Version of 14.9.09

Problem GR

1 The problem For $k \geq 1$, let G_k be the following $(2k+1)$-move game for two players. I chooses $n \in \mathbb{N}$ and $L_0 = [2n]^n$. When I has chosen L_i, where $i < k$, II chooses $S_i \subseteq [L_i]^2$ and I chooses $L_{i+1} \subseteq L_i$ such that $[L_{i+1}]^2$ is either included in S_i or disjoint from S_i. I wins if $\#(L_k) \geq 3$ and $\bigcap L_k = \emptyset$. Which player has a winning strategy?

(Even the case $k = 1$ seems to be non-trivial; all I know is that the first player loses if he chooses $n \leq 2$.)

2 Weakly Ramsey ultrafilters and measure-centering ultrafilters (a) A filter $\mathcal{F}$ on a set I is **weakly Ramsey** (Blass 74) if whenever S_0, S_1, S_2 are disjoint subsets of $[I]^2$ there is a $J \in \mathcal{F}$ such that $[J]^2$ is disjoint from at least one of S_0, S_1, S_2.

(b) A filter $\mathcal{F}$ on a set I is **measure-centering** (or 'property M') if whenever $(\mathfrak{A}, \bar\mu)$ is a probability algebra and $\langle a_i \rangle_{i \in I}$ is a family in $\mathfrak{A}$ such that $\inf_{i \in I} \bar\mu a_i > 0$, then there is an $A \in \mathcal{F}$ such that $\{a_i : i \in A\}$ is centered (Benedikt 98, Fremlin 08, §538, Fremlin p09).

3 The real problem (due to A.Blass) Must every weakly Ramsey ultrafilter be measure-centering?

4 Proposition Suppose that $\mathfrak{p} = \mathfrak{c}$. If player I has a winning strategy in G_k for every $k \geq 1$, then there is a weakly Ramsey ultrafilter on $\mathbb{N}$ which is not measure-centering.

proof (a)(i) Set $I_n = [2n]^n$, for each n, and $I = \bigcup_{n \in \mathbb{N}} I_n$. Note that $\langle I_n \rangle_{n \in \mathbb{N}}$ is disjoint; for $L \in I$ let $\pi(L)$ be that n such that $L \in I_n$. For $k \in \mathbb{N}$ define $\mathcal{L}_k$ by saying that

$$\mathcal{L}_0 = \bigcup_{n \in \mathbb{N}} \mathcal{P} I_n, \quad \mathcal{L}_1 = \{L : L \in \mathcal{L}_0, \bigcap L = \emptyset, \#(L) \geq 3\}$$

and for $k \geq 1$ say that

$$\mathcal{L}_{k+1} = \{L : L \in \mathcal{L}_0, \text{ for every } S \subseteq [I]^2 \text{ there is an } L' \in \mathcal{L}_k \text{ such that } L' \subseteq L \text{ and } [L']^2 \text{ is either included in } S \text{ or disjoint from } S\}.$$

It is easy to see that if $L \in \mathcal{L}_k$ and $L \subseteq L' \in \mathcal{L}_0$ then $L' \in \mathcal{L}_k$; consequently $\mathcal{L}_{k+1} \subseteq \mathcal{L}_k$ for every k.

(ii) For $L \in \mathcal{L}_0$, say that its **depth** $\mathrm{depth}(L)$ is the greatest k such that $L \in \mathcal{L}_k$. (Of course $\mathrm{depth}(L) \leq \#(L)$.)

I will say that a subset A of I is **deep** if $\sup_{n \in \mathbb{N}} \mathrm{depth}(A \cap I_n) = \infty$. We are supposing that I has a winning strategy in G_k for every k; that is, that for every $k \geq 1$ there is an $n \in \mathbb{N}$ such that $\mathrm{depth}(I_n) \geq k+1$; in this case, I itself is deep.

