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Non-Radon product spaces

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1 The problem A Hausdorff space X is **Radon** if every totally finite Borel measure is 'tight', that is, inner regular with respect to the (closed) compact sets. For basic facts concerning Radon spaces, see Schwartz 73 and Fremlin 03, §434. Here I address the question: must the product of two Radon Hausdorff spaces be Radon? In Wage 80 there is a counter-example depending on the continuum hypothesis. I give two further examples based on different special axioms.

2 The first example is a development of Wage's method to work with a weaker axiom.

2A Preliminaries (a) Set $Z=\{0,1\}^{\mathbb{N}}$, and write ν for its usual measure (Fremlin 01, §254). For $z\in Z$, $n\in\mathbb{N}$ set $H_n(z)=\{w:w\in Z,\,w{\restriction}n=z{\restriction}n\}$. Recall that if $F\subseteq Z$ is measured by ν then $\lim_{n\to\infty}2^n\nu(F\cap H_n(z))=\chi F(z)$ for ν-almost every z (Fremlin 01, 275I, or otherwise).

(b) Let $\preccurlyeq$ be the lexicographic ordering of Z. For $z\in Z$ set

$$E^+(z)=\{w:z\prec w\}=\bigcup_{n\in\mathbb{N},z(n)=0}\{w:w\in H_n(z),\,w(n)=1\},$$

$$E^-(z)=\{w:w\prec z\}=\bigcup_{n\in\mathbb{N},z(n)=1}\{w:w\in H_n(z),\,w(n)=0\},$$

so that $E^+(z)$, $E^-(z)$ are disjoint open sets with union $Z\setminus\{z\}$. Note that if $G\subseteq Z$ is open, then

$$A=\bigcup_{n\in\mathbb{N}}\{z:z\in Z\setminus G,\,E^+(z)\cap H_n(z)\subseteq G\},$$

$$A'=\bigcup_{n\in\mathbb{N}}\{z:z\in Z\setminus G,\,E^-(z)\cap H_n(z)\subseteq G\}$$

are both countable. **P** If $z\in A$, there is an $n\in\mathbb{N}$ such that A does not meet $E^+(z)\cap H_n(z)$. **?** If A is uncountable, there is an $n\in\mathbb{N}$ such that $B=\{z:A\cap E^+(z)\cap H_n(z)=\emptyset\}$ is uncountable. Now there must be distinct w, $z\in B$ with $H_n(w)=H_n(z)$, in which case $w\not\prec z$ and $z\not\prec w$, which is impossible. **X** Thus A is countable; similarly, A' is countable. **Q**

(c) Set $Z'=\{z:z\in Z,\,z\text{ is eventually constant}\}$. Then Z' is countable. For any $z\in Z\setminus Z'$,

$$\limsup_{n\to\infty}2^n\nu(E^+(z)\cap H_n(z))\ge\tfrac12,\quad \limsup_{n\to\infty}2^n\nu(E^-(z)\cap H_n(z))\ge\tfrac12.$$

2B An axiom Let (†) be the statement

there is a set $C\subseteq Z$ such that $\#(C\cap E)=\mathfrak{c}$ whenever $\nu E>0$ and $C\cap E$ is universally negligible whenever $\nu E=0$.

Observe that (†) is true if $\operatorname{non}\mathcal{N}=\operatorname{cov}\mathcal{N}=\mathfrak{c}$, where $\mathcal{N}$ is the ideal of Lebesgue negligible sets; in particular, it is a consequence of Martin's axiom. It is also true in any model obtained by adding κ random reals where (in the ground model) $\kappa\ge 2^\omega$ is a cardinal of uncountable cofinality. (Take $C_0\subseteq\{0,1\}^{\mathbb{N}}$ to be the set of random reals, and set $C=\{z+_2w:z\in C_0,\,w\in Z'\}$.)

2C Construction Assume that (†) is true, witnessed by $C\subseteq Z$.

