Version of 27.7.07

Problem GN

D.H.Fremlin

University of Essex, Colchester, England

1 The problem Suppose that ν is a non-zero Radon measure on $\mathbb{R}$ and that $A \subseteq \mathbb{R}$ is such that $\nu(A+x)=0$ for every $x \in \mathbb{R}$. Does it follow that A is Lebesgue negligible?

2 First answer If the uniformity $\mathrm{non}\,\mathcal{N}$ of the Lebesgue null ideal $\mathcal{N}$ is $\mathfrak{c}$, no.

proof Let ν be Cantor measure on $\mathbb{R}$, that is, the image of the usual measure on $\{0,1\}^{\mathbb{N}}$ under the map $x \mapsto \frac{2}{3}\sum_{n=0}^{\infty} 3^{-n}x(n)$ (Fremlin 01, 256Hc). Let C be the Cantor set, so that $\nu C = 1$. Enumerate $\mathbb{R}$ as $\langle t_\xi \rangle_{\xi<\mathfrak{c}}$ and the family of compact sets of non-zero Lebesgue measure as $\langle K_\xi \rangle_{\xi<\mathfrak{c}}$. By Fremlin n05, Theorem 4, $K_\xi \not\subseteq \bigcup_{\eta<\xi} t_\eta + C$; so we can choose $s_\xi \in K_\xi \setminus \bigcup_{\eta<\xi} t_\eta + C$ for every $\xi < \mathfrak{c}$. Set $A = \{s_\xi : \xi < \mathfrak{c}\}$. As $A \cap K_\xi \neq \emptyset$ for every ξ, A has full outer measure for Lebesgue measure, and is certainly not Lebesgue negligible. But $\#(C \cap (A+x)) = \#(A \cap (C-x)) < \mathfrak{c}$, so $\nu(A+x)=0$ for every $x \in \mathbb{R}$, by Fremlin 08?, 522Va.

3 Second answer If the covering number $\mathrm{cov}\,\mathcal{N}$ of $\mathcal{N}$ is equal to its cofinality $\mathrm{cf}\,\mathcal{N}$, no.

proof (a) Note first that there is a non-empty perfect set $K \subseteq \mathbb{R}$ such that $K-K$ is Lebesgue negligible. **P** Set $K = \{\frac{4}{5}\sum_{n=0}^{\infty} 5^{-n}x(n) : x \in \{0,1\}^{\mathbb{N}}\}$; then for each $n \in \mathbb{N}$, $K-K$ is covered by 4^n intervals of length $2 \cdot 5^{-n}$. **Q** Let ν be the corresponding Radon measure on $\mathbb{R}$, the image of the usual measure on $\{0,1\}^{\mathbb{N}}$ under the map $x \mapsto \frac{4}{5}\sum_{n=0}^{\infty} 5^{-n}x(n)$.

(b) Write Σ for the σ-algebra of Lebesgue measurable sets. Then $\mathrm{ci}(\Sigma \setminus \mathcal{N}) = \mathrm{cf}\,\mathcal{N}$ (Fremlin 08?, 524O). Let $\langle K_\xi \rangle_{\xi<\mathrm{cf}\,\mathcal{N}}$ run over a coinitial set in $\Sigma \setminus \mathcal{N}$. Choose $\langle t_\xi \rangle_{\xi<\mathrm{cf}\,\mathcal{N}}$ inductively so that $t_\xi \in K_\xi \setminus \bigcup_{\eta<\xi} t_\eta + K - K$; this is possible because $t_\eta + K - K$ is always Lebesgue negligible and $\mathrm{cov}\,\mathcal{N} = \mathrm{cf}\,\mathcal{N}$. Set $A = \{t_\xi : \xi < \mathrm{cf}\,\mathcal{N}\}$. Because $A \cap K_\xi$ is non-empty for every ξ, A has full outer Lebesgue measure. Also $x_\xi - x_\eta \notin K-K$ for all $\eta < \xi < \mathrm{cf}\,\mathcal{N}$, so $\#(K \cap (A+x)) \le 1$ for every $x \in \mathbb{R}$, and $\nu(K \cap (A+x)) = 0$ for every x.

4 Third answer If there is a Sierpiński set for Lebesgue measure, no.

proof If $A \subseteq \mathbb{R}$ is a Sierpiński set, it is non-negligible and meets $C-x$ in a countable set for every $x \in \mathbb{R}$, so $\nu(A+x)=0$ for every x, where ν is Cantor measure on $\mathbb{R}$, as in §2.

5 Supplementary problem Let X be a topological group carrying Haar measures (Fremlin 03, chap. 44). Suppose that $A \subseteq X$ and that ν is a non-zero quasi-Radon measure on X such that $\nu(xA)=0$ for every $x \in X$. Does it follow that A is Haar negligible? If A is Haar measurable, yes (Fremlin 03, 444L).

References

Fremlin D.H. [01] *Measure Theory, Vol. 2: Broad Foundations.* Torres Fremlin, 2001.

Fremlin D.H. [03] *Measure Theory, Vol. 4: Topological Measure Spaces.* Torres Fremlin, 2003.

Fremlin D.H. [08?] *Measure Theory, Vol. 5: Set-theoretic Measure Theory,* in preparation. *Partial drafts available via* `http://www.essex.ac.uk/maths/staff/fremlin/mt.htm`. *Note that paragraph references are subject to fluctuation.*

Fremlin D.H. [n05] 'On results of M.Elekes and G.Gruenhage', note of 14.1.05 (`http://www.essex.ac.uk/maths/staff/fremlin/preprints.htm`).