Supplement to 'Weakly α-favourable measure spaces'

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1. I describe a construction, due to G.Plebanek, related to questions in Fremlin p99.

1A A special axiom (Plebanek n99) Consider the statement

(†) There are a regular uncountable cardinal κ and a family $\langle x_\xi\rangle_{\xi<\kappa}$ in $[0,1]$ such that (i) $\{x_\eta:\eta<\xi\}$ is Lebesgue negligible for every $\xi<\kappa$ (ii) $\{x_\eta:\eta\in S\}$ is not Lebesgue negligible for any stationary set $S\subseteq\kappa$.

Observe that if there is any Sierpiński set $A\subseteq[0,1]$, then any enumeration of a subset of A of cardinal ω_1 will witness (†); more generally, if κ is regular and $A\subseteq[0,1]$ is a κ-Sierpiński set in the sense of Fremlin 84, A3Ed, then an enumeration of a subset of A will again witness (†). So, for instance, $\mathfrak{m}=\mathfrak{c}$ implies (†) (Fremlin 84, 32Cb and 21Eb).

1B The construction (Plebanek n99) Assume that $\langle x_\xi\rangle_{\xi<\kappa}$ witnesses (†). Set $X=\{x_\xi:\xi<\kappa\}$. For $\zeta<\kappa$ set

$$D_\zeta=\{(x_\xi,x_\eta):\zeta\le\eta<\xi\}\cup(([0,1]\setminus X)\times[0,1]).$$

Because $\mathrm{cf}(\kappa)>\omega$, $\mathcal{D}=\{D_\zeta:\zeta<\kappa\}$ is closed under countable intersections. Write μ_L for Lebesgue measure on $[0,1]$ and Σ_L for its domain. If $\zeta<\kappa$, then

$$[0,1]\setminus\pi_1[D_\zeta]\subseteq\{x_\xi:\xi\le\zeta\}$$

is μ_L-negligible, and $\mu_L^*(\pi_1[D_\zeta])=1$. Let Σ be the σ-algebra of subsets of $[0,1]^2$ generated by $\{E\times[0,1]:E\in\Sigma_L\}\cup\{D_\zeta:\zeta<\kappa\}$. Then there is a measure μ with domain Σ defined by saying that

$$\mu((E\times[0,1])\cap D_\zeta)=\mu_L E \text{ for every } E\in\Sigma_L,\ \zeta<\kappa.$$

Of course μ is of countable Maharam type.

1C Proposition (Plebanek n99) μ is perfect.

1D Proposition μ is not weakly α-favourable.

proof (a) It will be helpful to start with a couple of elementary facts.

(i) First, if $H\in\Sigma$, then $\mu H=\mu(H\cap D_0)=\mu_L(\pi_1[H\cap D_0])$, because $\mu_L(\pi_1[D_0\setminus D_\zeta])=0$ for every $\zeta<\kappa$. This means that if H is not conegligible then there is an $H'\in\Sigma$ such that $\mu H'>0$ and $\pi_1[H']\cap\pi_1[H]=\emptyset$. Another way of looking at this is to say that if $\mathcal{L}$ is coinitial with $\Sigma\setminus\mathcal{N}_\mu$, where $\mathcal{N}_\mu$ is the family of μ-negligible sets, then there is a countable set $\mathcal{L}'\subseteq\mathcal{L}$, with conegligible union, such that $\pi_1[L]\cap\pi_1[L']=\emptyset$ for all distinct L, $L'\in\mathcal{L}'$.

(ii) Next, if $E\in\Sigma_L$ is μ_L-conegligible, the set $\{\xi:x_\xi\notin E\}$ cannot be stationary, and there is a club $C\subseteq\kappa$ such that $x_\xi\in E$ for every $\xi\in C$.

(b) Let $\mathcal{K}$ be the family of sets of the form $(F\times[0,1])\cap D_\zeta$, where $F\subseteq[0,1]$ is closed and $\zeta<\kappa$; then μ is inner regular with respect to $\mathcal{K}$.

? Suppose, if possible, that μ is weakly α-favourable. Then $\mathcal{K}$ is a weakly α-favourable class (Fremlin p99, 1C); let $Q\subseteq\bigcup_{n\ge1}\mathcal{K}^n$ be a family as in Fremlin p99, 1D.

