$ does not include $\omega_1 \setminus \alpha$, so there is a least $\zeta_x \ge \alpha$ such that $\zeta_x \notin D_x$; let $\zeta \in \omega_1 \setminus \alpha$ be the least ordinal such that $Q = \{x : \zeta_x = \zeta\}$ is not negligible. Set $Y_1 = \{y : y \in Y, \zeta_y \ge \zeta\}$, so that $Y_1$ is conegligible. Express $\zeta$ as $\beta+n$ where $\beta\in\Omega$, $n\in\mathbb{N}$. Observe that $H_\xi$ is conegligible, so belongs to $\Sigma_\Gamma$, for every $\xi\in\beta\setminus\alpha$, so $H_\xi\in\Sigma_\Gamma$ for every $\xi<\beta$.

For $x\in X$, set $I_x=\{m:m\in\mathbb{N},\ D_x\cap C_x(\beta,m)\ne\emptyset\}$. Observe that

$$\{x:m\in I_x\}=\bigcup_{\xi<\beta}\{x:\xi\in D_x\cap C_x(\beta,m)\}\in\Sigma_{\Gamma\cup\beta}$$

for every $m\in\mathbb{N}$. If $y\in Y_1$, $D_y\cap\beta$ is cofinal with $\beta$, whether or not $\beta=\alpha$, because $D_y\cap\alpha$ is cofinal with $\alpha$ and $\beta\setminus\alpha\subseteq D_y$. Since

$$\sup C_y(\beta,m)\le\sup B_y(e_\beta(m))=e_\beta(m)<\beta$$

for every $m$, $I_y$ is infinite, for every $y\in Y_1$.

On the other hand, setting $F_m=\{x:2^{-n-1}\le\phi_{\beta m}(x)<2^{-n}\}$ for $m\in\mathbb{N}$, $\langle F_m\rangle_{m\in\mathbb{N}}$ is a stochastically independent sequence of sets of measure $2^{-n-1}$ all belonging to $\Sigma_{(\beta+\omega)\setminus\beta}$, which is independent from $\Sigma_{\Gamma\cup\beta}$. Set $J_x=\{m:x\in F_m\}$. Then

$$G=\{x:I_x\text{ is infinite, }I_x\cap J_x\text{ is finite}\}$$

is negligible. **P** Set $h_1(x)=I_x$, $h_2(x)=J_x$, $h(x)=(I_x,J_x)$, so that $h_1$ and $h_2$ are functions from $X$ to $\mathcal{P}\mathbb{N}$. Let $\mu_1$, $\mu_2$ be the Radon measures on $\mathcal{P}\mathbb{N}$ such that $h_1$ and $h_2$ are inverse-measure-preserving. Then $h_1$ is inverse-measure-preserving for $\nu\restriction\Sigma_{\Gamma\cup\beta}$ and $\mu_1$, and $h_2$ is inverse-measure-preserving for $\nu\restriction\Sigma_{(\beta\cup\omega)\setminus\beta}$ and $\mu_2$, so $h$ is inverse-measure-preserving for $\nu$ and the product measure $\mu_1\times\mu_2$ on $\mathcal{P}\mathbb{N}\times\mathcal{P}\mathbb{N}$. Now the set

$$\{(I,J):I\in[\mathbb{N}]^\omega,\ J\subseteq\mathbb{N},\ I\cap J\text{ is finite}\}$$

is a Borel set in $\mathcal{P}\mathbb{N}\times\mathcal{P}\mathbb{N}$ which has $\mu_2$-negligible vertical sections, so is $\mu_1\times\mu_2$-negligible, and its inverse image $G$ is $\nu$-negligible. **Q**

There is therefore some $y\in Q\cap Y_1\setminus G$. But in this case $\zeta\notin D_y$, so $B_y(\zeta)\cap D_y$ is a $\mathfrak{T}_y$-compact set not containing $\zeta$. Since

$$B_y(\zeta)=\{\zeta\}\cup\bigcup_{m\in J_y}C_y(\beta,m),$$

there must be a finite set $K\subseteq J_y$ such that $B_y(\zeta)\cap D_y\subseteq\bigcup_{m\in K}C_y(\beta,m)$. Since $I_y$ is infinite and $y\notin G$, there is an $m\in I_y\cap J_y\setminus K$, and $C_y(\beta,m)\cap D_y$ is not empty; which is impossible. **X**

This contradiction shows that for at least one $x\in X$, every $\mathfrak{T}_x$-closed set is either countable or cocountable.

**Remarks** This proposition is based on ideas of J.Moore.

Observe that, for a topology $\mathfrak{T}$ as described, $\omega_1$ is hereditarily separable. **P?** If $A\subseteq\omega_1$ is not separable, of course it is uncountable. For $\xi<\omega_1$, choose $\alpha_\xi$, $\beta_\xi$ in $A$, and countable open sets $G_\xi\subseteq\omega_1$ inductively, as follows. Given $\langle\alpha_\eta\rangle_{\eta<\xi}$, let $\beta_\xi\in A$ be such that $\beta_\xi\notin\overline{\{\alpha_\eta:\eta<\xi\}\cup(A\cap\xi)}$. Let $G_\xi$ be a countable open set containing $\beta_\xi$ and not meeting $\{\alpha_\eta:\eta<\xi\}$. Given $\langle G_\eta\rangle_{\eta\le\xi}$, take $\alpha_\xi\in A\setminus(\xi\cup\bigcup_{\eta\le\xi}G_\eta)$.

Now consider $F=\overline{\{\alpha_\xi:\xi<\omega_1\}}$. Then $\alpha_\xi\in F$ and $\beta_\xi\notin F$ for every $\xi$. Since $\alpha_\xi$, $\beta_\xi\ge\xi$ for every $\xi$, $F$ is neither countable nor cocountable, which is impossible. **XQ**

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## S7. Partially ordered sets

**S7A Cofinalities III: Lemma** Let $\kappa$ be a real-valued-measurable cardinal and $\langle\alpha_i\rangle_{i\in I}$ a countable family of ordinals less than $\kappa$ and of cofinality at least $\omega_2$. Then there is a set $F\subseteq\prod_{i\in I}\alpha_i$ such that (i) $F$ is cofinal with $\prod_{i\in I}\alpha_i$ (ii) if $\langle f_\xi\rangle_{\xi<\omega_1}$ is an increasing family in $F$ then $\sup_{\xi<\omega_1}f_\xi\in F$ (iii) $\#(F)<\kappa$.