(b)(i) Set $\Delta = \bigcup_{n \in \mathbb{N}} [I_n]^2$. If $A \subseteq I$ is deep and $S \subseteq [I]^2$, then there is a deep $A' \subseteq A$ such that $[A']^2 \cap \Delta$ is either included in S or disjoint from S. **P** For each $n \in \mathbb{N}$ such that $\mathrm{depth}(A \cap I_n) \geq 2$, there is an $L_n \subseteq A \cap I_n$ such that $\mathrm{depth}(L_n) \geq \mathrm{depth}(A \cap I_n) - 1$ and $[L_n]^2$ is either included in S or disjoint from S. Set

$$K = \{n : \mathrm{depth}(A \cap I_n) \geq 2, [L_n]^2 \subseteq S\},$$

$$K' = \{n : \mathrm{depth}(A \cap I_n) \geq 2, [L_n]^2 \cap S = \emptyset\};$$

then $\sup_{n \in K \cup K'} \mathrm{depth}(L_n)$ is infinite, so at least one of $\sup_{n \in K} \mathrm{depth}(L_n)$, $\sup_{n \in K'} \mathrm{depth}(L_n)$ is infinite. In the former case we can take $A' = \bigcup_{n \in K} L_n$, in the latter case we can take $A' = \bigcup_{n \in K'} L_n$. **Q**

(ii) If $A \subseteq I$ is deep and $B \subseteq A$, then at least one of B, $A \setminus B$ is deep. **P** Set $S = [B]^2 \cup [A \setminus B]^2$. By (i), there is a deep $A' \subseteq A$ such that $[A']^2 \cap \Delta$ is either included in S or disjoint from S. Set $A'' = \bigcup \{A' \cap I_n : n \in \mathbb{N}, \mathrm{depth}(A' \cap I_n) \geq 1\}$; then $\#(A'' \cap I_n) \geq 3$ whenever $A'' \cap I_n$ is non-empty, so $A'' \cap I_n$ is either included in B or disjoint from B. Of course A'' is deep. Set $K = \{n : A'' \cap I_n \subseteq B\}$. $K' = \{n : A'' \cap I_n \cap B = \emptyset\}$; then $\sup_{n \in K \cup K'} \mathrm{depth}(A'' \cap I_n)$ is infinite, so at least one of $\sup_{n \in K} \mathrm{depth}(I_n \cap B)$, $\sup_{n \in K'} \mathrm{depth}(I_n \cap A \setminus B)$ is infinite; in the former case B is deep, in the latter case $A \setminus B$ is deep. **Q**

It follows that if $A \subseteq I$ is deep and $A_0, \dots, A_m$ cover A, then there is some $j \leq m$ such that A_j is deep.

(iii) If $A \subseteq I$ is deep and $S \subseteq [I]^2$, there is a deep $A' \subseteq A$ such that $[A']^2 \setminus \Delta$ is either included in S or disjoint from S. **P** Choose $\langle n_i \rangle_{i\in\mathbb{N}}$ and $\langle A_i \rangle_{i\in\mathbb{N}}$ inductively, as follows. $A_0 = A$. Given A_i and $\langle n_j \rangle_{j<i}$ such that A_i is deep, let n_i be such that $n_j < n_i$ for every $j < i$ and $\operatorname{depth}(A_i \cap I_{n_i}) \geq i+2$; set $A'_i = A \cap \bigcup_{n>n_i} I_n$. For $L \in A'_i$ set $B_L = \{L' : L' \in \bigcup_{j\leq i} I_{nj},\ \{L, L'\} \in S\}$. By (ii), there is a deep $A_{i+1} \subseteq A'_i$ such that $B_L = B_{L'}$ for all L, $L' \in A_{i+1}$. Continue.

At the end of this induction, set $\hat{A} = \bigcup_{i\in\mathbb{N}} A_i \cap I_{n_i}$. Then $\hat{A} \subseteq A$ is deep, and for L, L', $L'' \in \hat{A}$, if $\pi(L) < \min(\pi(L'), \pi(L''))$ then $\{L, L'\} \in S$ iff $\{L, L''\} \in S$. Set $A^* = \{L : L \in \hat{A},\ \{L, L'\} \in S$ whenever $L' \in \hat{A}$ and $\pi(L') > \pi(L)\}$. If A^* is deep, set $A' = A^*$; otherwise set $A' = \hat{A} \setminus A^*$. **Q**