Choose inductively, for finite sequences $\boldsymbol{s}\in\bigcup_{n\in\mathbb{N}}\kappa^n$, countable sets $Q_{\boldsymbol{s}}\subseteq Q$, as follows. $Q_\emptyset$ is just any countable subset of $Q\cap\mathcal{K}^1$ such that $\{K:(K)\in Q_\emptyset\}$ has conegligible union and $\pi_1[K]\cap\pi_1[K']=\emptyset$ whenever (K), (K') are distinct members of $Q_\emptyset$; such a set exists by (a-i) above, because $\{K:(K)\in Q\}$ is coinitial with $\Sigma\setminus\mathcal{N}_\mu$. Given $\boldsymbol{s}\in\kappa^n$ and $Q_{\boldsymbol{s}}\subseteq Q\cap\mathcal{K}^{n+1}$, and $\zeta<\kappa$, choose $Q_{\boldsymbol{s}^\frown\zeta}\subseteq Q\cap\mathcal{K}^{n+2}$ in such a way that (i) whenever $(K_0,\dots,K_{n+1})\in Q_{\boldsymbol{s}^\frown\zeta}$, then $(K_0,\dots,K_n)\in Q_{\boldsymbol{s}}$ and $K_{n+1}\subseteq D_\zeta$ and $\mathrm{diam}(\pi_1[K_{n+1}])\le2^{-n-1}$ (ii) $Q_{\boldsymbol{s}^\frown\zeta}$ is countable and $\{K_{n+1}:(K_0,\dots,K_{n+1})\in Q_{\boldsymbol{s}^\frown\zeta}\}$ has conegligible union in $[0,1]$ (iii) if $(K_0,\dots,K_{n+1})$ and $(K'_0,\dots,K'_{n+1})$ are distinct members of $Q_{\boldsymbol{s}^\frown\zeta}$, then $\pi_1[K_{n+1}]$ and $\pi_1[K'_{n+1}]$ are disjoint.

For each $\boldsymbol{s}\in\bigcup_{n\in\mathbb{N}}\kappa^n$, let $H_{\boldsymbol{s}}$ be the conegligible set $\bigcup\{K_n:(K_0,\dots,K_n)\in Q_{\boldsymbol{s}}$. Then (a) tells us that there is a club $C_{\boldsymbol{s}}\subseteq\kappa$ such that $x_\xi\in\pi_1[H_{\boldsymbol{s}}]$ for every $\xi\in C_{\boldsymbol{s}}$. Setting $\tilde C_\alpha=\bigcap\{C_{\boldsymbol{s}}:s\in\bigcup_{n\in\mathbb{N}}\alpha^n\}$ for $\alpha<\kappa$, we see that every $\tilde C_\alpha$ is a club, being the intersection of fewer than κ clubs, so that the diagonal intersection C of $\langle C_\alpha\rangle_{\alpha<\kappa}$ is also a club, and there is an $\alpha\in C$ of cofinality ω; let $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ be a sequence in α with supremum α.

Because $\alpha\in C$, $x_\alpha\in\pi_1[H_{\boldsymbol{s}}]$ for every $\boldsymbol{s}\in\bigcup_{n\in\mathbb{N}}\alpha^n$; in particular, $x_\alpha\in\pi_1[H_{(\alpha_0,\dots,\alpha_n)}]$ for every n. We can therefore define $\langle K_n\rangle_{n\in\mathbb{N}}$ inductively by saying that $(K_0,\dots,K_{n+1})\in Q_{(\alpha_0,\dots,\alpha_n)}$ and $x_\alpha\in\pi_1[K_n]$ for every n. Now we are supposing that Q is a witness that $\mathcal{K}$ is weakly α-favourable, so we must have a point $(t,u)\in\bigcap_{n\in\mathbb{N}}K_n$. Because $\operatorname{diam}(\pi_1[K_n])\le 2^{-n}$ for every n, $t=x_\alpha$; because K_0, being a member of $\mathcal{K}$, is a subset of D_0, $u=x_\eta$ for some $\eta<\alpha$. But we also know that $K_{n+1}\subseteq D_{\alpha_n}$, so we must have $\alpha_n\le\eta$ for every n; which is impossible. **X**

This contradiction shows that μ is not weakly α-favourable.

