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Problem GI

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1 Definitions (a) The **centering number** of a Boolean algebra $\mathfrak{A}$ is the smallest cardinal of any family of centered subsets of $\mathfrak{A}$ covering $\mathfrak{A}\setminus\{0\}$; it is the topological density of the Stone space of $\mathfrak{A}$ (Fremlin 08?, 514C).

For $n\ge 2$, a Boolean algebra $\mathfrak{A}$ is **σ-n-linked** if $\mathfrak{A}\setminus\{0\}$ is expressible as $\bigcup_{k\in\mathbb{N}}A_k$ where $\inf_{i<n}a_i\ne 0$ for any family $\langle a_i\rangle_{i\in n}$ in any A_k.

(b) $\mathcal{N}$ will be the null ideal of Lebesgue measure on $\mathbb{R}$, and $\operatorname{cf}\mathcal{N}$ its cofinality (see Fremlin 08?, §522).

(c) I will say that **Gaifman's algebra** is the algebra constructed in Fremlin 02, 391N, based on ideas in Gaifman 64. To recapitulate: let $\mathcal{I}$ be the set of half-open intervals of the form $[q,q'[$ where q, $q'\in\mathbb{Q}$ and $q<q'$. Then $\mathcal{I}$ is countable; enumerate it as $\langle I_k\rangle_{k\in\mathbb{N}}$. For each $n\in\mathbb{N}$ let $\mathcal{J}_k\subseteq\mathcal{I}$ be a disjoint family of subintervals of I_k such that $\#(\mathcal{J}_k)=(k+1)^2$. Give $\mathcal{P}\mathbb{R}$ its usual topology, so that it is a compact Hausdorff space. Set

$$X=\bigcap_{k\in\mathbb{N}}\{x:x\in\mathcal{P}\mathbb{R},\,\#(\{J:J\in\mathcal{J}_k,\,x\cap J\ne\emptyset\})\le k+1\}.$$

Gaifman's algebra is the algebra of open-and-closed subsets of X. As noted in Fremlin 02, 391Yh, Gaifman's algebra is σ-n-linked for every n.

2 Theorem The centering number of Gaifman's algebra is at most $\operatorname{cf}\mathcal{N}$.

proof (a) For each $n\in\mathbb{N}$ set

$$\Phi_n=\prod_{4^n\le k<4^{n+1}}[\mathcal{J}_k]^{n+1},\qquad M_n=\#(\Phi_n)$$

and let $\langle\phi_{ni}\rangle_{i<M_n}$ be an enumeration of Φ_n; for $i<M_n$ set

$$K_{ni}=\bigcap_{4^n\le k<4^{n+1}}(\mathbb{R}\setminus\bigcup\mathcal{J}_k)\cup\bigcup\phi_{ni}(k)$$

so that $\{J:J\in\mathcal{J}_k,\,K_{ni}\cap J\ne\emptyset\}\subseteq\phi_{ni}(k)$ has at most $n+1$ members if $4^n\le k<4^{n+1}$.

For $n\in\mathbb{N}$, $f\in\prod_{k<4^n}[\mathcal{J}_k]^{k+1}$ set

$$L_f=\bigcap_{k<4^n}(\mathbb{R}\setminus\bigcup\mathcal{J}_k)\cup\bigcup f(k),$$

so that $\{J:J\in\mathcal{J}_k,\,L_f\cap J\ne\emptyset\}\subseteq f(k)$ has at most $k+1$ members if $k<4^n$.
Set

$$\mathcal{S}=\{S:S\subseteq\mathbb{N}\times\mathbb{N},\,S[\{n\}]\subseteq M_n\text{ and }\#(S[\{n\}])\le 2^n\text{ for every }n\in\mathbb{N}\},$$

and for $n\in\mathbb{N}$, $f\in\prod_{k<4^n}[\mathcal{J}_k]^{k+1}$ and $S\in\mathcal{S}$ set

