Problem FR

D.H.Fremlin

University of Essex, Colchester, England

Example Suppose that κ is an infinite cardinal such that the uniformity of the usual measure μ_κ on $\{0,1\}^\kappa$ is 2^κ and $2^\theta \le 2^\kappa$ for every $\theta < 2^\kappa$. Setting $\lambda = 2^\kappa$, there is a compact Hausdorff space X of cardinal at most λ with a family of 2^λ mutually singular Radon probability measures.

construction Enumerate $\{0,1\}^\kappa$ as $\langle t_\xi\rangle_{\xi<\lambda}$. For each $\xi < \lambda$, let $\langle F_{\xi n}\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of closed subsets of $\{0,1\}^\kappa$ such that $t_\eta \notin F_{\xi n}$ and $\mu_\kappa F_{\xi n} \ge 1 - 2^{-n}$ whenever $\eta < \xi$ and $n \in \mathbb{N}$. Set

$$X = \bigcap_{\xi<\lambda,n\in\mathbb{N}}\{(t,u) : t \in \{0,1\}^\kappa,\ u \in \{0,1\}^{\lambda\times\mathbb{N}},\ t \in F_{\xi n} \text{ or } u(\xi,n) = 0\}.$$

Then X is compact. For any $t \in \{0,1\}^\kappa$, there is a $\xi < \lambda$ such that $t \notin F_{\eta n}$ for any $\eta \ge \xi$, $n \in \mathbb{N}$, so that $\#(\{u : (t,u) \in X\}) \le \#(\mathcal{P}(\xi \times \mathbb{N})) \le \lambda$; accordingly $\#(X) \le \lambda$. For each $v \in \{0,1\}^\lambda$, define $f_v : \{0,1\}^\kappa \to X$ by setting $f_v(t) = (t,u)$ where $u(\xi,n) = v(\xi)$ if $t \in F_{\xi n}$, 0 otherwise. Then $f_v^{-1}[E]$ is a Borel set for every open-and-closed set $E \subseteq X$, so we have a Radon probability measure ν_v on X defined by saying that $\nu_v E = \mu_\kappa f_v^{-1}[E]$ for every open-and-closed subset E of X. If v, $v' \in \{0,1\}^\lambda$ and $\xi < \lambda$ are such that $v(\xi) = 0$ and $v'(\xi) = 1$, then, setting $E = \{(t,u) : (t,u) \in X,\ \exists\ k \in \mathbb{N},\ u(\xi,k) = 1\}$, we have $f_v^{-1}[E] = \emptyset$ while $f_{v'}^{-1}[E] = \bigcup_{k\in\mathbb{N}} F_{\xi k}$, so that $\nu_v E = 0$ and $\nu_{v'} E = 1$; thus ν_v and $\nu_{v'}$ are mutually singular. This shows that $\{\nu_v : v \in \{0,1\}^\lambda\}$ is a family of mutually singular Radon probability measures on X.

Remarks (a) This example is derived from Plebanek n01, in which it is supposed that $\mathfrak{m} = \mathfrak{c}$, so that ω satisfies the conditions above.

(b) If κ is a strong limit cardinal of countable cofinality such that $2^\kappa = \kappa^+$, then κ satisfies the conditions above (see Fremlin 05?, 522Hd).

Reference

Fremlin D.H. [05?] *Measure Theory, Vol. 5: Set-theoretic measure theory,* in preparation. *Partial drafts available via* `http://www.essex.ac.uk/maths/staff/fremlin/mt.htm`.

Plebanek G. [n01] 'On problem FR', note of 22.10.01.