(a) Enumerate as $\langle\langle A_{\xi n}\rangle_{n\in\mathbb{N}}\rangle_{\xi<\mathfrak{c}}$ the family of all sequences $\langle A_n\rangle_{n\in\mathbb{N}}$ of countable sets such that $\bigcap_{n\in\mathbb{N}}\overline{A}_n$ is not ν-negligible. Choose $\langle z_\xi\rangle_{\xi<\mathfrak{c}}$ inductively so that

$$z_\xi\in C\cap\bigcap_{n\in\mathbb{N}}\overline{A}_{\xi n}\setminus(Z_\xi\cup\bigcup_{n\in\mathbb{N}}A_{\xi n}\cup Z'),$$

where

$$Z_\xi=\{z_\eta:\eta<\xi\}\cup\bigcup_{\eta<\xi,n\in\mathbb{N}}A_{\eta n}.$$

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Observe that $\langle Z_\xi\rangle_{\xi<\mathfrak{c}}$ is a non-decreasing family with union Z and that $Z_\xi=\bigcup_{\eta<\xi}Z_\eta$ if $\xi<\mathfrak{c}$ is a limit ordinal. Set $C_1=\{z_\xi:\xi<\mathfrak{c}\}$.

(b) For each $\xi<\mathfrak{c}$ choose a sequence $\langle w_{\xi i}\rangle_{i\in\mathbb{N}}$ in $\bigcup_{n\in\mathbb{N}}A_{\xi n}$ such that $z_\xi=\lim_{i\to\infty}w_{\xi i}$, $w_{\xi i}\ne w_{\xi j}$ if $i\ne j$, and $\{i:w_{\xi i}\in A_{\xi n}\}$ is infinite for every $n\in\mathbb{N}$. Now choose inductively basic open-and-closed sets $G_{\xi i}\subseteq Z$ such that

$$w_{\xi i}\in G_{\xi i},$$

$$G_{\xi i}\cap(\bigcup_{j<i}G_{\xi j}\cup\{w_{\xi k}:k>i\}\cup\{z_\xi\})=\emptyset,$$

$$\sum_{j=0}^{i}\nu(G_{\xi j}\cap H_r(z_\xi))<4^{-r}\text{ for every }r\in\mathbb{N}$$

for every $i\in\mathbb{N}$; the last being possible because we can start by taking r_i such that $w_{\xi i}\notin H_{r_i}(z_\xi)$, and ensure that $G_{\xi i}\cap H_{r_i}(z_\xi)$ is empty, so that we only have to worry about $r<r_i$. Note that this will ensure that $\bigcap_{n\in\mathbb{N}}\overline{\bigcup_{i\ge n}G_{\xi i}}=\{z_\xi\}$, and that

$$\lim_{n\to\infty}2^n\nu(H_n(z_\xi)\cap\bigcup_{i\in\mathbb{N}}G_{\xi i})=0.$$

(c) Let X be the disjoint union of two copies of Z, expressed as $X=\{z^+:z\in Z\}\cup\{z^-:z\in Z\}$. Define $\pi:X\to Z$ by setting $\pi(z^+)=\pi(z^-)=z$ for every $z\in Z$.

For $x\in X$ define $V(x)\subseteq X$ as follows. If $z\in Z\setminus C_1$ set

$$V(z^+)=\{z^+\}\cup\pi^{-1}[E^+(z)],\quad V(z^-)=\{z^-\}\cup\pi^{-1}[E^-(z)].$$

For $\xi<\mathfrak{c}$ define V_ξ^+, V_ξ^-, $V(z_\xi^+)$ and $V(z_\xi^-)$ inductively by the rules

$$V_\xi^+=\bigcup_{i\in\mathbb{N}}V(w_{\xi i}^+)\cap\pi^{-1}[G_{\xi i}],$$

$$V_\xi^-=\bigcup_{i\in\mathbb{N}}V(w_{\xi i}^-)\cap\pi^{-1}[G_{\xi i}],$$

$$V(z_\xi^+)=\{z_\xi^+\}\cup V_\xi^+\cup(\pi^{-1}[E^+(z_\xi)]\setminus V_\xi^-),$$

$$V(z_\xi^-)=\{z_\xi^-\}\cup V_\xi^-\cup(\pi^{-1}[E^-(z_\xi)]\setminus V_\xi^+),$$

recalling that every $w_{\xi i}$ belongs to $\bigcup_{n\in\mathbb{N}}A_{\xi n}$, so that either $w_{\xi i}\notin C_1$ or $w_{\xi i}=z_\eta$ for some $\eta<\xi$.

