Postscript to 'Consequences of Martin's Axiom'

In these notes, references given simply in the form '42A' are to FREMLIN 84, and unexplained notation follows that book, as do the chapter headings.

Chapter 2

When $\mathfrak{p} > \omega_1$

S22 Category and measure

S22Pb Answer to 22Pb If $\mathfrak{p} > \omega_1$, there can be no atomlessly-measurable cardinal; see FREMLIN 93, P4.

S23 Descriptive set theory

S23Od Answer to 23Od It is relatively consistent with ZFC to suppose that $\mathfrak{m} = \mathfrak{c} = \omega_2$ and that there is a set $X \subseteq \mathbb{N}^{\mathbb{N}}$ such that $X \cap K$ is F_σ for every compact $K \subseteq \mathbb{N}^{\mathbb{N}}$, but X is not analytic (TODORČEVIĆ 98).

Chapter 3

When $\mathfrak{m}_K > \omega_1$

S32 Measure Theory I

S32Rc Answer to 32Rc If X is a first-countable compact Radon measure space, then $L^1(X)$ is separable. This was shown by G.Plebanek (PLEBANEK 95); see also S33Kb.

S33 Measure theory II

S33Jc Remark The supposed proof of 33Jc was catastrophically flawed. The result, however, is true; see S33Ka.

S33Ka Remark It *is* true that any probability algebra of cardinal at most $\mathfrak{c}$ can be regularly embedded as an order-closed subalgebra of $\mathcal{P}\mathbb{N}/\mathcal{Z}$; see FREMLIN 03, §491.

S33Kb Answer If $\mathfrak{m}_K > \omega_1$, and X is a compact Radon measure space such that $L^1(X)$ is not separable, then there is a continuous surjection from X onto $[0,1]^{\omega_1}$ (FREMLIN 97).

S35 Ideals of $\mathcal{P}\omega_1$

S35Nb Answer to 35Nb Foreman Magidor & Shelah 88 show that the axiom 'Martin's maximum' implies that the ideal of non-stationary subsets of ω_1 is ω_2-saturated.

Chapter 4

When $\mathfrak{m} > \omega_1$

S41 Combinatorics I

S41Pa Partial answer to 41Pa Writing H for the statement

> 'if P is an upwards-ccc partially ordered set and $A \subseteq P$ is uncountable, there is an uncountable partially ordered subset of A which is upwards-centered in P',

H implies that $\mathfrak{m} > \omega_1$ (Todorčević & Veličković 87). In particular, of course, H implies that $2^{\omega_1} = \mathfrak{c}$.

S41Pb Answer to 41Pb If $\mathfrak{m} > \omega_1$ then every uncountable Boolean algebra has an uncountable weak antichain (Losada & Todorčević p98).

S42 S-respecting partial orders

S42A Theorem [$\mathfrak{m} > \omega_1$] Let X be an an uncountable set and S a subset of $[X]^2$. Then *either* there is an uncountable $A \subseteq X$ such that $[A]^2 \cap S = \emptyset$ *or* there are an infinite $D \subseteq X$ and an uncountable $B \subseteq X$ such that $[D]^2 \subseteq S$ and $\{x, y\} \in S$ whenever $x \in D$, $y \in B$.

proof (a) It is enough to consider the case $X = \omega_1$. Throughout this proof I will suppose that the first alternative is false, that is, that $[A]^2 \cap S \neq \emptyset$ for every uncountable $A \subseteq \omega_1$, and seek to prove that the second alternative is true. Set

$$\tilde{S} = \{(\xi, \eta) : \eta < \xi < \omega_1,\ \{\xi, \eta\} \in S\},$$

so that $\tilde{S} \subseteq \omega_1 \times \omega_1$ has countable vertical sections. Let P_0 be $[\omega_1]^{<\omega} \times [\omega_1]^{<\omega}$ with the $\tilde{S}$-respecting partial order. **?** If P_0 is upwards-ccc, then ω_1 is expressible as $\bigcup_{n \in \mathbb{N}} Z_n$ with $[Z_n]^2 \cap S = \emptyset$ for every $n \in \mathbb{N}$ (42A, with $Y = Z = X = \omega_1$, $f(\xi) = \xi$ for every $\xi \in \omega_1$); but now every Z_n must be countable, so ω_1 is countable. **X**