(iv) If $A \subseteq I$ is deep and S_0, S_1, $S_2 \subseteq [I]^2$ are disjoint, there are a deep $A' \subseteq A$ and a $j \leq 2$ such that $[A']^2 \cap S_j = \emptyset$. **P** By (ii), applied first to S_0 and then to S_1, there is a deep $B \subseteq A$ such that $[B]^2 \cap \Delta$ meets at most one S_j. By (iii), there is a deep $A' \subseteq B$ such that $[A']^2 \setminus \Delta$ meets at most one further S_k. So $[A']^2$ misses at least one of the three sets. **Q**

(v) Suppose that $\xi < \mathfrak{p}$ and that $\langle A_\eta \rangle_{\eta<\xi}$ is a family of deep subsets of I such that $A_\zeta \setminus A_\eta$ is finite whenever $\eta \leq \zeta < \xi$. Then there is a deep subset A of I such that $A \setminus A_\eta$ is finite for every $\eta < \xi$. **P** Set $P = [I]^{<\omega} \times [\xi]^{<\omega}$; for $p = (K_p, M_p)$ and $q = (K_q, M_q) \in P$, say that $p \leq q$ if $K_p \subseteq K_q$, $M_p \subseteq M_q$ and $K_q \setminus K_p \subseteq A_\eta$ for every $\eta \in M_p$. Then $(P, \leq)$ is a partially ordered set which is σ-centered upwards. For each $\eta < \xi$, $Q_\xi = \{p : \eta \in M_p\}$ is cofinal with P; for each $k \in \mathbb{N}$, $Q'_k = \{p : \max_{n\in\mathbb{N}} \operatorname{depth}(K_p \cap I_n) \geq k\}$ is cofinal with P. As $\xi < \mathfrak{p}$, there is an upwards-directed $R \subseteq P$ meeting every Q_η and every Q'_k; set $A = \bigcup\{K_p : p \in R\}$. **Q**

(c) There is a weakly Ramsey ultrafilter $\mathcal{F}$ on I such that every member of $\mathcal{F}$ is a deep set. **P** Enumerate as $\langle (S_\xi, S'_\xi, S''_\xi) \rangle_{\xi<\mathfrak{c}}$ the family of all disjoint triples (S, S', S'') of subsets of $[I]^2$, and enumerate $\mathcal{P}I$ as $\langle C_\xi \rangle_{\xi<\mathfrak{c}}$. Choose $\langle A_\xi \rangle_{\xi<\mathfrak{c}}$ as follows. $A_0 = I$. The inductive hypothesis will be that A_η is deep for every $\eta \leq \xi$ and that $A_\zeta \setminus A_\eta$ is finite whenever $\eta \leq \zeta \leq \xi$. Given A_ξ, use (b-ii) and (b-iv) to choose a deep set $A_{\xi+1} \subseteq A_\xi$ such that $A_{\xi+1}$ is either included in C_ξ or disjoint from it, and $[A_{\xi+1}]^2$ is disjoint from at least one of S_ξ, S'_ξ and S''_ξ. Given $\langle A_\eta \rangle_{\eta<\xi}$ where $\xi < \mathfrak{c}$ is a non-zero limit ordinal, use (b-v) (and the hypothesis that $\mathfrak{p} = \mathfrak{c}$) to find a deep set A_ξ such that $A_\xi \setminus A_\eta$ is finite for every $\eta < \xi$. Continue. At the end of the induction, let $\mathcal{F}$ be the filter generated by $\{A_\xi : \xi < \mathfrak{c}\}$. **Q**

(d) $\mathcal{F}$ is not measure-centering. **P** Let $(\mathfrak{A}, \bar{\mu})$ be an atomless probability algebra. For each $n \geq 1$, let $\langle c_i \rangle_{i<2n}$ be a partition of unity in $\mathfrak{A}$ consisting of elements of measure $\dfrac{1}{2n}$; for $L \in I_n$, set $a_L = \sup_{i\in L} c_{in}$. For completeness, set $a_\emptyset = 1$. This defines a family $\langle a_L \rangle_{L\in I}$ of elements of $\mathfrak{A}$ of measure at least $\frac{1}{2}$. If $A \in \mathfrak{F}$, there is an $n \in \mathbb{N}$ such that $\operatorname{depth}(A \cap I_n) \geq 1$, that is, $\#(A \cap I_n) \geq 3$ and $\bigcap(A \cap I_n)$ is empty; but now $\inf_{L\in A\cap I_n} a_L = 0$. So $\{a_L : L \in A\}$ is not centered; as A is arbitrary, $\mathcal{F}$ is not measure-centering. **Q**

