Version of 9.1.95

Problem Z

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1. The problem The following question arose in the course of Bourgain Fremlin & Talagrand 78. Let X be a Polish space and B the space of Borel measurable functions from X to $\mathbb{R}$, endowed with the topology $\mathfrak{T}_p$ of pointwise convergence inherited from the product space $\mathbb{R}^X$. Suppose that $K \subseteq B$ is $\mathfrak{T}_p$-compact, and consider the closed convex hull $\overline{\Gamma(K)}$ of K, taken in $\mathbb{R}^X$. Must $\overline{\Gamma(K)}$ be included in B?

If K consists of functions of the first Baire class only, yes; this is Proposition 5H of Bourgain Fremlin & Talagrand 78. But for general K the question appears to be undecidable. If the continuum hypothesis is true, there is a compact K consisting of functions of the second Baire class such that $\overline{\Gamma(K)} \not\subseteq B$ (Talagrand 84, 10-1-1). I show in this note that if there is an atomlessly-measurable cardinal, then $\overline{\Gamma(K)} \subseteq B$ for every compact $K \subseteq B$ (Proposition 6 below).

2. Lemma Suppose that there is an atomlessly-measurable cardinal. Let (K,μ) be a Radon probability space with $\mathrm{w}(K) \le \mathfrak{c}$. Then there is an extension $\bar\mu$ of μ to a measure defined on every subset of K.

proof Fremlin 93, 3H.

3. Notation Let X be a metric space. Define $B_\alpha = B_\alpha(X)$, for ordinals α, by saying that B_0 is the set of continuous functions from X to $\mathbb{R}$, and for $\alpha > 0$ B_α is the set of functions form X to $\mathbb{R}$ expressible as limits of sequences in $\bigcup_{\beta<\alpha} B_\beta$. Then $B = B_{\omega_1} = \bigcup_{\alpha<\omega_1} B_\alpha$ is the set of Borel measurable real-valued functions on X.

4. Lemma Suppose that $(K, \mathcal{P}K, \bar\mu)$ is a probability space, X a metric space, and $\alpha < \omega_1$ an ordinal. Let $F : K \times X \to \mathbb{R}$ be a bounded function such that for $\bar\mu$-almost every $u \in K$, the vertical section $x \mapsto F(u,x) : X \to \mathbb{R}$ is in B_α. Then the function $x \mapsto g(x) = \int F(u,x)\bar\mu(du)$ is in B_α.

proof Induce on α. To start the induction, with $\alpha = 0$, we have almost every vertical section continuous. So if $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in X converging to $x \in X$,

$$\int F(u,x)\bar\mu(du) = \int \lim_{n\to\infty} F(u,x_n)\bar\mu(du) = \lim_{n\to\infty} \int F(u,x_n)\bar\mu(du)$$

by Lebesgue's theorem; thus g is continuous.

For the inductive step to $\alpha > 0$, let K' be the set of $u \in K$ such that $x \mapsto F(u,x)$ is of Baire class α. If $\alpha = \beta + 1$ is a successor ordinal, set $\alpha_n = \beta$ for every n; if α is a limit ordinal let $\langle \alpha_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in α with supremum α. For each $u \in K'$ let $\langle f_{un}\rangle_{n\in\mathbb{N}}$ be a sequence of functions such that $f(u,x) = \lim_{n\to\infty} f_{un}(x)$ for every $x \in X$ and every f_{un} is of Baire class α_n. Let M be such that $|F(u,x)| \le M$ for all $u \in K$, $x \in X$, and set

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$$\begin{aligned} F_n(u,x) &= \max(-M, \min(M, f_{un}(x))) \text{ if } u \in K',\ x \in X, \\ &= F(u,x) \text{ if } u \in K \setminus K',\ x \in X. \end{aligned}$$

By the inductive hypothesis, $x \mapsto g_n(x) = \int F_n(u,x)\bar{\mu}(du)$ is of Baire class $\alpha_n < \alpha$ for every n. But by Lebesgue's theorem again, $g(x) = \lim_{n\to\infty} g_n(x)$ for every x, so g is of Baire class α. Thus the induction proceeds.