(d) Now give X the topology $\mathfrak{T}$ generated by

$$\{V(x):x\in X\}\cup\{\pi^{-1}[G]:G\subseteq Z\text{ is open}\}.$$

2D Properties of X (a) For every $z\in Z$, $V(z^-)=X\setminus V(z^+)$. **P** For $z\in Z\setminus C_1$ this is just because $E^+(z)$, $E^-(z)$ part $Z\setminus\{z\}$. For $z=z_\xi$, induce on ξ. Given that $V(z_\eta^-)=X\setminus V(z_\eta^+)$ for every $\eta<\xi$, then $V(w_{\xi i}^-)=X\setminus V(w_{\xi i}^+)$ for every $i\in\mathbb{N}$, so $V_\xi^-\cap V_\xi^+=\emptyset$; also $z\notin\bigcup_{i\in\mathbb{N}}G_{\xi i}$, so neither z^+ nor z^- belongs to $V_\xi^+\cup V_\xi^-$; as $\pi^{-1}[E^+(z_\xi)]$, $\pi^{-1}[E^-(z_\xi)]$ part $X\setminus\{z_\xi^+,z_\xi^-\}$, we have all we need. **Q**

(b) If x, $y\in X$ and $y\in V(x)$, there is an $n\in\mathbb{N}$ such that $V(y)\cap\pi^{-1}[H_n(\pi(y))]\subseteq V(x)$. **P** (i) If $y=x$ this is trivial. Otherwise, $\pi(y)\ne\pi(x)$, since $z^+\notin V(z^-)$ and $z^-\notin V(z^+)$ for every z. (ii) If $y\ne x$ and $\pi(x)\notin C_1$ then $\pi(y)$ belongs to one of the open sets $E^+(\pi(x))$, $E^-(\pi(x))$ and the result is immediate. (iii) For the case $\pi(x)=z_\xi$, induce on ξ. Suppose that $x=z_\xi^+$. We have

$$y\in V(x)\setminus\{x\}=V_\xi^+\cup(\pi^{-1}[E^+(z_\xi)\setminus V_\xi^-).$$

(α) If $y\in V_\xi^+$ there is an $i\in\mathbb{N}$ such that $y\in V(w_{\xi i}^+)\cap\pi^{-1}[G_{\xi i}]$. By the inductive hypothesis and (ii), there is an $n\in\mathbb{N}$ such that $V(y)\cap\pi^{-1}[H_n(\pi(y))]\subseteq V(w_{\xi i}^+)$; increasing n if necessary, we can arrange that $H_n(\pi(y))\subseteq G_{\xi i}$, so that

$$V(y)\cap\pi^{-1}[H_n(\pi(y))]\subseteq V_\xi^+\subseteq V(x).$$

(β) If

$$y \in \pi^{-1}[E^+(z_\xi)] \setminus (V_\xi^+ \cup V_\xi^-) = \pi^{-1}[E^+(z_\xi) \setminus \bigcup_{i\in\mathbb{N}} G_{\xi i},$$

then $E^+(z_\xi) \setminus \bigcup_{i\in\mathbb{N}} G_{\xi i}$ is open (because $z_\xi \notin E^+(z_\xi)$, so there is an $n \in \mathbb{N}$ such that $H_n(\pi(y)) \subseteq E^+(z_\xi) \setminus \bigcup_{i\in\mathbb{N}} G_{\xi i}$ and $\pi^{-1}[H_n(\pi(y))] \subseteq V(x)$. The same arguments apply if $x = z_\xi^-$, so the induction proceeds. **Q**