5 A note on weakly Ramsey filters (a) Let $\mathcal{F}$ be a weakly Ramsey ultrafilter on $\mathbb{N}$ which is not Ramsey. Then there is a partition $\langle M_n \rangle_{n\in\mathbb{N}}$ of $\mathbb{N}$ into sets not belonging to $\mathcal{F}$ such that there is no selector for the partition. Set $f(i) = n$ for $i \in M_n$; then $f[[\mathcal{F}]]$ is a non-principal ultrafilter and $f{\restriction}A$ is not injective for any $A \in \mathcal{F}$. Set $\Delta = \bigcup_{n\in\mathbb{N}} [M_n]^2$.

(b)(i) If $S \subseteq [\mathbb{N}]^2$ then there is an $A \in \mathcal{F}$ such that $[A]^2 \cap \Delta$ is either included in S or disjoint from S. **P** Consider $S_0 = \Delta \cap S$, $S_1 = \Delta \setminus S$ and $S_2 = [\mathbb{N}]^2 \setminus \Delta$. There is an $A \in \mathcal{F}$ such that $[A]^2$ is disjoint from some S_j. $[A]^2$ cannot be included in Δ, so $[A]^2$ meets S_2. If $[A]^2$ does not meet S_1, then $[A]^2 \cap \Delta \subseteq S$; if $[A]^2$ does not meet S_0, then $[A]^2 \cap \Delta$ is disjoint from S. **Q**

(ii) If $S \subseteq [\mathbb{N}]^2$ then there is an $A \in \mathcal{F}$ such that $[A]^2 \setminus \Delta$ is either included in S or disjoint from S. **P** Consider $S_0 = \Delta$, $S_1 = S \setminus \Delta$ and $S_2 = ([\mathbb{N}]^2 \setminus S) \setminus \Delta$. There is an $A \in \mathcal{F}$ such that $[A]^2$ is disjoint from some S_j. $[A]^2$ cannot be disjoint from Δ, so $[A]^2$ meets S_0. If $[A]^2$ does not meet S_2, then $[A]^2 \setminus \Delta \subseteq S$; if $[A]^2$ does not meet S_1, then $[A]^2 \setminus \Delta$ is disjoint from S. **Q**

(c) $f[[\mathcal{F}]]$ is Ramsey. **P** If $S \subseteq [\mathbb{N}]^2$ set $S_0 = \Delta$, $S_1 = \{\{i, j\} : \{f(i), f(j)\} \in S\}$, $S_2 = \{\{i, j\} : \{f(i), f(j)\} \in [\mathbb{N}]^2 \setminus S\}$. There are an $A \in \mathcal{F}$ and a $k \leq 2$ such that $[A]^2 \cap S_k = \emptyset$. Since $f{\restriction}A$ is not injective, $k \neq 0$. If $k = 1$, $[f[A]]^2$ does not meet S. If $k = 2$, $[f[A]]^2 \subseteq S$. **Q**

There is an $A \in \mathcal{F}$ such that $f{\restriction}A$ is monotonic. **P** Set $S_0 = \Delta$, $S_1 = \{\{i, j\} : i < j,\ f(i) < f(j)\}$, $S_2 = \{\{i, j\} : i < j,\ f(i) > f(j)\}$. There are an $A \in \mathcal{F}$ and a $k \leq 2$ such that $[A]^2 \cap S_k = \emptyset$. Again, $k \neq 0$.

So f is monotonic. **Q** Since $f[A]$ is infinite, $f{\restriction}A$ must be non-decreasing. In particular, $A \cap M_n$ is finite for every n.