$$x_{fS}=L_f\cap\bigcap_{m\ge n}\bigcup_{i\in S[\{m\}]\cap M_m}K_{mi}.$$

(b) $x_{fS}\in X$ whenever $n\in\mathbb{N}$, $f\in\prod_{k<4^n}[\mathcal{J}_k]^{\le k+1}$ and $S\in\mathcal{S}$. **P** If $k<4^n$ then $x_S\subseteq L_f$ so

$$\{J:J\in\mathcal{J}_k,\,x_S\cap J\ne\emptyset\}\subseteq\{J:J\in\mathcal{J}_k,\,L_f\cap J\ne\emptyset\}$$

has at most $k+1$ members. If $k\ge 4^n$, let $m\ge n$ be such that $4^m\le k<4^{m+1}$; then $x_S\subseteq\bigcup_{i\in S[\{m\}]\cap M_m}K_{mi}$, so

$$\begin{aligned}\#(\{J:J\in\mathcal{J}_k,\,x_S\cap J\ne\emptyset\})&\le\sum_{i\in S[\{m\}]\cap M_m}\#(\{J:J\in\mathcal{J}_k,\,K_{mi}\cap J\ne\emptyset\})\\&\le 2^m(m+1)\le 4^m\le k+1.\ \mathbf{Q}\end{aligned}$$

(c) $\mathrm{cf}\mathcal{N}$ is the least cardinal of any $\mathcal{R} \subseteq \mathcal{S}$ such that for every $h \in \mathbb{N}^{\mathbb{N}}$ there is an $R \in \mathcal{R}$ such that $h \setminus R$ is finite. **P** FREMLIN 08?, 522M. **Q** Of course this is equal to the least cardinal of any $\mathcal{R} \subseteq \mathcal{S}$ such that for every $h \in \mathbb{N}^{\mathbb{N}}$ there is an $R \in \mathcal{R}$ covering h.

(d) If $x \in X$ is finite, there are $n \in \mathbb{N}$, $f \in \prod_{k<4^n}[\mathcal{J}_k]^{k+1}$ and $R \in \mathcal{R}$ such that $x \subseteq x_{fR}$. **P** For $k \in \mathbb{N}$ let $g(k) \in [\mathcal{J}_k]^{k+1}$ be such that $\{J : J \in \mathcal{J}_k,\ x \cap J \neq \emptyset\} \subseteq g(k)$. For $m \in \mathbb{N}$ let $h(m) < M_m$ be such that $g(k) = \phi_{m,h(m)}(k)$ for $4^m \le k < 4^{m+1}$; then there are $R \in \mathcal{R}$ and $n \ge \#(x)$ such that $h(m) \in R[\{m\}] \cap M_m$ for every $m \ge n$. Set $f = g\restriction 4^n$. Then $x \in L_f$. Also, for $m \ge n$ and $4^m \le k < 4^{m+1}$, $g(k) = \phi_{m,h(m)}(k)$; so

$$x \subseteq \bigcap_{4^m \le k < 4^{m+1}}(\mathbb{R} \setminus \bigcup \mathcal{J}_k) \cup \bigcup g(k) = K_{m,h(m)} \subseteq \bigcup_{i \in R[\{m\}] \cap M_m} K_{mi}.$$

So $x \subseteq x_{fR}$. **Q**

(e) For any $x \in X$, $\mathcal{P}x \subseteq X$, and $\mathcal{P}x$ is separable just because $\#(x) \le \mathfrak{c}$. So

$$Y = \bigcup\{\mathcal{P}x_{fR} : R \in \mathcal{R},\ n \in \mathbb{N},\ f \in \prod_{k<4^m} \mathcal{J}_k\}$$

has density at most $\mathrm{cf}\mathcal{N}$. But (d) tells us that Y includes $X \cap [\mathbb{R}]^{<\omega}$, so Y is dense in X and $d(X)$, the centering number of Gaifman's algebra, is at most $\mathrm{cf}\mathcal{N}$.