**proof** We have $\mathrm{cf}(\prod_{i\in I}\alpha_i)=\mathrm{cf}(\prod_{i\in I}\mathrm{cf}(\alpha_i))<\kappa$, by 7Kb. So we may find a cofinal set $F_0\subseteq\prod_{i\in I}\alpha_i$ of cardinal less than $\kappa$. Now for $0<\zeta\le\omega_2$ define $F_\zeta$ by saying that

$$F_{\zeta+1}=\{\sup_{\xi<\omega_1}f_\xi:\langle f_\xi\rangle_{\xi<\omega_1}\text{ is an increasing family in }F_\zeta\},$$
$$F_\zeta=\bigcup_{\eta<\zeta}F_\eta\text{ for non-zero limit ordinals }\zeta\le\omega_2.$$

Then $\#(F_\zeta)<\kappa$ for every $\zeta$. **P** Induce on $\zeta$. For the inductive step to $\zeta+1$, **?** suppose, if possible, that $\#(F_\zeta)<\kappa$ but $\#(F_{\zeta+1})\ge\kappa$. For each $h\in F_{\zeta+1}$ choose an increasing family $\langle f_{h\xi}\rangle_{\xi<\omega_1}$ in $F_\zeta$ with supremum $h$. The set $h[I]$ of values of $h$ is a countable subset of $Y=\bigcup_{i\in I}\alpha_i$. By 5A, taking $X=F_{\zeta+1}$ and $\mu$ a non-trivial $\kappa$-additive measure on $X$, there is a set $H\subseteq F_{\zeta+1}$, of cardinal $\kappa$, such that $M=\bigcup_{h\in H}h[I]$ is countable. Now, for each $h\in H$, there is a $\gamma(h)<\omega_1$ such that whenever $i\in I$ and $\beta\in M$ then $h(i)>\beta$ iff

$f_{h,\gamma(h)}(i) > \beta$. If $g$, $h \in H$ and $i \in I$ and $g(i) < h(i)$, then $f_{g,\gamma(g)}(i) \le g(i) < f_{h,\gamma(h)}(i)$, because $g(i) \in M$. Thus $h \mapsto f_{h,\gamma(h)} : H \to F_\zeta$ is injective; but $\#(F_\zeta) < \kappa = \#(H)$. **X**

Thus $\#(F_{\zeta+1}) < \kappa$ if $\#(F_\zeta) < \kappa$. At limit ordinals $\zeta$ the induction proceeds without difficulty because $\mathrm{cf}(\kappa) > \zeta$. **Q**

So $\#(F_{\omega_2}) < \kappa$ and we may take $F = F_{\omega_2}$.

**S7B Theorem** Let $\kappa$ be a real-valued-measurable cardinal.

(a) For any cardinal $\theta$, $\mathrm{cf}([\kappa]^{<\theta}) \le \kappa$.

(b) For any cardinal $\lambda < \kappa$, any $\theta$, $\mathrm{cf}([\lambda]^{<\theta}) < \kappa$.

**proof (a)(i)** Consider first the case $\theta = \omega_1$. Write $G_1$ for the set of ordinals less than $\kappa$ of cofinality less than or equal to $\omega_1$; for $\delta \in G_1$ let $\psi_\delta : \mathrm{cf}(\delta) \to \delta$ enumerate a cofinal subset of $\delta$. Next, write $G_2$ for $\kappa \setminus G_1$, and for every countable set $A \subseteq G_2$ let $F(A) \subseteq \prod_{\alpha \in A} \alpha$ be a cofinal set, of cardinal less than $\kappa$, closed under suprema of increasing families of length $\omega_1$; such exists by S7A above.

**(ii)** It is worth observing at this point that if $\langle A_\zeta \rangle_{\zeta<\omega_1}$ is any family of countable subsets of $G_2$, $D = \bigcup_{\zeta<\omega_1} A_\zeta$, and $g \in \prod_{\alpha \in D} \alpha$, then there is an $f \in \prod_{\alpha \in D} \alpha$ such that $f \ge g$ and $f \restriction A_\zeta \in F(A_\zeta)$ for every $\zeta < \omega_1$. **P** Let $\langle \phi(\xi) \rangle_{\xi<\omega_1}$ run over $\omega_1$ taking every value uncountably often. Choose an increasing family $\langle f_\xi \rangle_{\xi<\omega_1}$ in $\prod_{\alpha \in D} \alpha$ in such a way that $f_0 = g$ and $f_{\xi+1} \restriction A_{\phi(\xi)} \in F(A_{\phi(\xi)})$ for every $\xi$; this is possible because $F(A)$ is cofinal with $\prod_{\alpha \in A} \alpha$ for every $A$. Set $f = \sup_{\xi<\omega_1} f_\xi$; this works because every $F(A)$ is closed under suprema of increasing families of length $\omega_1$. **Q**

**(iii)** We can now find a family $\mathcal{A}$ of countable subsets of $\kappa$ such that

($\alpha$) $\{\alpha\} \in \mathcal{A}$ for every $\alpha < \kappa$;

($\beta$) whenever $A$, $A' \in \mathcal{A}$, $\zeta < \omega_1$ then $A \cup A'$, $A \cap G_2$, $\{\psi_\alpha(\xi) : \alpha \in A \cap G_1,\ \xi < \min(\zeta, \mathrm{cf}(\alpha))\}$ all belong to $\mathcal{A}$;

($\gamma$) whenever $A \in \mathcal{A}$, $A \subseteq G_2$ then $f[A] \in \mathcal{A}$ for every $f \in F(A)$;

($\delta$) $\#(\mathcal{A}) \le \kappa$.