6 More games For $k \ge 1$ and $\epsilon > 0$ let $G_{k\epsilon}$ be the following $2k+1$-move game: I chooses a Boolean algebra $\mathfrak{A}$, an additive functional $\mu : \mathfrak{A} \to [0,1]$, and a finite set $\langle a_i \rangle_{i \in X_0}$ of elements of $\mathfrak{A}$ such that $\mu a_i \ge \epsilon$ for every i; given X_j, II chooses $S_j \subseteq [X_j]^2$ and I chooses $X_{j+1} \subseteq X_j$ such that $[X_{j+1}]^2$ is either included in S_j or disjoint from S_j; I wins if $\#(X_k) \ge 3$ and $\mu(\inf_{i \in X_k} a_i) = 0$.

7 Lemma If II has a winning strategy for G_k for some $k \ge 1$, then for every $\epsilon > 0$ there is an $l \ge 1$ such that II has a winning strategy for $G_{l\epsilon}$.

proof (a) To begin with (down to the end of (c) below) consider the case in which I starts by choosing an atomless probability algebra $(\mathfrak{A}, \bar\mu)$. Induce on $r = \lfloor \frac{1}{\epsilon} \rfloor$.

(b) Suppose that $r = 1$, that is, $\epsilon > \frac{1}{2}$. In this case II has a winning strategy for $G_{k\epsilon}$. **P?** Otherwise, because the game has only finitely many moves, I has a winning strategy for $G_{k\epsilon}$. Take $\epsilon' \in \left]\frac{1}{2}, \epsilon\right[$. Suppose that I plays $(\mathfrak{A}, \bar\mu)$ and $\langle a_i \rangle_{i \in X_0}$. II starts by taking $b_i \subseteq a_i$ such that $\bar\mu b_i \ge \epsilon'$ for each i and all the b_i are different, and considers the finite subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ generated by $\{b_i : i \in X_0\}$. Let C be the set of atoms of $\mathfrak{C}$, and let $n \ge 1$ be so large that $\frac{1}{2n} \sum_{c \in C} \lfloor 2n\bar\mu c \rfloor \ge \frac{3}{2} - \epsilon'$ and $\frac{1}{n} \le \bar\mu c$ for every $c \in C$. Let D be a partition of unity in $\mathfrak{A}$ such that every member of D has measure $\frac{1}{2n}$ and $\#(\{d : d \in D,\ d \subseteq c\}) = \lfloor 2n\bar\mu c \rfloor$ for every $c \in C$; setting $c' = \sup\{d : d \in D,\ d \subseteq c\}$ for $c \in C$, $\sum_{c \in C} \bar\mu(c \setminus c') \le \epsilon' - \frac{1}{2}$. Consequently, for each $i \in X_0$, $\#(\{d : d \in D,\ d \subseteq b_i\}) \ge n$ and we have a set $D_i \subseteq D$, with $\#(D_i) = n$, such that $\sup D_i \subseteq b_i$; moreover, since every element of C includes at least two elements of D, we can take it that $\sup D_i$ meets c whenever $c \in C$ and $c \subseteq b_i$, so that all the D_i are different.

Having done this, II now plays a game in G_k while I plays in $G_{k\epsilon}$, each following his winning strategy, and keeping in step as follows. To begin with, II supposes that in G_k I played n and $X_0' = \mathcal{P}[D]^n$, and replies with his winning move $S_0' \subseteq [X_0']^2$. I interprets this in $G_{k\epsilon}$ as the move $S_0 = \{\{i,j\} : \{D_i, D_j\} \in S_0'\}$, and replies with an S_0-homogeneous set $X_1 \subseteq X_0$. II interprets this as the S_0'-homogeneous set $X_1' = \{D_i : i \in X_0'\}$, and reacts with his prepared move S_1'. They carry on until I has chosen X_k and II can examine $X_k' = \{D_i : i \in X_k\}$. At this point, $\inf_{i \in X_k} a_i = 0$ and $\#(X_k) \ge 3$, because I was playing a winning strategy in $G_{k\epsilon}$; so $\inf_{i \in X_k} \sup D_i = 0$ and $\bigcap X_k' = \emptyset$, while $\#(X_k') = \#(X_k) \ge 3$, so II has lost the play in G_k, despite following a winning strategy there. **XQ**

