Distributivity of quotients of Borel algebras

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1 Proposition (Zapletal 04) Let $\mathcal{B}$ be the Borel σ-algebra of a Polish space X, and $\mathcal{I}$ a σ-ideal of $\mathcal{B}$. Then the following are equiveridical:

(i) the quotient algebra $\mathfrak{A} = \mathcal{B}/\mathcal{I}$ is weakly σ-distributive;

(ii) for every Borel measurable $f : X \to \mathbb{R}$ and $E \in \mathcal{B} \setminus \mathcal{I}$ there is a compact set $K \subseteq E$ such that $K \notin \mathcal{I}$ and $f \restriction K$ is continuous.

proof (i)$\Rightarrow$(ii) Let $\mathfrak{S}$ be a Polish topology on X, finer than the given topology $\mathfrak{T}$, such that E is $\mathfrak{S}$-closed and f is $\mathfrak{S}$-continuous. Recall that $\mathcal{B}$ is the algebra of $\mathfrak{S}$-Borel sets. Let ρ be a metric on X, defining $\mathfrak{S}$, for which X is complete. In this case, for each $n \in \mathbb{N}$, there is a cover of X by a sequence $\langle F_{ni} \rangle_{i \in \mathbb{N}}$ of Borel sets of diameter at most 2^{-n}. Set $G_{ni} = F_{ni} \setminus \bigcup_{j<i} F_{nj}$; then $\{G^\bullet_{ni} : i \in \mathbb{N}\}$ is a partition of unity in $\mathfrak{A}$ (because $\mathcal{I}$ is a σ-ideal). Let C be a partition of unity in $\mathfrak{A}$ such that $\{i : c \cap G^\bullet_{ni} \neq 0\}$ is finite for each n; let $c \in C$ be such that $c \cap E \neq 0$. For $n \in \mathbb{N}$ let $m_n \in \mathbb{N}$ be such that $c \cap G^\bullet_{ni} = 0$ for $i > m_n$, so that $c \subseteq \sup_{i \le m_n} G^\bullet_{ni}$.

Consider

$$K = E \cap \bigcap_{n\in\mathbb{N}} \bigcup_{i \le m_n} \overline{G}_{ni}.$$

Then K is closed and ρ-totally bounded, therefore compact, and of course $K \subseteq E$. Also

$$0 \neq E^\bullet \cap c \subseteq E^\bullet \cap \inf_{n\in\mathbb{N}} \sup_{i \le m_n} G^\bullet_{ni} \subseteq K^\bullet,$$

so $K \notin \mathcal{I}$. Since $\mathfrak{S}$ and $\mathfrak{T}$ give the same subspace topology on K, and $f \restriction K$ is $\mathfrak{S}$-continuous, $f \restriction K$ is $\mathfrak{T}$-continuous.

(ii)$\Rightarrow$(i) Let $\langle A_n \rangle_{n\in\mathbb{N}}$ be a sequence of countable maximal antichains in $\mathfrak{A}$. For each $n \in \mathbb{N}$, we have a partition $\langle E_{ni} \rangle_{i\in\mathbb{N}}$ of X into Borel sets such that $\{E^\bullet_{ni} : i \in \mathbb{N}\} = A_n \cup \{0\}$. Define $f : X \to \mathbb{N}^\mathbb{N}$ by setting $f(x)(n) = i$ if $x \in E_{ni}$; then f is Borel measurable. Set $D = \{K^\bullet : K \subseteq X$ is compact, $f \restriction K$ is continuous$\}$. Then (ii) tells us that D is order-dense in $\mathfrak{A}$ (remember that $\mathbb{N}^\mathbb{N}$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$). So there is a maximal antichain $B \subseteq D$. If $b \in B$, express it as $K^\bullet$ where K is compact and $f \restriction K$ is continuous. For each n, $\{f(x)(n) : x \in K\}$ is bounded, that is, there is some m such that $K \subseteq \bigcup_{i \le m} E_{ni}$; now $b \subseteq \sup_{i\le m} E^\bullet_{ni}$ so $\{a : a \in A_n,\ b \cap a \neq 0\}$ is finite. As $\langle A_n \rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly σ-distributive.