**(iv) ?** Suppose, if possible, that $\mathrm{cf}([\kappa]^{\le\omega}) > \kappa$. Because $[\kappa]^{\le\omega} = \bigcup_{\lambda<\kappa} [\lambda]^{\le\omega}$, there is a cardinal $\lambda < \kappa$ such that $\mathrm{cf}([\lambda]^{\le\omega}) > \kappa$. We can therefore choose inductively a family $\langle a_\xi \rangle_{\xi<\kappa}$ of countable subsets of $\lambda$ such that

$$a_\xi \not\subseteq \bigcup_{\eta \in A \cap \xi} a_\eta$$

whenever $\xi < \kappa$, $A \in \mathcal{A}$. By 5A, there is a set $W \subseteq \kappa$, of cardinal $\kappa$, such that $\bigcup_{\xi \in W} a_\xi$ is countable. Let $\delta < \kappa$ be such that $W \cap \delta$ is cofinal with $\delta$ and of order type $\omega_1$.

**(v)** I choose a family $\langle A_{k\zeta} \rangle_{\zeta<\omega_1, k \in \mathbb{N}}$ in $\mathcal{A}$ as follows. Start by setting $A_{0\zeta} = \psi_\delta[\zeta]$ for every $\zeta < \omega_1$; then $A_{0\zeta} \in \mathcal{A}$ by (iii)($\alpha$-$\beta$). Given $\langle A_{k\zeta} \rangle_{\zeta<\omega_1}$, set $A'_{k\zeta} = A_{k\zeta} \cap G_2$ for each $\zeta < \omega_1$. For $\alpha \in D_k = \bigcup_{\zeta<\omega_1} A'_{k\zeta}$, set $g_k(\alpha) = \sup(\alpha \cap W \cap \delta) < \alpha$; choose $f_k \in \prod_{\alpha \in D_k} \alpha$ such that $g_k \le f_k$ and $f_k \restriction A'_{k\zeta} \in F(A'_{k\zeta})$ for every $\zeta$; this is possible by (ii) above. Set

$$A_{k+1,\zeta} = A_{k\zeta} \cup f_k[A'_{k\zeta}] \cup \{\psi_\alpha(\xi) : \alpha \in A_{k\zeta} \cap G_1,\ \xi < \min(\zeta, \mathrm{cf}(\alpha))\} \in \mathcal{A}$$

for each $\zeta < \omega_1$, and continue. An easy induction on $k$ shows that $\langle A_{k\zeta} \rangle_{\zeta<\omega_1}$ is increasing for every $k$; also, every $A_{k\zeta}$ is a subset of $\delta$.

**(vi)** Set $V_k = \bigcup_{\zeta<\omega_1} A_{k\zeta}$, $b_k = \bigcup\{a_\xi : \xi \in W \cap V_k\}$; then $b_k$ is countable and there is a $\beta(k) < \omega_1$ such that $b_k = \bigcup\{a_\xi : \xi \in W \cap A_{k,\beta(k)}\}$. Now $\bigcup_{k \in \mathbb{N}} A_{k,\beta(k)}$ is a countable subset of $\delta$, so there is a member $\gamma$ of $W \cap \delta$ greater than its supremum. We have

$$a_\gamma \not\subseteq \bigcup\{a_\eta : \eta \in A_{k,\beta(k)}\}$$

for every $k$, so $a_\gamma \not\subseteq b_k$ and $\gamma \notin V_k$, for each $k$.

Set $V = \bigcup_{k \in \mathbb{N}} V_k$, $\gamma_0 = \min(W \cap \delta \setminus V)$. Because $V_0 = \psi_\delta[\omega_1]$ is cofinal with $\delta$, $V \setminus \gamma_0 \ne \emptyset$; let $\gamma_1$ be its least member. Then $\gamma_1 > \gamma_0$. Suppose $\gamma_1 \in A_{k\zeta}$. Observe that if $\alpha \in V \cap G_1$ then $V \cap \alpha$ is cofinal with $\alpha$; but $V \cap \gamma_1 \subseteq \gamma_0$, so $\gamma_1 \notin G_1$ and $\gamma_1 \in D_k$. But now $f_k(\gamma_1) \in A_{k+1,\zeta} \subseteq V$ and $\gamma_0 \le g_k(\gamma_1) \le f_k(\gamma_1) < \gamma_1$, so $\gamma_1 \ne \min(V \setminus \gamma_0)$. **X**

**(vii)** This contradiction shows that $\mathrm{cf}([\kappa]^{\le\omega}) \le \kappa$. Now consider $\mathrm{cf}([\kappa]^{\le\delta})$, where $\delta < \kappa$ is an infinite cardinal. Then

$$\mathrm{cov}_{\mathrm{Sh}}(\kappa, \delta^+, \delta^+, \omega_1) = \sup_{\lambda<\kappa} \mathrm{cov}_{\mathrm{Sh}}(\lambda, \delta^+, \delta^+, \omega_1) \le \kappa$$

by 7Kd. (See A1J for the definition of $\mathrm{cov}_{\mathrm{Sh}}$.) So there is a family $\mathcal{B} \subseteq [\kappa]^{\le\delta}$, of cardinal at most $\kappa$, such that every member of $[\kappa]^{\le\delta}$ is covered by a sequence in $\mathcal{B}$. But now there is a family $\mathfrak{C}$ of countable subsets of $\mathcal{B}$ which is cofinal with $[\mathcal{B}]^{\le\omega}$ and of cardinal at most $\kappa$; setting $\mathcal{D} = \{\bigcup \mathcal{C} : \mathcal{C} \in \mathfrak{C}\}$, we have $\mathcal{D}$ cofinal with $[\kappa]^{\le\delta}$ and of cardinal at most $\kappa$. So $\mathrm{cf}([\kappa]^{\le\delta}) \le \kappa$.

Finally, of course, $[\kappa]^{<\theta} = \bigcup_{\delta<\theta}[\kappa]^{\le\delta}$, so $\mathrm{cf}([\kappa]^{<\theta}) \le \sup_{\delta<\theta} \mathrm{cf}([\kappa]^{\le\delta}) \le \kappa$ whenever $\theta \le \kappa$. For $\theta > \kappa$ we have $\mathrm{cf}([\kappa]^{<\theta}) = 1$, so $\mathrm{cf}([\kappa]^{<\theta}) \le \kappa$ for every $\theta$.