(c) For the inductive step to $r+1 \ge 2$, given that $\epsilon > \frac{1}{r+1}$, set $\epsilon' = 2\epsilon - \epsilon^2 > \frac{1}{r}$. Let l be such that II has a winning strategy in $G_{l\epsilon'}$. Then II has a winning strategy in $G_{l+1,\epsilon}$. **P** Suppose that, playing in $G_{l+1,\epsilon}$, I starts with $(\mathfrak{A}, \bar\mu)$ and $\langle a_i \rangle_{i \in X_0}$. Let $(\mathfrak{C}, \bar\lambda)$ be the probability algebra free product $(\mathfrak{A}, \bar\mu) \widehat{\otimes}_{\mathrm{loc}} (\mathfrak{A}, \bar\mu)$, with canonical maps ε_0, $\varepsilon_1 : \mathfrak{A} \to \mathfrak{C}$ (Fremlin 02, §325). Set $c_i = \varepsilon_0 a_i \cup \varepsilon_1 a_i$, so that $\bar\lambda c_i = 2\bar\mu a_i - (\bar\mu a_i)^2 \ge \epsilon'$ for every $i \in X_0$. This time, I continues to play in $G_{l+1,\epsilon}$ while II plays in $G_{l,\epsilon'}$, the plays linked as follows. II treats $(\mathfrak{C}, \bar\lambda)$ and $\langle c_i \rangle_{i \in X_0}$ as a first move by his opponent in $G_{l,\epsilon'}$, and responds according to his winning strategy in this game with $S_0 \subseteq [X_0]^2$. I responds with an S_0-homogeneous set $X_1 \subseteq X_0$, and they continue.

When they reach X_l, II has won the play in $G_{l,\epsilon'}$ and can turn his attention to the state of play in $G_{l+1,\epsilon}$. If $\#(X_l) \le 2$, II plays $S_l = \emptyset$ and wins. Otherwise, $\inf_{i \in X_l} \varepsilon_0 a_i \cup \varepsilon_1 a_i$ is non-zero, so there is a $J \subseteq X_l$ such that $\inf_{i \in J} \varepsilon_0 a_i \cap \inf_{i \in X_l \setminus J} \varepsilon_1 a_i$ is non-zero, that is, $\inf_{i \in J} a_i$ and $\inf_{i \in X_l \setminus J} a_i$ are both non-zero. So II plays $S_l = [J]^2 \cup [X_l \setminus J]^2$. I has to pick an S_l-homogeneous set $X_{l+1} \subseteq X_l$; now either $\#(X_{l+1}) \le 2$ or $X_{l+1} \subseteq J$ or $X_{l+1} \subseteq X_l \setminus J$, and in all cases II wins. **Q**

(d) This deals with the case in which I chooses an atomless probability algebra on his first move. For the general case, if I starts with $(\mathfrak{A}, \mu)$ such that $\mu 1 = 0$ then X_0 must be empty and II has only to set $S_i = \emptyset$ for every i and wait for the end. Otherwise, there are an atomless probability algebra $(\mathfrak{B}, \bar\nu)$ and a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $\bar\nu\pi a = \frac{\mu a}{\mu 1}$ for every $a \in \mathfrak{A}$. Setting $\epsilon' = \frac{\epsilon}{\mu 1}$, II plays as if he were playing in an appropriate $G_{l\epsilon'}$ against the first move $(\mathfrak{B}, \bar\nu, \langle \pi a_i \rangle_{i \in X_0})$. At the end, he will have a position

X_l in which either $\#(X_l) \le 2$ or $\inf_{i\in X_l} \pi a_i \ne 0$; but in the latter case $\mu(\inf_{i\in X_l} a_i) > 0$, so II has also won in the game starting with $(\mathfrak{A}, \mu, \langle a_i\rangle_{i\in X_0})$.