2. Corresponding to Marczewski's theorem that countably compact finitely additive measures have countably additive extensions, I describe a game Γ_m played on a set X with a finitely additive measure ν such that if the second player has a winning strategy in Γ_m, then ν has an extension to a weakly α-favourable measure on X (2H).

2A In the contexts which are the principal concern of this paper, we have a useful structure associated with a family Q of the type described in 1D.

Lemma Suppose that $\mathcal{K}$ is a weakly α-favourable class with the property that

(†) whenever $\mathcal{H}$ is a maximal disjoint subset of $\mathcal{K}$ and $K\in\mathcal{K}$, then there is an $H\in\mathcal{H}$ such that K and H have a common lower bound in $\mathcal{K}$.

Let $Q\subseteq\bigcup_{n\ge1}\mathcal{K}^n$ be a family as in Lemma 1D. For $n\in\mathbb{N}$, let $\mathcal{R}_n$ be the family of subsets R of Q such that, $R\subseteq\mathcal{K}^{n+1}$ and

(i) if $(K_0,\dots,K_n)$, $(K'_0,\dots,K'_n)$ are distinct members of R, then there is an $i\le n$ such that $K_j=K'_j$ for $j<i$ and $K_i\cap K'_i=\emptyset$,

(ii) $\mathcal{H}_R=\{K_n:(K_0,\dots,K_n)\in R\}$ is a maximal disjoint subset of $\mathcal{K}$.

For $R\in\mathcal{R}_m$, $R'\in\mathcal{R}_n$, where $m\le n$, say that $R\preccurlyeq R'$ if $R=\{(K_0,\dots,K_m):(K_0,\dots,K_n)\in R'\}$. Now whenever $R\in\mathcal{R}_n$ and $E\in\mathcal{K}$, there is an $R'\in\mathcal{R}_{n+1}$ such that $R\preccurlyeq R'$ and every member of $\mathcal{H}_{R'}$ is either included in E or disjoint from it.

proof Condition (i) here ensures that each member H of $\mathcal{H}_R$ is the last term of exactly one sequence $\boldsymbol{s}_H$ in R. Now, for $H\in\mathcal{H}_R$, let $\mathcal{E}_H$ be a maximal disjoint set in $\{K:K\subseteq H\setminus E,\ \boldsymbol{s}^\frown K\in Q\}$. Then $\mathcal{E}_H$ is a maximal disjoint set in $\{K:K\in\mathcal{K},\ K\subseteq H\setminus E\}$, by condition (iii) of 1D. Similarly, let $\mathcal{E}'_H$ be a maximal disjoint set in $\{K:K\subseteq H\cap E,\ \boldsymbol{s}^\frown K\in Q\}$; then this is also a maximal disjoint set in $\{K:K\in\mathcal{K},\ K\subseteq H\cap E\}$. Set $\mathcal{E}=\bigcup\{\mathcal{E}_H:H\in\mathcal{H}_R\}$, $\mathcal{E}'=\bigcup\{\mathcal{E}'_H:H\in\mathcal{H}_R\}$.

Observe that $\{E\}\cup\mathcal{E}'$ is a disjoint subset of $\mathcal{K}$. Also it is maximal. **P** If $K\in\mathcal{K}$ does not meet E, then, by (†), there is an $H\in\mathcal{H}_R$ such that K and H have a common lower bound K_1 say. Because $\mathcal{E}'_H$ is maximal, there is a $K'\in\mathcal{E}'_H\subseteq\mathcal{E}'$ meeting K_1 and therefore K. **Q** It follows that $\mathcal{E}\cup\mathcal{E}'$ is also a maximal disjoint subset of $\mathcal{K}$. **P** This time, if $K\in\mathcal{K}$ does not meet $\bigcup\mathcal{E}'$, (†) tells us that K and E have a common lower bound $K_1\in\mathcal{K}$, and that there is some $H\in\mathcal{H}_R$ such that K_1 and H have a common lower bound K_2, which must meet some member of $\mathcal{E}_H\subseteq\mathcal{E}$ because $\mathcal{E}_H$ is disjoint. **Q**

Setting

$$R'=\{\boldsymbol{s}_H^\frown K:H\in\mathcal{H}_R,\ K\in\mathcal{E}_H\cup\mathcal{E}'_H\},$$

we have the result.