**(b)** If $\mathcal{A}$ is cofinal with $[\kappa]^{<\theta}$ then $\{A \cap \lambda : A \in \mathcal{A}\}$ is cofinal with $[\lambda]^{<\theta}$, so $\mathrm{cf}([\lambda]^{<\theta}) \le \mathrm{cf}([\kappa]^{\theta}) \le \kappa$, by (a). But by 7Id and 7Jb, $\mathrm{cf}([\lambda]^{<\theta}) \ne \kappa$. Thus $\mathrm{cf}([\lambda]^{<\theta}) < \kappa$, as claimed.

**Remark** This result is taken from Shelah #430.

**S7C Corollary** Let $\kappa$ be a real-valued-measurable cardinal. Let $\langle P_\zeta\rangle_{\zeta<\lambda}$ be a family of partially ordered sets such that $\lambda < \mathrm{add}(P_\zeta) \le \mathrm{cf}(P_\zeta) < \kappa$ for every $\zeta < \kappa$. Then $\mathrm{cf}(\prod_{\zeta<\lambda} P_\zeta) < \kappa$.

**proof** For each $\zeta < \lambda$ let $Q_\zeta$ be a cofinal subset of $P_\zeta$ of cardinal less than $\kappa$. Set $P = \prod_{\zeta<\kappa} P_\zeta$, $Z = \bigcup_{\zeta<\lambda} Q_\zeta$; then $\#(Z) < \kappa$ so $\mathrm{cf}([Z]^{\le\lambda}) < \kappa$, by S7Bb. Let $\mathcal{A}$ be a cofinal subset of $[Z]^{\le\lambda}$ with $\#(\mathcal{A}) < \kappa$. For each $A \in \mathcal{A}$ choose $f_A \in P$ such that $f_A(\zeta)$ is an upper bound for $A \cap P_\zeta$ for every $\zeta$; this is possible because $\mathrm{add}(P_\zeta) > \#(A)$. Set $F = \{f_A : A \in \mathcal{A}\}$.

If $g \in P$, then there is an $h \in \prod_{\zeta<\lambda} Q_\zeta$ such that $g \le h$. Now $h[\lambda] \in [Z]^{\le\lambda}$ so there is an $A \in \mathcal{A}$ such that $h[\lambda] \subseteq A$. In this case $h \le f_A$. Accordingly $F$ is cofinal with $P$ and $\mathrm{cf}(P) \le \#(F) < \kappa$, as required.

**S7D** The techniques of 7C can be adapted to give a further combinatorial result.

**Theorem** Let $\kappa$ be an atomlessly-measurable cardinal, and $\lambda < \kappa$ an infinite cardinal. Then there is a function $f : [\lambda^+]^2 \to \mathbb{N}$ such that whenever $\mathcal{I}$ is a disjoint family of finite subsets of $\lambda^+$ with $\#(\mathcal{I}) = \lambda^+$, and $k \in \mathbb{N}$, there are distinct $I, J \in \mathcal{I}$ such that $f(\{\xi,\eta\}) = k$ for every $\xi \in I$, $\eta \in J$.

**proof (a)** For $\xi < \lambda^+$ let $e_\xi : \xi \to \lambda$ be an injective function. Give $\mathbb{N}$ the Radon measure $\mu_0$ defined by saying that $\mu_0\{n\} = 2^{-n-1}$ for every $n \in \mathbb{N}$, and give $\mathbb{N}^\lambda$ the product measure $\mu$. Then $\mu$ is an atomless Radon probability measure with Maharam type $\lambda$, so there is a $\kappa$-additive measure $\nu$, defined on every subset of $\mathbb{N}^\lambda$, extending $\mu$ (3I).

For $w \in \mathbb{N}^\lambda$, define $f_w : [\lambda^+]^2 \to \mathbb{N}$ by setting

$$f_w(\{\xi,\eta\}) = w(e_\eta(\xi))$$

whenever $\xi < \eta < \lambda^+$.

**(b) ?** Suppose, if possible, that no $f_w$ witnesses the truth of the theorem. Then for every $w \in \mathbb{N}^\lambda$ there are a $k_w \in \mathbb{N}$ and a disjoint $\mathcal{I}_w \subseteq [\lambda^+]^{<\omega}$ such that $\#(\mathcal{I}_w) = \lambda^+$ and whenever $I$, $J$ are distinct members of $\mathcal{I}_w$ there are $\xi \in I$, $\eta \in J$ such that $f_w(\{\xi,\eta\}) \ne k_w$. Take $k \in \mathbb{N}$ such that $\nu R > 0$, where

$$R = \{w : w \in \mathbb{N}^\lambda,\ k_w = k\}.$$

For every $\alpha < \lambda^+$, $w \in \mathbb{N}^\lambda$ there is an $I \in \mathcal{I}_w$ such that $I \cap \alpha = \emptyset$; because $\#([\lambda^+ \setminus \alpha]^{<\omega}) < \kappa$, there is an $I \subseteq \lambda^+ \setminus \alpha$ such that $\nu\{w : w \in R,\ I \in \mathcal{I}_w\} > 0$. We may therefore choose $\langle I_\alpha\rangle_{\alpha<\lambda^+}$ inductively so that

$$\xi < \eta \text{ whenever } \xi \in I_\alpha,\ \eta \in I_\beta \text{ and } \alpha < \beta < \lambda^+,$$

$$\nu S_\alpha > 0 \text{ for every } \alpha < \lambda,$$
$$\text{where } S_\alpha = \{w : w \in R,\ I_\alpha \in \mathcal{I}_w\}.$$

Take $\delta > 0$, $n \in \mathbb{N}$ such that $\#(A) = \lambda^+$, where

$$A = \{\alpha : \alpha < \lambda^+,\ \#(I_\alpha) = n,\ \nu S_\alpha \ge \delta\}.$$

Take $l \in \mathbb{N}$ such that $(1 - 2^{-(k+1)n^2})^l \le \frac{1}{3}\delta$.