(b) Accordingly P_0 must fail to be upwards-ccc. By Szentmiklóssy's Lemma (42B), there are disjoint families $\langle K_\xi \rangle_{\xi < \omega_1}$, $\langle L_\xi \rangle_{\xi < \omega_1}$ of non-empty finite subsets of ω_1 such that

> for every uncountable $C \subseteq \bigcup_{\xi < \omega_1} L_\xi$ there is a countable $D \subseteq C$ and an $\alpha < \omega_1$ such that $D \cap \bigcap_{\xi \in M} \tilde{S}[K_\xi]$ is infinite for every finite $M \subseteq [\alpha, \omega_1[$.

Set $W = \bigcup_{\xi < \omega_1} L_\xi$.

(c) Set $\mathcal{S} = \{I : I \in [W]^{<\omega},\ [I]^2 \subseteq S\}$. For $I \in \mathcal{S}$, set

$$E_I = \{\xi : \xi \in W,\ I \cup \{\xi\} \in \mathcal{S}\},$$

$$E_I^* = \{\xi : \xi \in E_I,\ E_{I\cup\{\xi\}} \text{ is uncountable}\}.$$

Then if $I \in \mathcal{S}$, $E_I \setminus E_I^*$ is countable. **P?** Otherwise, we can choose inductively a strictly increasing family $\langle\alpha_\xi\rangle_{\xi<\omega_1}$ in $E_I \setminus E_I^*$ such that $\alpha_\xi \notin E_{I\cup\{\alpha_\eta\}}$ whenever $\eta < \xi < \omega_1$. But this means just that $[I \cup \{\alpha_\eta, \alpha_\xi\}]^2 \not\subseteq S$ whenever $\eta < \xi$; since $[I \cup \{\alpha_\xi\}]^2 \subseteq S$ and $[I \cup \{\alpha_\eta\}]^2 \subseteq S$, this means that $\{\alpha_\eta, \alpha_\xi\} \notin S$, whenever $\eta < \xi < \omega_1$. But now $A = \{\alpha_\xi : \xi < \omega_1\}$ is uncountable and $[A]^2 \cap S = \emptyset$, contrary to hypothesis. **XQ** In particular, E_I^* must be uncountable whenever E_I is.

Write

$$\mathcal{S}^* = \{I : I \in \mathcal{S},\ E_I \text{ is uncountable}\}.$$

Note that $\emptyset \in \mathcal{S}^*$ because all singleton subsets of W belong to $\mathcal{S}$.

(d) For every $I \in \mathcal{S}^*$, let δ_I be such that $E_I^* \cap \delta_I \cap \bigcap_{\xi\in M} \tilde{S}[K_\xi]$ is infinite whenever M is a finite subset of $[\delta_I, \omega_1[$. Let $\delta < \omega_1$ be such that $\delta_I \le \delta$ whenever $I \in \mathcal{S}^* \cap [\delta]^{<\omega}$.

Let P_1 be the partially ordered set $(\mathcal{S}^* \cap [\delta]^{<\omega}) \times [\omega_1 \setminus \delta]^{<\omega}$, with the partial ordering $\le$ given by setting

$$(I, J) \le (I', J') \text{ if } I \subseteq I',\ J \subseteq J',\ I' \setminus I \subseteq \tilde{S}[K_\xi]\ \ \forall\ \xi \in J.$$

Evidently P_1 is σ-centered upwards, and if we set

$$Q_\zeta = \{(I, J) : (I, J) \in P,\ \zeta \in J\}$$

then Q_ζ is cofinal with P for every $\zeta \in \omega_1 \setminus \delta$. Next, if $(I, J) \in P$ then there is an I', strictly including I, such that $(I, J) \le (I', J) \in P$. **P** Take any $\eta \in E_I^* \cap \delta \cap \bigcap_{\xi\in J} \tilde{S}[K_\xi] \setminus I$; then $(I, J) \le (I \cup \{\eta\}, J) \in P$. **Q** It follows that

$$\tilde{Q}_n = \{(I, J) : (I, J) \in P,\ \#(I) \ge n\}$$

is cofinal with P for every $n \in \mathbb{N}$. Because $\mathfrak{p} \ge \mathfrak{m} > \omega_1$, there is an upwards-directed $R \subseteq P$ meeting every Q_ζ and every $\tilde{Q}_n$. Set $D_0 = \bigcup\{I : (I, J) \in R\}$.