2 Proposition There is a σ-ideal $\mathcal{I}$ of the Borel σ-algebra $\mathcal{B}$ of $[0,1]^\mathbb{N}$ such that the quotient algebra $\mathfrak{A} = \mathcal{B}/\mathcal{I}$ is weakly σ-distributive but not weakly (σ,∞)-distributive.

proof (a) Let $\mathcal{K}$ be the family of all sets of the form $\prod_{i\in\mathbb{N}} K_i$ where $K_i \subseteq [0,1]$ is compact and not empty for each i and $\{i : K_i$ is countable$\}$ is finite. Let $\mathcal{I}$ be the family of Borel sets not including any member of $\mathcal{K}$. Then $\mathcal{I}$ is a σ-ideal. **P** Of course $E \in \mathcal{I}$

whenever $E \in \mathcal{B}$ and $E \subseteq F \in \mathcal{I}$. Now let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{I}$ with union $E \notin \mathcal{I}$. Let $K = \prod_{i\in\mathbb{N}} K_i$ be a member of $\mathcal{K}$ included in E; trimming K down if need be, we may suppose that $K_i = \{t_i\}$ is a singleton for $i < m$ and uncountable for $i \ge m$. For $i \ge m$, let μ_i be an atomless probability measure on K_i; let μ be the product Radon measure on $K' = \prod_{i\ge m} K_i$. Then there must be some $n \in \mathbb{N}$ such that $\mu E'_n > 0$, where $E'_n = \{x : x \in K,\ \langle t_i\rangle_{i<m}{}^\frown x \in E_n\}$. By Cieselski & Pawlikowski 03 or Fremlin 03, 494C, there is a family $\langle L_i\rangle_{i\ge m}$ such that each L_i is an uncountable compact subset of K_i and $\prod_{i\ge m} L_i \subseteq E'_n$. But now, setting $L_i = \{t_i\}$ for $i < m$, $L = \prod_{i\in\mathbb{N}} L_i \in \mathcal{K}$ and $L \subseteq E_n$, so $E_n \notin \mathcal{I}$. As $\langle E_n\rangle_{n\in\mathbb{N}}$ is arbitrary,$\mathcal{I}$ is a σ-ideal of $\mathcal{B}$. **Q**

(b) Let $f : [0,1]^{\mathbb{N}}$ be a Borel measurable function. Let μ be the usual measure on $[0,1]^{\mathbb{N}}$. Then there is a compact set K such that $\mu K > 0$ and $f\restriction K$ is continuous. Now there is a family $\langle L_i\rangle_{i\ge m}$ such that each L_i is an uncountable compact subset of $[0,1]$ and $\prod_{i\ge m} L_i \subseteq K$, so $K \notin \mathcal{I}$. Thus $\mathcal{I}$ satisfies the condition of Proposition 1 and $\mathfrak{A}$ is weakly σ-distributive.

3 Proposition Let $\mathcal{B}$ be the Borel σ-algebra of a Hausdorff space X, and $\mathcal{I}$ a σ-ideal of $\mathcal{B}$ such that the quotient algebra $\mathfrak{A} = \mathcal{B}/\mathcal{I}$ is weakly (σ,∞)-distributive. Suppose that $E \subseteq X$ is K-analytic and that there is an $H \in \mathcal{B}\setminus\mathcal{I}$ such that $F \in \mathcal{I}$ whenever F is a Borel subset of $H \setminus E$. Then there is a compact set $K \subseteq E$ such that $K \notin \mathcal{I}$.

proof (a) Let $\widehat{\mathfrak{A}}$ be the Dedekind completion of $\mathfrak{A}$; note that $\widehat{\mathfrak{A}}$ is weakly (σ,∞)-distributive. Let $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ be an usco-compact relation such that $R[\mathbb{N}^{\mathbb{N}}] = E$. For $\sigma \in S^* = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ set $I_\sigma = \{\alpha : \sigma \subseteq \alpha \in \mathbb{N}^{\mathbb{N}}\}$, $A_\sigma = \{F^\bullet : F \in \mathcal{B},\ F \cap R[I_\sigma] = \emptyset\}$ and $b_\sigma = 1 \setminus \sup A_\sigma \in \widehat{\mathfrak{A}}$. Then $b_\sigma = \sup_{i\in\mathbb{N}} b_{\sigma^\frown i}$. **P?** Otherwise, since $\sup(\{b_{\sigma^\frown i}\} \cup A_{\sigma^\frown i}) = 1$ for each i, and $A_{\sigma^\frown i}$ is upwards-directed, we can find $a_i \in A_{\sigma^\frown i}$, for $i \in \mathbb{N}$, such that