**(b)** By A2S, there is a set $B \subseteq A$, of cardinal $\lambda^+$, such that $\mu^*(\bigcap_{\alpha\in L} S_\alpha) \ge \frac{2}{3}\delta$ whenever $L \subseteq B$ and $\#(L) \le l+1$. Take $\beta \in B$ such that $B \cap \beta$ is infinite. Now for each $\eta \in I_\beta$, $e_\eta : \eta \to \lambda$ is injective. Accordingly we can find a set $L \subseteq B \cap \beta$ such that $\#(L) = l$ and

$$e_\eta[I_\alpha] \cap e_{\eta'}[I_{\alpha'}] = \emptyset$$

whenever $\alpha$, $\alpha'$ are distinct members of $L$ and $\eta$, $\eta' \in I_\beta$. For $\alpha \in L$ set

$$K_\alpha = \{e_\eta(\xi) : \xi \in I_\alpha,\, \eta \in I_\beta\};$$

then $\langle K_\alpha\rangle_{\alpha\in L}$ is disjoint and $\#(K_\alpha) \le n^2$ for every $\alpha$.

Consider

$$E = \{w : w \in \mathbb{N}^\lambda,\, \forall\, \alpha \in L\, \exists\, \zeta \in K_\alpha \text{ such that } w(\zeta) \ne k\}.$$

Then

$$\mu E = \prod_{\alpha\in L}(1 - 2^{-(k+1)\#(K_\alpha)}) \le \tfrac{1}{3}\delta.$$

On the other hand, if $w \in S_\beta \cap \bigcap_{\alpha\in L} S_\alpha$, $I_\beta \in \mathcal{I}_w$ and $I_\alpha \in \mathcal{I}_w$ for every $\alpha \in L$. So for every $\alpha \in L$ there are $\xi \in I_\alpha$, $\eta \in I_\beta$ such that $f_w(\{\xi,\eta\}) \ne k_w$, that is, $w(e_\eta(\xi)) \ne k$; and we have a $\zeta \in K_\alpha$ such that $w(\zeta) \ne k$. Thus

$$\bigcap_{\alpha\in L\cup\{\beta\}} S_\alpha \subseteq E,$$

and (by the choice of $B$) we get

$$\tfrac{2}{3}\delta \le \mu^*(\bigcap_{\alpha\in L\cup\{\beta\}} S_\alpha) \le \mu E \le \tfrac{1}{3}\delta,$$

which is absurd. **X**

**(d)** Thus there is a $w \in \mathbb{N}^\lambda$ for which $f_w$ is a suitable function.

**S7E Corollary** Let $\kappa$ be an atomlessly-measurable cardinal, and $\lambda < \kappa$ an infinite cardinal. Then there is a function $g : [\lambda^+]^2 \to \mathbb{N}$ such that whenever $\mathcal{I}$ is a disjoint family of finite subsets of $\lambda^+$ with $\#(\mathcal{I}) = \lambda^+$, and $h : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ is a function, there are distinct $I$, $J \in \mathcal{I}$ such that $g(\{\xi,\eta\}) = h(\#(I\cap\xi), \#(J\cap\eta))$ whenever $\xi \in I$, $\eta \in J$ and $\xi < \eta$.

**proof** Take $f : [\lambda^+]^2 \to \mathbb{N}$ from S7D. Enumerate as $\langle h_k\rangle_{k\in\mathbb{N}}$ the set of all maps from finite products $\{0,1\}^m \times \{0,1\}^m$ to $\mathbb{N}$; take the domain of $h_k$ to be $\{0,1\}^{m_k}\times\{0,1\}^{m_k}$. Choose a family $\langle w_\xi\rangle_{\xi<\lambda^+}$ of distinct elements of $\{0,1\}^{\mathbb{N}}$. Now, given $\xi < \eta < \lambda^+$, set $k = f(\{\xi,\eta\})$ and try $g(\{\xi,\eta\}) = h_k(w_\xi{\restriction}m_k, w_\eta{\restriction}m_k)$.

Suppose that $\mathcal{I}$ is a disjoint family of finite subsets of $\lambda^+$ with cardinal $\lambda^+$, and that $h : \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ is any function. Let $m$, $n$ be such that

$$\mathcal{J} = \{I : I \in \mathcal{I},\, \#(I) = n,\, w_\xi{\restriction}m \ne w_\eta{\restriction}m \text{ for all distinct } \xi,\, \eta \in I\}$$

has cardinal $\lambda^+$, and $\langle v_i\rangle_{i<n}$ such that

$$\mathcal{K} = \{I : I \in \mathcal{J},\, w_\xi{\restriction}m = v_i \text{ whenever } \xi \in I,\, \#(\xi\cap I) = i < n\}$$

has cardinal $\lambda^+$. Take $k \in \mathbb{N}$ such that $m_k = m$ and $h_k(v_i, v_j) = h(i,j)$ for all $i$, $j < n$. (Note that $v_i \ne v_j$ if $i \ne j$, because $w_\xi{\restriction}m \ne w_\eta{\restriction}m$ if $\xi$, $\eta$ are distinct members of any $I \in \mathcal{J}$.) Then there are distinct $I$, $J \in \mathcal{K}$ such that $f(\{\xi,\eta\}) = k$ for $\xi \in I$, $\eta \in J$. So if $\xi \in I$, $\eta \in J$, $\xi < \eta$ we have

$$g(\{\xi,\eta\}) = h_k(w_\xi{\restriction}m, w_\eta{\restriction}m) = h_k(v_i, v_j) = h(i,j),$$

where $i = \#(I\cap\xi)$, $j = \#(J\cap\eta)$. And this is what we need.

**S7F Corollary** If there is an atomlessly-measurable cardinal, there is a ccc Hausdorff abelian topological group whose square is not ccc.

**proof (a)** Let $g : [\omega_1]^2 \to \mathbb{N}$ be such that whenever $\mathcal{I}$ is an uncountable disjoint family of finite subsets of $\omega_1$, and $h : \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ is a function, there are distinct $I$, $J \in \mathcal{I}$ such that $g(\{\xi,\eta\}) = h(\#(I\cap\xi), \#(J\cap\eta))$ whenever $\xi \in I$, $\eta \in J$ and $\xi < \eta$. Set $X = \omega_1 \times \{0,1\}$, and let $G$ be the family of finite subsets of $X$ with the group operation $a + b = a\triangle b$. For $\beta < \omega_1$ let $H_\beta$ be the set of those $a \in G$ such that

$(\beta,0)$, $(\beta,1) \notin a$,

if $\alpha < \beta$ and $g(\{\alpha,\beta\}) = i \le 1$ then $(\alpha,i) \notin a$,

if $\alpha < \beta$ and $g(\{\alpha,\beta\}) \ge 2$ then $(\alpha,0) \in a$ iff $(\alpha,1) \in a$.