3 Example For an example of a Boolean algebra, σ-n-linked for every n, with centering number exactly $\mathrm{cf}\mathcal{N}$, take $\mathcal{S}$ as in the proof above. Let P be the set of triplets (I, m, F) where $n \in \mathbb{N}$, $F : m \to [\mathbb{N}]^{<\omega}$ is a function, $I \in [\mathbb{N}^{\mathbb{N}}]^{\le m}$ is a finite set, $f(i) \in F(i)$ whenever $i < n$ and $f \in I$, and $\#(F(i)) \le 2^i$ whenever $i < m$. Say $(I, l, F) \le (J, m, G)$ if $I \subseteq J$ and G extends F. If $R \subseteq P$ is upwards-directed, then there is an $S \in \mathcal{S}$ such that $f \subseteq S$ whenever $(I, m, F) \in R$ and $f \in I$. Conversely, any expression of P as a union of upwards-directed sets leads to a family in $\mathcal{S}$ covering every function. So the upwards-centering number $d^{\uparrow}(P)$ is $\mathrm{cf}\mathcal{N}$ precisely, by FREMLIN 08?, 522M again.

On the other hand, given $n \ge 2$, and $G : mn \to [\mathbb{N}]^{<\omega}$ such that $\#(G(i)) \le 2^i$ for every $i < mn$, then

$$\{(I, mn, G) : I \in [\mathbb{N}^{\mathbb{N}}]^{\le m},\ f(i) \in G(i) \text{ whenever } f \in I \text{ and } i < mn\}$$

is upwards-n-linked, and the union of these n-linked sets is cofinal with P; so P is σ-n-linked upwards.

If we now take $\mathfrak{A}$ to be the regular open algebra of P, then $\mathfrak{A}$ is σ-n-linked for every $n \ge 2$ and has centering number $\mathrm{cf}\mathcal{N}$ (FREMLIN 08?, 514O).

4 Example (See TODORČEVIĆ 86, Theorem 11.) Let $n \ge 2$. Then there is a partially ordered set P such that P is σ-n-linked upwards but $\mathrm{link}^{\uparrow}_{n+1}(P) = \mathfrak{c}$. So there is a σ-n-linked Boolean algebra $\mathfrak{A}$ which has $(n+1)$-linking number $\mathfrak{c}$. In particular, there is a σ-linked Boolean algebra with centering number $\mathfrak{c}$.

proof (a) There is an injective function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\mathbb{N}^{\mathbb{N}}$ cannot be covered by fewer than $\mathfrak{c}$ sets on which f is continuous. **P** Enumerate $\mathbb{N}^{\mathbb{N}}$ as $\langle x_\xi \rangle_{\xi<\mathfrak{c}}$ and $[\mathbb{N}^{\mathbb{N}}]^{\le\omega}$ as $\langle D_\xi \rangle_{\xi<\mathfrak{c}}$ in such a way that $D_\xi \subseteq \{x_\eta : \eta < \xi\}$ for every ξ. Choose $\langle f(x_\xi) \rangle_{\xi<\mathfrak{c}}$ inductively in such a way that, for every $\eta < \xi < \mathfrak{c}$,

$f(x_\xi) \neq f(x_\eta)$,

if $x_\xi \in \overline{D}_\eta$ and $\lim_{x \in D_\eta, x \to x_\xi} f(x)$ is defined, then $f(x_\xi)$ is not equal to this limit.

If now $\mathcal{A}$ is a cover of $\mathbb{N}^{\mathbb{N}}$ by fewer than $\mathfrak{c}$ sets, take η_A, for $A \in \mathcal{A}$, such that D_{η_A} is dense in A, and let $\xi < \mathfrak{c}$ be such that $\eta_A < \xi$ for every $A \in \mathcal{A}$. Then there is an $A \in \mathcal{A}$ such that $x_\xi \in A$; but now $f\restriction D_{\eta_A} \cup \{x_\xi\}$ cannot be continuous, so $f\restriction A$ is not continuous. **Q**

(b) Give $\mathbb{N}^{\mathbb{N}}$ the metric ρ where $\rho(x, y) = 2^{-m}$ if $x\restriction m = y\restriction m$ and $x(m) \neq y(m)$. Let P be the family of all finite sets $p \subseteq \mathbb{N}^{\mathbb{N}}$ such that

$$\text{if } x \in p \text{ and } m \in \mathbb{N} \text{ then } \#(\{\rho(x, y) : y \in p,\ \rho(f(x), f(y)) = 2^{-m}\}) < n,$$

ordered by inclusion.