$$0 \ne (b_\sigma \setminus \sup_{i\in\mathbb{N}} b_{\sigma^\frown i}) \cap \inf_{i\in\mathbb{N}} (b_{\sigma^\frown i} \cup a_i) \subseteq b_\sigma \cap \inf_{i\in\mathbb{N}} a_i = b$$

say. For each i, let $F_i \subseteq X \setminus R[I_{\sigma^\frown i}]$ be a Borel set such that $F_i^\bullet = a_i$, and set $F = \bigcap_{i\in\mathbb{N}} F_i$; then $F \cap R[I_\sigma] = \emptyset$, so $F^\bullet \in A_\sigma$; but $b_\sigma \cap F^\bullet = b \ne 0$. **XQ**

(b) We can therefore find a family $\langle c_\sigma\rangle_{\sigma\in S^*}$ in $\widehat{\mathfrak{A}}$ such that $c_\emptyset = b_\emptyset$ and, for each $\sigma \in S^*$, $c_\sigma \subseteq b_\sigma$, $\langle c_{\sigma^\frown i}\rangle_{i\in\mathbb{N}}$ is disjoint and $\sup_{i\in\mathbb{N}} c_{\sigma^\frown i} = c_\sigma$. For $n, m \in \mathbb{N}$, set $d_{nm} = \sup\{c_\sigma : \sigma \in \mathbb{N}^{n+1},\ \sigma(n) \le m\}$; then $\sup_{m\in\mathbb{N}} d_{nm} = b_\emptyset$. Note that $b_\emptyset \supseteq H^\bullet \ne 0$. Because $\widehat{\mathfrak{A}}$ is weakly (σ,∞)-distributive, there is an $f : \mathbb{N} \to \mathbb{N}$ such that $d = \inf_{n\in\mathbb{N}} d_{n,f(n)} \ne 0$. Now, for any $n \in \mathbb{N}$,

$$\inf\nolimits_{k\le n} d_{k,f(k)} = \sup\{c_\sigma : \sigma \in \mathbb{N}^{n+1},\ \sigma(k) \le f(k) \text{ for } k \le n\}$$

(induce on n). Observe now that

$$c_\sigma \subseteq b_\sigma \subseteq \overline{R[I_\sigma]}^\bullet$$

for each σ, so $\inf_{k\le n} d_{k,f(k)} \subseteq \overline{R[V_n]}^\bullet$ for each n, where $V_n = \{\alpha : \alpha \in \mathbb{N}^{\mathbb{N}},\ \alpha(k) \le f(k)$ for $k \le n\}$. Now (because R is usco-compact) $\bigcap_{n\in\mathbb{N}} \overline{R[V_n]} = R[L]$, where $L = \bigcap_{n\in\mathbb{N}} V_n$ is compact. But this means that $K = R[L] \subseteq E$ is compact and that $K^\bullet = \inf_{k\in\mathbb{N}} d_{k,f(k)} \ne 0$, so $K \notin \mathcal{I}$.

4 Corollary Let X be an analytic Hausdorff space, $\mathcal{B}$ its Borel σ-algebra and $\mathcal{I}$ a σ-ideal of $\mathcal{B}$ such that $\mathfrak{A} = \mathcal{B}/\mathcal{I}$ is weakly (σ, ∞)-distributive. If $f : X \to \mathbb{R}$ is Borel measurable and $E \in \mathcal{B} \setminus \mathcal{I}$, there is an $F \in \mathcal{B} \setminus \mathcal{I}$ such that $F \subseteq E$ and $f \restriction F$ is continuous.

proof Let $\mathfrak{S}$ be a finer analytic topology on X for which f is continuous. By Proposition 3, there is an $\mathfrak{S}$-compact $K \subseteq E$ such that $K \notin \mathcal{I}$; now $f \restriction K$ is continuous for the original topology.

References

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