Note that $H_\beta$ is a subgroup of $G$. Give $G$ the topology generated by the subgroups $H_\beta$ and their cosets for $\beta < \omega_1$. This is a topological-group topology, and because $\bigcap_{\beta<\omega_1} H_\beta = \{\emptyset\}$ it is Hausdorff.

**(b)** It may make the idea behind this definition clearer if I show first that $G\times G$ is not ccc. **P** Consider

$$W_\xi = (\{(\xi,0)\}+H_\xi)\times(\{(\xi,1)\}+H_\xi)$$

for $\xi<\omega_1$.

If $\xi<\eta<\omega_1$ and $a\in(\{(\xi,0)\}+H_\xi)\cap(\{(\eta,0)\}+H_\eta)$, then $a+\{(\xi,0)\}\in H_\xi$, so $(\xi,0)\in a$ and $(\xi,1)\notin a$. But now $(\xi,0)\in a+\{(\eta,0)\}$ and $(\xi,1)\notin a+\{(\eta,0)\}$; since $a+\{(\eta,0)\}\in H_\eta$, we must have $g(\{\xi,\eta\})=1$.

If $\xi<\eta<\omega_1$ and $b\in(\{(\xi,1)\}+H_\xi)\cap(\{(\eta,1)\}+H_\eta)$, then $b+\{(\xi,1)\}\in H_\xi$, so $(\xi,0)\notin b$ and $(\xi,1)\in b$. But now $(\xi,0)\notin b+\{(\eta,1)\}$ and $(\xi,1)\in b+\{(\eta,1)\}$; since $b+\{(\eta,1)\}\in H_\eta$, we must have $g(\{\xi,\eta\})=0$.

Since these cannot happen together, $W_\xi\cap W_\eta=\emptyset$ whenever $\xi<\eta<\omega_1$, and we have an uncountable disjoint family of open subsets of $G\times G$. **Q**

**(c)** $G$ is ccc. **P** Let $\langle U_\xi\rangle_{\xi<\omega_1}$ be a family of non-empty open subsets of $G$. For each $\xi<\omega_1$ take $a_\xi\in U_\xi$, $I_\xi\in[\omega_1]^{<\omega}$ such that $a_\xi+\bigcap_{\beta\in I_\xi}H_\beta\subseteq U_\xi$. Let $A\subseteq\omega_1$ be an uncountable set such that $\langle a_\xi\rangle_{\xi\in A}$, $\langle I_\xi\rangle_{\xi\in A}$ are $\Delta$-systems with roots $a$, $I$ respectively; set $a'_\xi=a_\xi\setminus a$ for $\xi\in A$. Set

$$J=I\cup\{(\alpha,0):\alpha\in a\}\cup\{(\alpha,1):\alpha\in a\},$$

$$J_\xi=(I_\xi\setminus I)\cup\{\alpha:(\alpha,0)\in a'_\xi\}\cup\{\alpha:(\alpha,1)\in a'_\xi\},$$

$$m_\xi=\#(J_\xi),$$

and enumerate $J_\xi$ in ascending order as $\langle\alpha_{\xi i}\rangle_{i<m_\xi}$. Define $h_\xi:m_\xi\to\{0,1,2\}$ by setting

$$\begin{aligned}h_\xi(i)&=0\text{ if }(\alpha_{\xi i},0)\notin a'_\xi,\ (\alpha_{\xi i},1)\in a'_\xi,\\&=1\text{ if }(\alpha_{\xi i},0)\in a'_\xi,\ (\alpha_{\xi i},1)\notin a'_\xi,\\&=2\text{ otherwise.}\end{aligned}$$

Let $m$, $h:m\to\{0,1,2\}$ be such that

$$B=\{\xi:\xi\in A,\ m_\xi=m,\ h_\xi=h\}$$

is uncountable. Because any $\alpha<\omega_1$ can belong to at most three of the $J_\xi$, there is an uncountable $C\subseteq B$ such that (i) $\alpha<\beta$ whenever $\alpha\in J$ and $\beta\in J_\xi$ for some $\xi\in C$ (ii) $\alpha<\beta$ whenever $\xi$, $\eta\in C$, $\xi<\eta$, $\alpha\in J_\xi$ and $\beta\in J_\eta$. Now, by the choice of $g$, there must be $\xi$, $\eta\in C$ such that $\xi<\eta$ and $g(\{\alpha_{\xi i},\beta\})=h(i)$ whenever $i<m$, $\beta\in J_\eta$.

What this means is that for $\alpha\in J_\xi$, $\beta\in J_\eta$,

$$\begin{aligned}g(\{\alpha,\beta\})&=0\text{ if }(\alpha,0)\notin a'_\xi,\ (\alpha,1)\in a'_\xi,\\&=1\text{ if }(\alpha,0)\in a'_\xi,\ (\alpha,1)\notin a'_\xi,\\&=2\text{ otherwise.}\end{aligned}$$

Set

$$b=a_\xi\cup a_\eta=a_\xi+a'_\eta=a_\eta+a'_\xi.$$

I seek to show that $b\in U_\xi\cap U_\eta$; it will be enough if I can show that $a'_\eta\in H_\beta$ for every $\beta\in I_\xi$ and $a'_\xi\in H_\beta$ for every $\beta\in I_\eta$.

(i) Suppose $\beta\in I$. In this case $\beta\in J$ so $\beta<\alpha$ whenever either $(\alpha,0)$ or $(\alpha,1)$ belongs to $a_\xi\triangle a_\eta=a'_\xi\cup a'_\eta$; consequently $a'_\xi$, $a'_\eta$ belong to $H_\beta$.

(ii) Suppose $\beta\in I_\xi\setminus I$. In this case $\beta<\alpha$ whenever either $(\alpha,0)$ or $(\alpha,1)$ belongs to $a'_\eta$. So $a'_\eta\in H_\beta$.