Remark This is a trivial generalization of Lemma 6 of Laczkovich & Miller p94.

5. Lemma Suppose that $(K, \mathcal{P}K, \bar{\mu})$ is a probability space and X a metric space. Let $F : K \times X \to \mathbb{R}$ be a bounded function such that (i) for every $x \in X$, the horizontal section $u \mapsto f(u,x)$ is bounded (ii) for $\bar{\mu}$-almost every $u \in K$, the vertical section $x \to F(u,x) : X \to \mathbb{R}$ is Borel. Then the function $x \mapsto g(x) = \int F(u,x)\bar{\mu}(du)$ is Borel.

proof For each $\alpha < \omega_1$ let K_α be the set of those $u \in K$ such that $x \mapsto F(u,x)$ belongs to B_α. Then $\langle K_\alpha \rangle_{\alpha<\omega_1}$ is a non-decreasing family with union K. But $\bar{\mu}$ is ω_2-additive (Fremlin 93, [1]), so there is an α such that $K \setminus K_\alpha$ is negligible. Now, for each $m \in \mathbb{N}$, set $F_m(u,x) = \max(-m, \min(m, F(u,x))$ for $u \in K$, $x \in X$. Then $x \mapsto F_m(u,x)$ is of Baire class α for each $m \in \mathbb{N}$, $u \in K_\alpha$, so $x \mapsto g_m(x) = \int F_m(u,x)\bar{\mu}(du)$ is of Baire class α, by Lemma 4. Now recall that for each $x \in X$ the function $u \mapsto F(u,x)$ is bounded, so that there is some m_0 such that $F_m(u,x) = F(u,x)$ whenever $u \in K$, $m \geq m_0$; consequently $g(x) = \int F(u,x)\bar{\mu}(du)$ is equal to $g_m(x)$ for every $m \geq m_0$. Thus $g = \lim_{n\to\infty} g_m$ is Borel, as required.

6. Proposition Assume that there is a real-valued-measurable cardinal. Let X be a metric space of cardinal at most $\mathfrak{c}$ and K a $\mathfrak{T}_p$-compact subset of the space B of Borel measurable real-valued functions on X. Let $\overline{\Gamma(K)}$ be the closed convex hull of K in $\mathbb{R}^X$. Then $\overline{\Gamma(K)} \subseteq B$.

proof Take any $g \in \overline{\Gamma(K)}$.

(a) There is a Radon probability measure μ on K such that $g(x) = \int f(x)\mu(df)$ for every $x \in X$. **P** Let M be the convex set of Radon probability measures on K. Then M is compact for the topology $\mathfrak{T}_s(M, C(K))$. For $\mu \in M$ set $g_\mu(x) = \int f(x)\mu(df)$ for every $x \in X$; this is defined because $f \mapsto f(x) : K \to \mathbb{R}$ is continuous. The function $\mu \to g_\mu : M \to \mathbb{R}^X$ is affine and continuous for $\mathfrak{T}_s(M, C(X))$ and the product topology of $\mathbb{R}^X$, so the image of M is a closed convex set in $\mathbb{R}^X$. But also it includes K, because if δ_f is the point mass concentrated at $f \in K$ then $g_{\delta_f} = f$. So it includes $\overline{\Gamma(K)}$ and contains g. **Q** (See Bourgain Fremlin & Talagrand 78, 5E.)

(b) By Lemma 2, there is an extension $\bar{\mu}$ of μ to a measure defined on every subset of K. Set $F(f,x) = f(x)$ for $f \in K$, $x \in X$. Then

$$g(x) = \int F(f,x)\bar{\mu}(df)$$

for every $x \in X$. By Lemma 5, applied to F, $g \in B$. As g is arbitrary, we have the result.

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[1] query

7. Problems (a) In Proposition 6, what happens if $\#(X)$ is large?
(b) What happens if $\mathfrak{m} = \mathfrak{c} > \omega_1$?

References

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