8 Proposition Suppose that II has a winning strategy in G_k for some $k \ge 1$. Then every weakly Ramsey ultrafilter on $\mathbb{N}$ is measure-centering.

proof (a) Let $\mathcal{F}$ be a weakly Ramsey ultrafilter. If $\mathcal{F}$ is Ramsey, it is certainly measure-centering (Benedikt 98, or Fremlin 08, §538); so we may suppose henceforth that $\mathcal{F}$ is not Ramsey. In this case, by Note 5 above, there are an $A_0 \in \mathcal{F}$ and a function $f : \mathbb{N} \to \mathbb{N}$ such that $f[[\mathcal{F}]]$ is a non-principal Ramsey ultrafilter and $f\restriction A_0$ is non-decreasing. Set $M_n = f^{-1}[\{n\}]$ for each n, $\Delta = \bigcup_{n\in\mathbb{N}}[M_n]^2$. Note that if $A \in \mathcal{F}$, then $f\restriction A$ cannot be injective, because $\mathcal{F}$ is not Ramsey; so $A \cap \bigcup\{M_n : n \in \mathbb{N},\ \#(A \cap M_n) \ge 3\} \in \mathcal{F}$.

Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\langle a_i\rangle_{i\in\mathbb{N}}$ a sequence in $\mathfrak{A}$ such that $\epsilon = \inf_{i\in\mathbb{N}} a_i$ is non-zero. For $a \in \mathfrak{A}$ set $\lambda a = \lim_{i\to\mathcal{F}} \bar\mu(a \cap a_i)$; then $\lambda : \mathfrak{A} \to [0, 1]$ is an additive functional, and $\lambda 1 > 0$.

(b) If $a \in \mathfrak{A}$ and $\lambda a > 0$, there is a $\delta > 0$ such that $\{i : i \in \mathbb{N},\ \lambda(a \cap a_i) \ge \delta\} \in \mathcal{F}$. **P** Set $\delta = \frac{(\lambda a)^2}{3\bar\mu a}$, and consider $S = \{\{i, j\} : i < j,\ \bar\mu(a \cap a_i \cap a_j) \ge \delta\}$. Then there is an $A \in \mathcal{F}$ such that $\bar\mu(a \cap a_i) \ge \frac{2}{3}\lambda a$ for every $i \in A$, $A \subseteq A_0$ and $[A]^2 \setminus \Delta$ is either included in S or disjoint from S. Since $\frac{\bar\mu(a\cap a_i)}{\bar\mu a} \ge \frac{2}{3}\frac{\lambda a}{\bar\mu a}$ for every $i \in A$, there is an infinite $B \subseteq A$ such that $\frac{\bar\mu(a\cap a_i\cap a_j)}{\bar\mu a} \ge \frac{1}{3}\big(\frac{\lambda a}{\bar\mu a}\big)^2$, that is, $\bar\mu(a \cap a_i \cap a_j) \ge \delta$, for all $i, j \in B$. As $A\cap M_n$ is finite for every n, $[B]^2 \not\subseteq \Delta$ and $[A]^2\cap S\setminus\Delta$ is non-empty; accordingly $[A]^2 \setminus \Delta \subseteq S$. Now if $i \in A$, the set $A' = \{j : j \in A,\ f(j) > f(i)\}$ belongs to $\mathcal{F}$ and $\bar\mu(a \cap a_i \cap a_j) \ge \delta$ for every $j \in A'$, so $\lambda(a \cap a_i) \ge \delta$. Thus $\{i : \lambda(a \cap a_i) \ge \delta\}$ includes A and belongs to $\mathcal{F}$. **Q**