Remark The condition (†) here is satisfied, in particular, whenever $\mathcal{K}$ is a coinitial subset of $\Sigma\setminus\mathcal{N}_\mu$ for a σ-finite measure space (X,Σ,μ).

2B A variant on $\Gamma(\mathcal{K})$ One of the most important uses of countably compact classes in measure theory concerns the extension of finitely additive measures to countably additive measures. If X is a set, T is an algebra of subsets of X, $\mathcal{K} \subseteq \mathrm{T}$ is a countably compact class, and $\nu : \mathrm{T} \to [0, \infty[$ is a finitely additive measure which is inner regular with respect to $\mathcal{K}$, then ν has an extension to a countably additive measure (Marczewski 53, §4, or Fremlin p00*, 413R). To find a corresponding result in terms of infinite games we need to modify the idea of 'weakly α-favourable', as follows.

Let X be a set, $\mathcal{K}$ a family of subsets of X, and $\Delta : \mathcal{K} \to]0, \infty[$ a function which is non-decreasing in the sense that $\Delta(K) \le \Delta(K')$ whenever $K \subseteq K'$ in $\mathcal{K}$. The game $\Gamma_m(\mathcal{K}, \Delta)$ is played as follows. The first player plays pairs (K, δ), where $K \in \mathcal{K}$ and $\delta < \Delta(K)$; the second player plays members of $\mathcal{K}$; the rules are that in any play $((K_0, \delta_0), K'_0, (K_1, \delta_1), K'_1, \dots)$, $K'_n \subseteq K_n$, $\Delta(K'_n) \ge \delta_n$, and $K_{n+1} \subseteq K'_n$ for every n; and the second player wins iff $\bigcap_{n \in \mathbb{N}} K_n \ne \emptyset$.

2C I had better show that we do not get a new class of measure spaces by looking at this new game, at least in its simplest forms.

Proposition Let (X, Σ, μ) be a totally finite measure space. Then it is weakly α-favourable iff the second player has a winning strategy in the game $\Gamma_m(\mathcal{K}, \Delta)$, where $\mathcal{K} = \Sigma \setminus \mathcal{N}_\mu$ and $\Delta = \mu \restriction \mathcal{K}$.

proof (a) If the second player has a winning strategy σ in $\Gamma_m(\mathcal{K}, \Delta)$, then they can turn this directly into a winning strategy σ' in $\Gamma(\mathcal{K})$ by setting

$$\sigma'(E_0, \dots, E_n) = \sigma((E_0, 0), (E_1, 0), \dots, (E_n, 0))$$

for all $E_0, \dots, E_n \in \mathcal{K}$. So in this case μ is weakly α-favourable.

(b) For the rest of this proof, therefore, I suppose that the second player has a winning strategy in $\Gamma(\mathcal{K})$ and seek a winning strategy in $\Gamma_m(\mathcal{K}, \Delta)$. Let $Q \subseteq \bigcup_{n \ge 1} \mathcal{K}^{n+1}$ satisfy the conditions (i)-(iv) of Lemma 1D. For $n \in \mathbb{N}$, $\boldsymbol{s} \in \mathcal{K}^{n+1}$ choose $Q_{\boldsymbol{s}} \subseteq Q$ as follows. If $\boldsymbol{s} = (E)$ is a single-term sequence, then set $\mathcal{E} = \{F : (F) \in Q, F \subseteq E \text{ or } F \cap E = \emptyset\}$, let $\mathcal{E}'$ be a maximal disjoint subset of $\mathcal{E}$, and set $Q_{\boldsymbol{s}} = \{(F) : F \in \mathcal{E}'\}$. Now suppose that $Q_{\boldsymbol{t}} \subseteq Q$ has been defined for $\boldsymbol{t} \in (\mathcal{K})^{n+1}$, and take $\boldsymbol{s} \in (\Sigma \setminus \mathcal{N}_\mu)^{n+2}$. Express $\boldsymbol{s}$ as $\boldsymbol{t}^\frown E$. For each $q = (F_0, \dots, F_n) \in Q_{\boldsymbol{t}}$, set $\mathcal{E}_q = \{F : q^\frown F \in Q, F \subseteq E \text{ or } F \cap E = \emptyset\}$ and let $\mathcal{E}'_q \subseteq \mathcal{E}_q$ be a maximal disjoint set; set $Q_{\boldsymbol{s}} = \{q^\frown F : q \in Q_{\boldsymbol{t}}, F \in \mathcal{E}'_q\}$. Continue.