(iii) Suppose $\beta\in I_\eta\setminus I$. In this case $\alpha<\beta$ whenever either $(\alpha,0)$ or $(\alpha,1)$ belongs to $a'_\xi$. In particular, neither $(\beta,0)$ nor $(\beta,1)$ belongs to $a'_\xi$. If $\alpha<\beta$ and $g(\{\alpha,\beta\})=i\le1$, then either $\alpha\notin J_\xi$, in which case $(\alpha,i)$ certainly does not belong to $a'_\xi$, or $\alpha\in J_\xi$, in which case again $(\alpha,i)\notin a'_\xi$. While if $\alpha<\beta$ and $g(\{\alpha,\beta\})\ge2$, then either $\alpha\notin J_\xi$ and neither $(\alpha,0)$ nor $(\alpha,1)$ belongs to $a'_\xi$, or $\alpha\in J_\xi$ and $(\alpha,0)\in a'_\xi$ iff $(\alpha,1)\in a'_\xi$. Thus $a'_\xi\in H_\beta$.

Putting these together, we see that all the requirements for $b\in U_\xi\cap U_\eta$ are met, so that $U_\xi\cap U_\eta\ne\emptyset$. As $\langle U_\xi\rangle_{\xi<\omega_1}$ is arbitrary, $G$ is ccc. **Q**

**Remark** S7D-S7F are taken from §2 of TODORČEVIĆ 93.

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## Appendix: Useful Facts

### SA1 Combinatorics

**SA1A Lemma** There is a family $\langle e_\alpha\rangle_{\alpha<\omega_1}$ such that

(i) for each $\alpha$, $e_\alpha$ is an injective function from $\alpha$ to $\mathbb{N}$;

(ii) if $\alpha \le \beta < \omega_1$, $\{\xi : \xi < \alpha,\, e_\alpha(\xi) \ne e_\beta(\xi)\}$ is finite.

**proof** Construct the $e_\alpha$ inductively. The inductive hypothesis must include a third condition

(iii) $\mathbb{N} \setminus e_\alpha[\alpha]$ is infinite for every $\alpha$.

The induction starts trivially with $e_0 = \emptyset$. For the inductive step to $\beta + 1$, take any $n \in \mathbb{N} \setminus e_\beta[\beta]$, and set

$$e_{\beta+1}(\xi) = e_\beta(\xi) \text{ if } \xi < \beta,\ e_{\beta+1}(\beta) = n.$$

For the inductive step to a non-zero limit ordinal $\beta < \omega_1$, take a strictly increasing sequence $\langle\beta_n\rangle_{n\in\mathbb{N}}$ with supremum $\beta$, starting with $\beta_0 = 0$. Observe that, for any $n$,

$$A_n = \mathbb{N} \setminus \bigcup_{i\le n} e_{\beta_i}[\beta_i] \supseteq (\mathbb{N} \setminus e_{\beta_n}[\beta_n]) \setminus \{e_{\beta_i}(\xi) : i \le n,\, \xi < \beta_i,\, e_{\beta_i}(\xi) \ne e_{\beta_n}(\xi)\}$$

is infinite. So we can choose a sequence $\langle m_n\rangle_{n\in\mathbb{N}}$ of distinct integers such that $m_n \in A_n$ for every $n$. Set $B = \{m_n : n \in \mathbb{N}\}$; then $B$ is infinite and $B \cap e_{\beta_n}[\beta_n]$ is finite for every $n$.

Set $C_{mn} = \{\xi : \xi < \beta_m,\, e_{\beta_m}(\xi) \ne e_{\beta_n}(\xi)\}$ for $m \le n \in \mathbb{N}$. These sets are all finite, so

$$C_n = e^{-1}_{\beta_{n+1}}[B \cup \bigcup_{m\le n} e_{\beta_m}[C_{m,n+1}]]$$

is always finite. If $\xi \in D_n = \beta_{n+1} \setminus (\beta_n \cup C_n)$, then $e_{\beta_{n+1}}(\xi) \notin \bigcup_{m\le n} e_{\beta_m}[\beta_m]$. **P** If $m \le n$ and $\eta < \beta_m$, then either $\eta \notin C_{m,n+1}$, so that $e_{\beta_m}(\eta) = e_{\beta_{n+1}}(\eta) \ne e_{\beta_{n+1}}(\xi)$, or $\eta \in C_{m,n+1}$ so that $e_{\beta_{n+1}}(\xi) \ne e_{\beta_m}(\eta)$ because $\xi \notin e^{-1}_{\beta_{n+1}}[e_{\beta_m}[C_{m,n+1}]]$. **Q**

Set $C = \bigcup_{n\in\mathbb{N}} C_n \setminus \beta_n = \beta \setminus \bigcup_{n\in\mathbb{N}} D_n$, and let $f : C \to B$ be an injective function such that $B \setminus f[C]$ is infinite. Now define $e_\beta : \beta \to \mathbb{N}$ by setting

$$\begin{aligned} e_\beta(\alpha) &= e_{\beta_{n+1}}(\alpha) \text{ if } \alpha \in D_n,\\ &= f(\alpha) \text{ if } \alpha \in C. \end{aligned}$$

I check the three requirements of the inductive hypothesis in turn.

(i) Because $f$ and all the $e_{\beta_n}$ are injective and

$$e_{\beta_{n+1}}[D_n] \cap e_{\beta_m}[\beta_m] = e_{\beta_{n+1}}[D_n] \cap B = \emptyset$$

whenever $m \le n$, $e_\beta$ is injective.

(ii) For $n \in \mathbb{N}$, set $E_n = \{\xi : \xi < \beta_n,\, e_{\beta_n}(\xi) \ne e_\beta(\xi)\}$. Then

$$E_{n+1} \subseteq \bigcup_{m\le n}(E_m \cup C_{m,n+1}) \cup C_n$$

for each $n$, so every $E_n$ is finite. If $\alpha < \beta$, then there is some $n$ such that $\alpha \le \beta_n$, so that

$$\{\xi : \xi < \alpha,\, e_\alpha(\xi) \ne e_\beta(\xi)\} \subseteq E_n \cup \{\xi : \xi < \alpha,\, e_\alpha(\xi) \ne e_{\beta_n}(\xi)\}$$

is finite.

(iii) Because $e^{-1}_{\beta_{n+1}}[B] \subseteq C_n$, $e_{\beta_{n+1}}[D_n] \cap B = \emptyset$ for every $n$, so $e_\beta[\beta]$ does not meet $B \setminus f[C]$ is $\mathbb{N} \setminus e_\beta[\beta]$ is infinite.