(i) P is σ-n-linked upwards. **P** For $k \in \mathbb{N}$, $K \in [\mathbb{N}^k]^{<\omega}$ and $h : K \to \mathbb{N}^k$, set

$$A(k, K, h) = \{p : p \in P,\ f(x)\restriction k \neq f(y)\restriction k \text{ for all distinct } x,\ y \in p,$$
$$f(x)\restriction k = h(x\restriction k) \text{ for every } x \in p\}.$$

If $p_i \in A(k, K, h)$ for $i < n$, set $p = \bigcup_{i<n} p_i$. Take $x \in p$ and $m \in \mathbb{N}$; let $j < n$ be such that $x \in p_j$. If $m \le k$ then

$$\{\rho(x,y) : y \in p,\ \rho(f(x),f(y)) = 2^{-m}\} = \{\rho(x,y) : y \in p_j,\ \rho(f(x),f(y)) = 2^{-m}\}$$

has fewer than n members. If $m > k$ then $\{\rho(x,y) : y \in p,\ \rho(f(x),f(y)) = 2^{-m}\}$ does not meet p_j and meets p_i in at most one point for each $i \neq j$, so again has fewer than n members. So $p \in P$.

This shows that each $A(k,K,h)$ is upwards-n-linked; as P is covered by countably many sets of this type, it is σ-n-linked upwards. **Q**

(ii) If $A \subseteq P$ is upwards-$(n+1)$-linked, then $f\restriction\bigcup A$ is continuous. **P?** Otherwise, there are $x \in p \in A$ and sequences $\langle p_n\rangle_{n\in\mathbb{N}}$, $\langle x_n\rangle_{n\in\mathbb{N}}$ such that $x_n \in p_n \in A$ and $\langle x_n\rangle_{n\in\mathbb{N}} \to x$ and all the x_n are distinct, but $\langle f(x_n)\rangle_{n\in\mathbb{N}} \not\to f(x)$. Taking a subsequence if necessary, we can suppose that there is an $m \in \mathbb{N}$ such that $\rho(f(x_n),f(x)) = 2^{-m}$ for every n and $0 < \rho(x,x_{n+1}) < \rho(x,x_n)$ for every n. But now $p, p_0, \ldots, p_{n-1}$ have no upper bound in P. **XQ**

By the choice of f, $\mathrm{link}^{\uparrow}_{n+1}(P) = \mathfrak{c}$.

(c) Now set $\mathfrak{A} = \mathrm{RO}^{\uparrow}(P)$.

5 Problems (a) Problem GI asks: can we find, in ZFC alone, a Boolean algebra which is σ-n-linked for every n and has centering number exactly $\mathfrak{c}$? In Example 3 we have one with centering number $\mathrm{cf}\,\mathcal{N}$. Note that the measure algebra $\mathfrak{B}_{\mathfrak{c}}$ of the usual measure $\nu_{\mathfrak{c}}$ on $\{0,1\}^{\mathfrak{c}}$ is σ-n-linked for every $n \geq 2$ (Dow & Steprāns 94, or Fremlin 08?, 524M), and its centering number is the uniformity $\mathrm{non}\,\mathcal{N}_{\mathfrak{c}}$ of the null ideal of $\nu_{\mathfrak{c}}$ (Fremlin 08?, 524N). If $\mathrm{non}\,\mathcal{N} = \omega_1$ and $\mathrm{cf}\,\mathcal{N} = \mathfrak{c} = \omega_2$ then $\mathrm{non}\,\mathcal{N}_{\mathfrak{c}} = \omega_1 < \mathrm{cf}\,\mathcal{N}$ (Kraszewski 01, or Fremlin 08?, 523J). I do not know whether $\mathrm{non}\,\mathcal{N}_{\mathfrak{c}}$ can be greater than $\mathrm{cf}\,\mathcal{N}$.

(b) In Theorem 2 I have not used the full strength of Bartoszyński's result as described in Fremlin 08?, 522M; so perhaps a change in (c) of the proof can give a stronger result.

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References

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