(e) Because $I \in \mathcal{S}$ whenever $(I, J) \in R$, and R is upwards-directed, $[D_0]^{<\omega} \subseteq \mathcal{S}$. If $\zeta \in \omega_1 \setminus \delta$, there is an $(I, J) \in R \cap Q_\zeta$; now $D_0 \setminus \tilde{S}[K_\zeta] \subseteq I$ is finite. Because R meets every $\tilde{Q}_n$, D_0 is infinite.

Let $\mathcal{F}$ be any non-principal ultrafilter on D_0. Then for any $\zeta \in \omega_1 \setminus \delta$,

$$\bigcup_{\beta\in K_\zeta} D_0 \cap \tilde{S}[\{\beta\}] = D_0 \cap \tilde{S}[K_\zeta] \in \mathcal{F},$$

so there is a $\beta_\zeta \in K_\zeta$ such that $D_0 \cap \tilde{S}[\{\beta_\zeta\}] \in \mathcal{F}$. Once again because $\mathfrak{p} > \omega_1$, there is an infinite $D_1 \subseteq D_0$ such that $D_1 \setminus \tilde{S}[\{\beta_\zeta\}]$ is finite for every $\zeta \in \omega_1 \setminus \delta$. But now there must be a finite set $I^* \subseteq D_1$ such that

$$B' = \{\zeta : \delta \le \zeta < \omega_1,\ D_1 \setminus I^* \subseteq \tilde{S}[\{\beta_\zeta\}]\}$$

is uncountable. Set $D = D_1 \setminus I^*$, $B = \{\beta_\zeta : \zeta \in B'\}$; these serve.

S42B Theorem $[\mathfrak{m} > \omega_1]$ $\omega_1 \to (\omega_1, \omega^2)$, where ω^2 is the ordinal power.

proof Let $S \subseteq [\omega_1]^2$ be any set. If there is an uncountable $A \subseteq \omega_1$ such that $[A]^2 \cap S = \emptyset$, we can stop. Otherwise, construct inductively sequences $\langle D_n \rangle_{n\in\mathbb{N}}$, $\langle B_n \rangle_{n\in\mathbb{N}}$ of subsets of ω_1 such that

D_n is infinite, B_n is uncountable, $\eta < \xi$ whenever $\eta \in D_n$, $\xi \in B_n$,

$[D_n]^2 \subseteq S$, $\{\eta, \xi\} \in S$ whenever $\eta \in D_n$, $\xi \in B_n$,

$D_{n+1} \cup B_{n+1} \subseteq B_n$

for every $n \in \mathbb{N}$; using S42A, this is easy. Now consider $D = \bigcup_{n\in\mathbb{N}} D_n$; we have $[D]^2 \subseteq S$ and $\operatorname{otp}(D) \geq \omega^2$, as required.

S42C Sources S42A-B come from Todorčević c (see 42Lc). I wrote out the proof here, presumably using the same ideas as the original, in November 1993.

S42D Problem Does $\mathfrak{m} > \omega_1$ imply that $\omega_1 \rightarrow (\omega_1, \alpha)^2$ for every $\alpha < \omega_1$? It is known that the Proper Forcing Axiom does imply that $\omega_1 \rightarrow (\omega_1, \alpha)^2$ for every $\alpha < \omega_1$ (Todorčević 83, Theorem 5, or Fremlin n86, 6E).

S43 Martin's axiom and separability

S43Rb Answer to 43Rb If $\mathfrak{m} > \omega_1$, then every ccc countably tight Čech-complete space has a countable π-base. (S.Todorčević, letter of October 1990.)

S43Rc Answer to 43Rc We can construct in ZFC a non-separable ccc compact Hausdorff space X such that $\pi\chi(x, X)$ is countable for every $x \in X$ (Todorčević 00, 8.4).

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