Thus the induction continues.

**Remark** This is due to .

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### SA2. Measure theory

**SA2A $\tau$-additive Borel measures** Let $X$ be a regular topological space and $\mu$ a totally finite $\tau$-additive Borel measure on $X$. Then the completion $\hat{\mu}$ of $\mu$ is a quasi-Radon measure on $X$. **P** FREMLIN P00*, 415Cb. **Q**

**SA2B Disintegrations** Let $(X, \mathfrak{T})$ be a compact Hausdorff space and $(Y, \mathrm{T}, \nu)$ a complete probability space. Let $\mu$ be a probability on $X \times Y$ such that (i) $\mu(E \times F)$ is defined whenever $E \in \Sigma$, the Baire $\sigma$-algebra of $X$, and $F \in \mathrm{T}$ (ii) $\mu(X \times F) = \nu F$ for every $F \in \mathrm{T}$. Then there is a family $\langle \mu_y \rangle_{y \in Y}$ of Radon probabilities on $X$ such that $\int_F \mu_y E \nu(dy)$ exists and is equal to $\mu(E \times F)$ whenever $E \in \Sigma$ and $F \in \mathrm{T}$.

**proof** Write $\mathcal{L}^\infty$ for the space of bounded T-measurable functions from $Y$ to $\mathbb{R}$. Because $(Y, \mathrm{T}, \nu)$ is complete, there is a 'linear lifting' $\rho : \mathcal{L}^\infty \to \mathcal{L}^\infty$ such that

$\rho(f + g) = \rho(f) + \rho(g)$, $\rho(\alpha f) = \alpha\rho(f)$ for all $f$, $g \in \mathcal{L}^\infty$ and $\alpha \in \mathbb{R}$;

if $f = 0$ $\nu$-a.e. then $\rho(f) = 0$;

$\rho(f) = f$ $\nu$-a.e. for every $f \in \mathcal{L}^\infty$;

if $f \geq 0$ then $\rho(f) \geq 0$;

if $f$ is constant then $\rho(f) = f$

(IONESCU TULCEA & IONESCU TULCEA 69, p. 46, Theorem 3, or FREMLIN P00*, §341). Write $C(X)$ for the set of continuous functions $g : X \to \mathbb{R}$. For each $g \in C(X)$, we have a functional $\nu_g : \mathrm{T} \to [0, 1]$ given by the formula $\nu_g F = \int_{X \times F} g(x)\mu(dxdy)$ for every $F \in \mathrm{T}$; now $\nu_g$ is additive and dominated by $\nu$, so by the Radon-Nikodým theorem (FREMLIN 74, 63J) there is a bounded T-measurable function $f_g : Y \to \mathbb{R}$ such that $\nu_g F = \int_F f_g d\nu$ for every $F \in \mathrm{T}$. Set $f'_g = \rho(f_g)$. If $g$, $h \in C(X)$, then $\nu_{g+h} = \nu_g + \nu_h$ so $f_{g+h} = f_g + f_h$ $\nu$-a.e. and $f'_{g+h} = f'_g + f'_h$. Similarly, $f'_{\alpha g} = \alpha f_g$ for every $g \in C(X)$, $\alpha \in \mathbb{R}$. Also, if $g \geq 0$ in $C(X)$, then $\nu_g \geq 0$ so $f_g \geq 0$ a.e. and $f'_g \geq 0$.

For each $y \in Y$, define $\phi_y : C(X) \to \mathbb{R}$ by setting $\phi_y(g) = f'_g(y)$ for every $g \in C(X)$. Then $\phi_y$ is a positive linear functional. Also, writing $\mathbf{1}$ for the function with constant value 1, $\nu_{\mathbf{1}} = \nu$ so $f_{\mathbf{1}} = 1$ a.e. and $f'_{\mathbf{1}} = 1$ everywhere; thus $\phi_y(\mathbf{1}) = 1$ for every $y$. By the Riesz representation theorem (FREMLIN 74, 73D) there is for each $y \in Y$ a Radon probability $\mu_y$ on $X$ such that $\int g d\mu_y = \phi_y(g)$ for every $g \in C(X)$. Consequently, if $g \in C(X)$ and $F \in \mathrm{T}$,

$$\int_F (\int g d\mu_y)\nu(dy) = \int_F f'_g(y)\nu(dy) = \int_F f_g d\nu = \nu_g F = \int_{X \times F} g(x)\mu(dxdy).$$

Consider the set $H$ of functions $h : X \to [0, 1]$ such that

$$\int_F \bigl(\int h(x)\mu_y(dx)\bigr)\nu(dy) = \int_{X \times F} h(x)\mu(dxdy) \ \ \forall \ F \in \mathrm{T}.$$

$H$ contains all continuous functions from $X$ to $[0, 1]$ and $\lim_{n\to\infty} h_n \in H$ whenever $\langle h_n \rangle_{n \in \mathbb{N}}$ is a pointwise convergent sequence in $H$. Consequently $H$ contains the characteristic function of any set $E \in \Sigma$; that is,

$$\int_F \mu_y E \mu(dy) = \mu(E \times F)$$

for every $E \in \Sigma$, $F \in \mathrm{T}$, as required. (Compare IONESCU TULCEA & IONESCU TULCEA 69, chap. IX, and FREMLIN P00*, §452.)

**SA2C Universally negligible sets** If $X$ is a Polish space (that is, a separable metrizable space in which the topology can be defined by a complete metric) a set $A \subseteq X$ is **universally negligible** if it is negligible for every Radon measure on $X$.

**SA2D Theorem** Let $X$ be a Hausdorff space and suppose that there is a Radon probability measure $\mu$ on $X$ with Maharam type $\kappa \geq \omega$. Then whenever $\omega \leq \lambda < \kappa$ there is a Maharam homogeneous Radon probability measure $\mu_1$ on $X$ with Maharam type $\lambda$.

**proof**

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The list here comprises only works referred to in this supplement, but shows where they have been used in the main text as well as the supplement.


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## Index

This is a joint index to both the present supplement and the original 'Real-valued-measurable cardinals'. References in **bold** type indicate definitions; references in *italics* indicate passing references.