An easy induction shows that $Q_{\boldsymbol{s}}$ is always a countable set such that $\mathcal{H}_{\boldsymbol{s}} = \{E_n : (E_0, \dots, E_n) \in Q_{\boldsymbol{s}}\}$ is a disjoint family with conegligible union.

Let $\mathcal{L}$ be the set of pairs (K, δ) with $K \in \mathcal{K}$ and $\delta < \mu K$. Let $\sigma : \bigcup_{n \in \mathbb{N}} \mathcal{L}^{n+1} \to \mathcal{K}$ be a function such that

> whenever $(E_0, \delta_0), \dots, (E_n, \delta_n) \in \mathcal{L}$, then $\sigma((E_0, \delta_0), \dots, (E_n, \delta_n))$ is a finite union of members of $\mathcal{H}_{(E_0, \dots, E_n)}$ included in E_n, with measure at least δ_n;

such a finite union can always be found because $\mathcal{H}_{\boldsymbol{s}}$ is always a countable family of sets with conegligible union in X, and every member of $\mathcal{H}_{(E_0, \dots, E_n)}$ is either included in E_n or disjoint from it.

Of course σ is a strategy for the second player in the game $\Gamma_m(\mathcal{K}, \Delta)$. To see that it is a winning strategy, take any sequence $\langle (E_n, \delta_n) \rangle_{n \in \mathbb{N}}$ in $\mathcal{L}$ such that $E_{n+1} \subseteq \sigma((E_0, \delta_0), \dots, (E_n, \delta_n))$ for every $n \in \mathbb{N}$. For each n, set $\mathcal{E}_n = \mathcal{H}_{(E_0, \dots, E_n)}$; then we have a finite set $\mathcal{E}'_n \subseteq \mathcal{E}_n$ such that $E_{n+1} \subseteq \bigcup \mathcal{E}'_n \subseteq E_n$. Now observe that, by the construction of the $\mathcal{H}_{\boldsymbol{s}}$, if $m \le n$ then every member of $\mathcal{E}_n$ meets just one member of $\mathcal{E}_m$, and is included in it. This means that if $\mathcal{F}$ is any non-principal ultrafilter on $\mathcal{E} = \bigcup_{n \in \mathbb{N}} \mathcal{E}'_n$, then for each $m \in \mathbb{N}$ there is a unique $E'_m \in \mathcal{E}'_m$ such that $\{E : E \in \mathcal{E}, E \subseteq E'_m\}$ belongs to $\mathcal{F}$. Since $E'_m \in \mathcal{E}_m$ for every m, we have $(E'_0, \dots, E'_n) \in Q_{(E_0, \dots, E_n)} \subseteq Q$ for every n, and $\bigcap_{n \in \mathbb{N}} E'_n \ne \emptyset$, so $\bigcap_{n \in \mathbb{N}} E_n \ne \emptyset$. As $\langle (E_n, \delta_n) \rangle_{n \in \mathbb{N}}$ is arbitrary, σ is a winning strategy, as required.

2D For finitely additive measures, however, the game Γ_m has something new to say, as follows.