(c) X is Hausdorff. **P** If x, $y \in X$ are distinct, either $\pi(x) \neq \pi(y)$ and they can be separated by open sets of the form $\pi^{-1}[G]$, $\pi^{-1}[G']$, or $\pi(x) = \pi(y) = z$ say and x, y are separated by $V(z^+)$, $V(z^-)$. **Q**

(d) By (b), $\{V(x) \cap \pi^{-1}[H_n(\pi(x))] : n \in \mathbb{N}\}$ is a base of neighbourhoods of x, for any $x \in X$. So X is first-countable. Moreover, it is compact. **P** Let $\mathcal{F}$ be an ultrafilter on X. Then $z = \lim \pi[[\mathcal{F}]]$ (definition: Fremlin 01, 2A1Ib) is defined in Z. Now either $V(z^+) \in \mathcal{F}$ and $\mathcal{F} \to z^+$ or $V(z^-) \in \mathcal{F}$ and $\mathcal{F} \to z^-$. **Q**

(e) Define $Q : Z \to [Z]^{\le\omega}$ by saying that

$$Q(z) = \{z\} \text{ if } z \in Z \setminus C_1,$$

$$Q(z_\xi) = \{z_\xi\} \cup \bigcup_{i\in\mathbb{N}} Q(\{w_{\xi i}) \text{ for } \xi < \mathfrak{c}.$$

If $z \in Z$, $x \in V(z^+)$, $y \in V(z^-)$ and $\pi(x) = \pi(y) = w$, then $w \in Q(z)$, $x = w^+$ and $y = w^-$. **P** If $z \notin C_1$, then $w = z$, $x = z^+$ and $y = z^-$, because $E^+(z)$ and $E^-(z)$ are disjoint. For $z = z_\xi$, induce on ξ. If $w \in G_{\xi i}$ for some $i \in \mathbb{N}$, then (because $V(w_{\xi i}^+) \cup V(w_{\xi i}^-) = X$) both x and y belong to $V_\xi^+ \cup V_\xi^-$. We must therefore have $x \in V(w_{\xi i}^+)$ and $y \in V(w_{\xi i}^-)$, and the inductive hypothesis tells us that $x = w^+$ and $y = w^-$, with $w \in Q(w_{\xi i}) \subseteq Q(z)$. Otherwise, $x \in \{z_\xi^+ \cup \pi^{-1}[E^+(z_\xi)]$ and $y \in \{z_\xi^+ \cup \pi^{-1}[E^-(z_\xi)]$, so $x = z_\xi^+$ and $y = z_\xi^-$. **Q**

2E Proposition X is a Radon space.

proof (a) Let μ be a Borel probability measure on X. Since $\pi : X \to Z$ is continuous, there is a Borel probability measure λ on Z defined by setting $\lambda E = \mu\pi^{-1}[E]$ for every Borel set $E \subseteq Z$. Now Z is hereditarily Lindelöf, so λ is τ-additive and has a support K, and $\mu\pi^{-1}[K] = 1$.

Let $\mathcal{V}$ be the family of μ-negligible open sets in X, and $W = \bigcup \mathcal{V}$. For each $x \in W$ let $m(x) \in \mathbb{N}$ be such that $V(x) \cap \pi^{-1}[H_{m(x)}(\pi(x))] \in \mathcal{V}$. Set

$$A = W \cap \pi^{-1}[K \setminus C_1], \quad B = W \cap \pi^{-1}[K \cap C_1],$$

$$D = \{z : z \in K, \limsup_{n\to\infty} 2^n \nu(H_n(z) \cap K) > 0\},$$

so that $\nu D = 0$ (2Aa).

(b) If $x \in A$ and $z = \pi(x)$, then either

$$\lambda(E^+(z) \cap H_{m(x)}(z)) \le \mu(V(x) \cap \pi^{-1}[H_{m(x)}(\pi(x))]) = 0$$

so $E^+(z) \cap H_{m(x)}(z) \subseteq Z \setminus K$, or

$$\lambda(E^-(z) \cap H_{m(x)}(z)) \le \mu(V(x) \cap \pi^{-1}[H_{m(x)}(\pi(x))]) = 0$$

so $E^-(z) \cap H_{m(x)}(z) \subseteq Z \setminus K$. By 2Ab above, $\pi[A]$ must be countable, so A is countable and $\mu A = 0$.