(c) If $a \in \mathfrak{A}$ and $\lambda a > 0$, there is an $A \in \mathcal{F}$ such that $\lambda(a \cap \inf\{a_i : a_i \in A \cap M_n\}) > 0$ for every $n \in \mathbb{N}$. **P** Let $\delta > 0$ be such that $A_1 = \{i : i \in A_0,\ \lambda(a \cap a_i) \ge \delta\}$ belongs to $\mathcal{F}$. By Lemma 7, there is an $l \in \mathbb{N}$ such that II has a winning strategy in $G_{l\delta}$. We call on this strategy in a simultaneous performance of infinitely many plays of the game, as follows. In the nth play, I starts with $(\mathfrak{A}, \lambda, \langle a\cap a_i\rangle_{i\in X_{n0}})$ where $X_{n0} = A_1 \cap M_n$, and II replies with $S_{n0} \subseteq [X_{n0}]^2$. Given $\langle X_{nj}\rangle_{n\in\mathbb{N}}$ and $\langle S_{nj}\rangle_{n\in\mathbb{N}}$ where $j < l$, with $B_j = \bigcup_{n\in\mathbb{N}} X_{nj} \in \mathcal{F}$, I considers $S_j = \bigcup_{n\in\mathbb{N}} S_{nj}$ and takes $B_{j+1} \in \mathcal{F}$ such that $B_{j+1} \subseteq B_j$ and $[B_{j+1}]^2 \cap \Delta$ is either included in S_j or disjoint from S_j; this is possible by 5(b-i). Now he plays $X_{n,j+1} = M_n \cap B_{j+1}$ for each n, so that $X_{n,j+1} \subseteq X_{nj}$ and $[X_{n,j+1}]^2$ is either included in S_{nj} or disjoint from it. If $j + 1 < l$, then II calls on his strategy to choose $S_{n,j+1} \subseteq [X_{n0}]^2$ for each $n \in \mathbb{N}$, and they continue.

At the end, II has won every play, so we have $B_l \in \mathcal{F}$ and $X_{nl} = B_l \cap M_n$ such that, for each n, either $\#(X_{nl}) \le 2$ or $\lambda(\inf_{i\in X_{nl}} a \cap a_i) > 0$. Take $A = B_l \cap \bigcup\{n : \#(B_l \cap M_n) \ge 3\}$; then $A \in \mathcal{F}$ and $\lambda(a \cap \inf_{i\in A\cap M_n} a_i) > 0$ for every $n \in \mathbb{N}$.

(d) For each $n \in \mathbb{N}$, let $\mathfrak{B}_n$ be the finite subalgebra of $\mathfrak{A}$ generated by $\{a_i : i \in A_0 \cap \bigcup_{m<n} M_n\}$; then there is a $C_n \in \mathcal{F}$ such that $\lambda(a \cap \inf_{i\in C_n\cap M_m} a_i) > 0$ whenever $m \in \mathbb{N}$, $a \in \mathfrak{B}_n$ and $\lambda a > 0$. Consider $S = \{\{i, j\} : f(i)+1 < f(j),\ j \in C_{f(i)+1}\}$. Then there is an $A_1 \in \mathcal{F}$ such that $A_1 \subseteq A_0\cap C_0$ and $[A_1]^2 \setminus \Delta$ is either included in S or disjoint from S. We certainly have $i, j \in A_1$ such that $f(i)+1 < f(j)$ and $j \in C_{f(i)+1}$, so we must have $[A]^2 \setminus \Delta \subseteq S$. Now $\lambda(\inf\{a_i : i \in A_1 \cap \bigcup_{m<n} M_m) > 0$ for every n. **P** Induce on n. The start $n = 0$, $\lambda 1 > 0$ is trivial. For the inductive step to $n+1 > 0$, set $a = \inf\{a_i : i \in A_1\cap\bigcup_{m<n} M_m\} \in \mathfrak{B}_n$, and consider $A_1 \cap M_n$. If $a = 1$, then $A_1 \cap M_n \subseteq C_0 \cap M_n$ so $\lambda(\inf_{i\in A_1\cap M_n} a_i) > 0$ and we can continue. Otherwise, take an $i \in A_1 \cap \bigcup_{m<n} A_m$ for which $f(i)$ is maximal; setting $m = f(i) + 1$, $a \in \mathfrak{B}_m$, while $A_1 \cap M_n \subseteq C_m$ (because either $A_1 \cap M_n$ is empty or $m < n$) and

$$\lambda(a \cap \inf_{i\in A_1\cap M_n} a_i) \ge \lambda(a \cap \inf_{i\in C_m\cap M_n} a_i) > 0,$$

so again the induction proceeds.

(e) Thus $\{a_i : i \in A_1\}$ is centered; as $\langle a_i\rangle_{i\in\mathbb{N}}$ is arbitrary, $\mathcal{F}$ is measure-centering.

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