Proposition Let X be a set, T an algebra of subsets of X, and $\nu : \mathrm{T} \to [0, \infty[$ a non-negative finitely additive functional on T. Suppose that the second player has a winning strategy in the game $\Gamma_m(\mathcal{K}_0, \Delta)$, where $\mathcal{K}_0 = \{E : E \in \mathrm{T}, \nu E > 0\}$ and $\Delta = \nu \restriction \mathcal{K}_0$. Then ν has an extension to a weakly α-favourable countably additive measure μ.

proof If $\nu X = 0$ the result is trivial; I suppose henceforth that $\nu X > 0$. Set $\mathcal{L} = \{(E, \delta) : E \in \mathrm{T}, 0 \le \delta < \nu E\}$, and let σ be a winning strategy for the second player in the game $\Gamma_m(\mathcal{K}_0, \Delta)$, that is, a function from $\bigcup_{n \ge 1} \mathcal{L}^{n+1}$ to $\mathcal{K}_0$ such that if $E = \sigma((E_0, \delta_0), \dots, (E_n, \delta_n))$ then $E \subseteq E_n$ and $\mu E \ge \delta_n$, and whenever

$\langle(E_n,\delta_n)\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}$ such that $E_{n+1}\subseteq\sigma((E_0,\delta_0),\dots,(E_n,\delta_n))$ for every n, then $\bigcap_{n\in\mathbb{N}}E_n$ is non-empty.

(a) The first step is to construct the measure μ. This is easy. For $A\subseteq X$, set

$$\theta A=\inf\{\sum_{n=0}^\infty\nu E_n:\langle E_n\rangle_{n\in\mathbb{N}}\text{ is a sequence in }\mathrm{T},\ A\subseteq\bigcup_{n\in\mathbb{N}}E_n\}.$$

Then θ is an outer measure on X; let μ be the measure defined from θ by Carathéodory's method, and Σ its domain. It is easy to check that (because ν is additive) $\theta A=\theta(A\cap E)+\theta(A\setminus E)$ whenever $A\subseteq X$ and $E\in\mathrm{T}$, so that $\mathrm{T}\subseteq\Sigma$, and of course $\nu E\le\theta E=\mu E$ for every $E\in\mathrm{T}$.

? Suppose, if possible, that $E\in\mathrm{T}$ and $\theta E<\nu E$. Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence in T such that $E\subseteq\bigcup_{n\in\mathbb{N}}E_n$ and $\sum_{n=0}^\infty\nu E_n<\nu E$. Define $\langle(F_n,\delta_n)\rangle_{n\in\mathbb{N}}$, $\langle F'_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. Start with $F_0=E$ and δ_0 such that $\sum_{n=0}^\infty\nu E_n<\delta_0<\nu E$. Given that $F_m\in\mathcal{K}$, $F_m\subseteq E$ and $\sum_{i=m}^\infty\nu E_i<\delta_m<\nu F_m$ for $m\le n$, set $F_{n+1}=\sigma((F_0,\delta_0),\dots,(F_n,\delta_n))\setminus E_n$. Then $\nu F_{n+1}\ge\delta_n-\mu E_n>\sum_{i=n+1}^\infty\mu E_i$, so we can find δ_{n+1} such that $\sum_{i=n+1}^\infty\mu E_i<\delta_{n+1}<\nu F_{n+1}$. Continue.

Since $F_{n+1}\subseteq\sigma((F_0,\delta_0),\dots,(F_n,\delta_n))$ for every n, $\bigcap_{n\in\mathbb{N}}F_n$ is non-empty, because σ is a winning strategy; but $\bigcap_{n\in\mathbb{N}}F_n\subseteq E\setminus\bigcup_{n\in\mathbb{N}}E_n$, so this is impossible. **X**

Thus $\mu E=\theta E=\nu E$ for every $E\in\mathrm{T}$, and μ extends ν.

(b) To see that μ is weakly α-favourable, I argue as follows. First, write $\mathcal{K}\subseteq\Sigma$ for the family of sets of non-zero measure for μ which are expressible as the intersection of a sequence in T. Then μ is inner regular with respect to $\mathcal{K}$. **P** If $H\in\Sigma$ and $0\le\delta<\mu H$, let $\langle F_n\rangle_{n\in\mathbb{N}}$ be a sequence in T such that $X\setminus H\subseteq\bigcup_{n\in\mathbb{N}}F_n$ and $\sum_{n=0}^\infty\nu F_n\ge\mu X-\delta$; then $K=\bigcap_{n\in\mathbb{N}}F_n$ belongs to $\mathcal{K}$ and is included in H, and $\mu K\ge\delta$. **Q**

For each $K\in\mathcal{K}$, express K as $\bigcap_{n\in\mathbb{N}}F_n(K)$, where $F_n(K)\in\mathrm{T}$ for $n\in\mathbb{N}$.