(c) $\pi[B] \subseteq D$. **P** If $x \in B$, then $\pi(x) = z_\xi$ for some $\xi < \mathfrak{c}$. Set

$$\begin{aligned} E &= E^+(z_\xi) \setminus \bigcup_{i\in\mathbb{N}} G_{\xi i} \text{ if } x = z_\xi^+, \\ &= E^-(z_\xi) \setminus \bigcup_{i\in\mathbb{N}} G_{\xi i} \text{ if } x = z_\xi^-, \end{aligned}$$

so that $E \subseteq Z$ is Borel and $\pi^{-1}[E] \subseteq V(x)$ and $E \cap H_{m(x)}(z_\xi) \cap K = \emptyset$. Now

$$\lim_{n\to\infty} 2^n \nu(H_n(z_\xi) \cap \bigcup_{i\in\mathbb{N}} G_{\xi i}) = 0$$

(see 2Cb), while

$$\limsup_{n\to\infty} 2^n \nu(H_n(z_\xi) \cap E^+(z_\xi), \quad \limsup_{n\to\infty} 2^n \nu(H_n(z_\xi) \cap E^-(z_\xi)$$

are both at least $\frac{1}{2}$ (2Ac); so

$$\limsup_{n\to\infty} 2^n\nu(H_n(z_\xi)\setminus K) \geq \limsup_{n\to\infty} 2^n\nu(E\cap H_n(z_\xi)) \geq \frac{1}{2}$$

and $z_\xi \in D$. **Q**

Now C was chosen so that $C\cap D$ would be universally negligible. If we look at the Borel measure λ_1 on Z defined by setting $\lambda_1 E = \mu^*(B\cap\pi^{-1}[E])$ for every Borel set $E\subseteq Z$, we see that $\lambda_1\{z\} = 0$ for every $z\in Z$, so $\lambda_1^*(C\cap D) = 0$ and there is a Borel set $E\supseteq C\cap D$ with

$$0 = \lambda_1 E = \mu^*(B\cap\pi^{-1}[E]) = \mu^* B.$$

(d) Putting these together, we have

$$\mu W \leq \mu\pi^{-1}[Z\setminus K] + \mu A + \mu^* B = 0.$$

As μ is arbitrary, X is Borel-measure-complete (Fremlin 03, 434Ia). But X is compact, so this is enough to ensure that X is a Radon space (Fremlin 03, 434Jf and 434Ka).

2F Proposition $X\times X$ is not a Radon space.

proof (a) Define $f : Z\to X\times X$ by setting $f(z) = (z^+, z^-)$ for $z\in Z$. Then $f[Z]$ is a Borel subset of $X\times X$. **P** The set

$$\Delta = \{(x,y) : \pi(x) = \pi(y)\} \subseteq X\times X$$

is closed. If $z\in Z$, then $V(z^+)\times V(z^-)$ is a neighbourhood of $f(z)$, and by 2Ae $\Delta\cap(V(z^+)\times V(z^-))\subseteq f[Z]$. So $f[Z]$ is relatively open in Δ and is a Borel subset of $X\times X$. **Q**

(b) Let $\mathfrak{S}$ be the usual topology on Z, and $\mathfrak{S}_1$ the coarsest topology such that f is continuous. Then

$$\lim^{\mathfrak{S}_1}_{i\to\infty} w_{\xi i} = z_\xi$$

for every $\xi < \mathfrak{c}$, just because $w^+_{\xi i}\in V(z^+_\xi)$ and $w^-_{\xi i}\in V(z^-_\xi)$ for every i, while $\langle w_{\xi i}\rangle_{i\in\mathbb{N}}$ converges to z for $\mathfrak{S}$. It follows that if $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence of $\mathfrak{S}_1$-closed sets with empty intersection, $\bigcap_{n\in\mathbb{N}}\overline{F}_n^{(\mathfrak{S})}$ is ν-negligible. **P** (Compare Fremlin 03, 439K). **?** Otherwise, let $\xi<\mathfrak{c}$ be such that $A_{\xi n}$ is an $\mathfrak{S}$-dense subset of F_n for each n. Then $\{i : w_{\xi i}\in A_{\xi n}\}$ is infinite for every n, so