(c) Define $\phi:\bigcup_{n\in\mathbb{N}}\mathcal{K}^{n+1}\to\bigcup_{n\in\mathbb{N}}\mathcal{L}^{n+1}$ as follows. Start by enumerating $\mathbb{N}\times\mathbb{N}$ as $\langle(i_n,j_n)\rangle_{n\in\mathbb{N}}$ in such a way that $i_n\le n$ for every $n\in\mathbb{N}$. Next, for any single-term sequence (K) in $\mathcal{K}$, set $\phi(K)=(X,\delta)$ for some δ such that $\mu(X\setminus K)<\delta<\mu X$.

Suppose that $\boldsymbol{s}=(K_0,\dots,K_n)\in\mathcal{K}^{n+1}$ and that we have defined $\phi(\boldsymbol{s})$ in $\mathcal{L}^{n+1}$. Take any further $K\in\mathcal{K}$. Set $E=F_{j_n}(K_{i_n})\cap\sigma(\phi(\boldsymbol{s}))$. If $E\cap K$ is not negligible, take δ such that $\mu(E\setminus K)<\delta<\mu E$, and set $\phi(\boldsymbol{s}^\frown K)=\phi(\boldsymbol{s})^\frown(E,\delta)$. If $E\cap K$ is negligible, set $\phi(\boldsymbol{s}^\frown K)=\phi(\boldsymbol{s})^\frown(X,0)$. Continue.

Now define $\sigma':\bigcup_{n\in\mathbb{N}}\mathcal{K}^{n+1}\to\mathcal{K}$ by setting $\sigma'(K_0,\dots,K_n)=K_n\cap\sigma(\phi(K_0,\dots,K_n))$ if this is not negligible, K_n otherwise. Then σ' is a strategy for the second player in the game $\Gamma(\mathcal{K})$. If $\langle K_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{K}$ such that $K_{n+1}\subseteq\sigma'(K_0,\dots,K_n)$ for every n, then we have a corresponding sequence $\langle(E_n,\delta_n)\rangle_{n\in\mathbb{N}}$ in $\mathcal{L}$ such that $\phi(K_0,\dots,K_n)=((E_0,\delta_0),\dots,(E_n,\delta_n))$ for every n. Now, writing $E'_n=\sigma((E_0,\delta_0),\dots,(E_n,\delta_n))$ for each n, we find that

$$\mu(E_n\setminus K_n)<\delta_n,\quad\sigma'(K_0,\dots,K_n)=E'_n\cap K_n,\quad E_{n+1}\subseteq F_{j_n}(K_{i_n})$$

for every n. **P** Induce on n. The induction starts with the observation that $\mu(E_0\setminus K_0)<\delta_0$. Given that $\mu(E_n\setminus K_n)<\delta_n$, then, because $E'_n\subseteq E_n$ and $\mu E'_n\ge\delta_n$, $E'_n\cap K_n$ is not negligible, and $\sigma'(K_0,\dots,K_n)=E'_n\cap K_n$; next,

$$K_{n+1}\subseteq K_n\subseteq K_{i_n}\subseteq F_{j_n}(K_{i_n}),$$

so $K_{n+1}\subseteq F_{j_n}(K_{i_n})\cap E'_n$ and $K_{n+1}\cap F_{j_n}(K_{i_n})\cap E'_n$ is not negligible. This means that the construction makes $E_{n+1}=F_{j_n}(K_{i_n})\cap E'_n$ and δ_{n+1} greater than $\mu(E_{n+1}\setminus K_{n+1})$, so the induction proceeds. **Q**

We shall therefore have

$$\bigcap_{n\in\mathbb{N}}K_n=\bigcap_{n\in\mathbb{N}}F_{j_n}(K_{i_n})\supseteq\bigcap_{n\in\mathbb{N}}E_n\ne\emptyset.$$

As $\langle K_n\rangle_{n\in\mathbb{N}}$ is arbitrary, σ' is a winning strategy, and μ is weakly α-favourable.

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