$$z_\xi \in \overline{A}_{\xi n}^{\mathfrak{S}_1} \subseteq F_n$$

for every n. **XQ**

(c) There is an $\mathfrak{S}_1$-topological measure μ on Z, extending ν, which is inner regular for the $\mathfrak{S}_1$-closed sets. **P** (Again, I use the ideas from Fremlin 03 439K). Let $\mathcal{K}$ be the family of $\mathfrak{S}_1$-closed sets. For $F\in\mathcal{K}$, set $\phi F = \nu^* F$.

(i) If E, F are disjoint $\mathfrak{S}_1$-closed sets, then $\overline{E}^{\mathfrak{S}}\cap\overline{F}^{\mathfrak{S}}$ must be ν-negligible. So

$$\begin{aligned}\phi(E\cup F) = \nu^*(E\cup F) &= \nu^*((E\cup F)\cap\overline{E}^{\mathfrak{S}}) + \nu^*((E\cup F)\setminus\overline{E}^{\mathfrak{S}})\\ &= \nu^* E + \nu^* F = \phi E + \phi F.\end{aligned}$$

(ii) If E, $F\in\mathcal{K}$, $E\subseteq F$ and $\epsilon>0$, there is an $\mathfrak{S}$-open set $G\supseteq E$ such that $\nu G\leq \nu^* E + \epsilon$. Now $F\setminus G\in\mathcal{K}$ and

$$\phi F = \nu^*(F\cap G) + \nu^*(F\setminus G) \leq \nu G + \phi(F\setminus G) \leq \phi E + \phi(F\setminus G) + \epsilon.$$

Putting this together with (i), we see that

$$\phi F = \phi E + \sup\{\phi E' : E'\in\mathcal{K},\, E'\subseteq F\setminus E\}.$$

(iii) If $\langle F_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection, then

$$\lim_{n\to\infty}\phi F_n \leq \lim_{n\to\infty}\nu\overline{F}_n^{\mathfrak{S}} = \nu(\bigcap_{n\in\mathbb{N}}\overline{F}_n^{\mathfrak{S}}) = 0.$$

(iv) Thus $\mathcal{K}$ and ϕ satisfy all the conditions of Fremlin 03, 413I, and there is a measure μ, extending ϕ, which is defined on every member of $\mathcal{K}$ and inner regular with respect to $\mathcal{K}$, and therefore is (for $\mathfrak{S}_1$) a topological measure inner regular with respect to the closed sets. **Q**

(d) We therefore have a Borel measure λ on $X \times X$ defined by setting $\lambda W = \mu f^{-1}[W]$ for every Borel set $W \subseteq X \times X$. Now observe that, for any $z \in Z$,

$$f^{-1}[V(z^+) \times V(z^-)] \subseteq Q(z)$$

is countable, so

$$\lambda(V(z^+) \times V(z^-) = \nu f^{-1}[V(z^+) \times V(z^-)] = 0.$$

Thus every point of Δ has a λ-negligible neighbourhood; since $\lambda((X \times X) \setminus \Delta) = 0$, λ cannot be τ-additive, and $X \times X$ is not a Radon space.

3 Second example If $\mathfrak{c}$ is not measure-free, we have a much simpler method, due to R.G.Haydon. Let $I^{\|}$ be the split interval. Being a hereditarily Lindelöf compact Hausdorff space, it is Radon (Fremlin 03, 434Ke). But $I^{\|} \times I^{\|}$ has a discrete Borel subset $\{(z^+, z^-) : z \in [0,1]\}$ of cardinal $\mathfrak{c}$, so if there is a non-trivial measure with domain $\mathcal{P}\mathfrak{c}$ there is a non-τ-additive Borel probability measure on $I^{\|} \times I^{\|}$.

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