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Problem FN

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1. Proposition Let X be a Hausdorff space, and $\mathcal{K}$ the family of K-analytic subsets of X. Let $\mathcal{I}$ be a family of subsets of X containing $\emptyset$ and closed under countable unions. Then $\mathcal{K}\setminus\mathcal{I}$ is a weakly α-favourable class.

proof Fremlin 87, 2R.

2. Proposition Let (X,Σ,μ) be a probability space and $\mathcal{N}$ the ideal of negligible sets. Let $\mathcal{K}$ be a weakly α-favourable class of subsets of X, containing X, such that whenever $K\in\mathcal{K}$, $E\in\Sigma$ and $E\cap\mathcal{K}\notin\mathcal{N}$ then there is a $K'\in\mathcal{K}\setminus\mathcal{N}$ such that $K'\subseteq E\cap\mathcal{K}$. Then μ is weakly α-favourable.

proof By Proposition 1, there is a winning strategy σ for the game on $\mathcal{K}\setminus\mathcal{N}$. Define a strategy σ' for the game on $\Sigma\setminus\mathcal{N}$ as follows. For $E\in\Sigma\setminus\mathcal{N}$ let $\rho(E)\in\mathcal{K}\setminus\mathcal{N}$ be such that $\rho(E)\subseteq E$, and take $\sigma(E)$ to be a measurable envelope of $\rho(E)$ included in E. Now suppose that $\rho(E_0,\dots,E_i)=K_i\in\mathcal{K}\setminus\mathcal{N}$ for $i\le n$, that $K_{i+1}\subseteq\sigma(K_0,\dots,K_i)$ for $i<n$, and that $\sigma'(E_0,\dots,E_n)$ is a measurable envelope of $\sigma(K_0,\dots,K_n)$. Let $E\in\Sigma\setminus\mathcal{N}$ be such that $E\subseteq\sigma(E_0,\dots,E_n)$. Then $E\cap\sigma(K_0,\dots,K_n)\notin\mathcal{N}$; let $K\subseteq E\cap\sigma(K_0,\dots,K_n)$ be such that $K\in\mathcal{K}\setminus\mathcal{N}$, set $\rho(E_0,\dots,E_n,E)=K$, and let $\sigma'(E_0,\dots,E_n,E)$ be a measurable envelope of $\sigma(K_0,\dots,K_n,K)$ included in E. Continue.

If now we have a sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\Sigma\setminus\mathcal{N}$ such that $E_{n+1}\subseteq\sigma'(E_0,\dots,E_n)$ for every n, then we have a corresponding sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}\setminus\mathcal{N}$ such that $K_n=\rho(E_0,\dots,E_n)$ and $K_{n+1}\subseteq\sigma(K_0,\dots,K_n)$ for every n; also $K_n\subseteq E_n$ for every n. But this means that $\bigcap_{n\in\mathbb{N}}E_n\supseteq\bigcap_{n\in\mathbb{N}}K_n\ne\emptyset$. As $\langle E_n\rangle_{n\in\mathbb{N}}$ is arbitrary, σ' is a winning strategy in the game on $\Sigma\setminus\mathcal{N}$.

Remark Compare Fremlin 87, 2S.

3. Proposition Let Σ be any σ-subalgebra of the Borel σ-algebra of $\mathbb{R}$, and μ a probability measure with domain Σ.

(a) μ is weakly α-favourable.

(b) If the additivity of Lebesgue measure is $\mathfrak{c}$ and μ has countable Maharam type, then μ is countably compact.

(c) If $\mathfrak{c}=\omega_1$ then μ is countably compact.

proof (a) Immediate from Props. 1 and 2.

(b), (c) Use Fremlin 00, 3E.

Remark Borodulin-Nadzieja & Plebanek p04 have shown that if Σ and μ are as in Proposition 3, then there are a probability space (Y,T,ν) with a (ν,μ)-inverse-measure-preserving function $f:Y\to\mathbb{R}$ such that ν is **regularly monocompact**, that is, inner regular with respect to some family $\mathcal{K}$ of sets such that any non-increasing sequence in $\mathcal{K}$ has non-empty intersection.

4. Example Write $\mathcal{B}_2$ for the Borel σ-algebra of $[0,1]^2$. Let $\mathcal{I}$ be the ideal of subsets of $[0,1]^2$ generated by those $H\in\mathcal{B}_2$ such that $\{x:H[\{x\}]$ is not Lebesgue negligible$\}$ is meager. In the language of Fremlin p02, $\mathcal{I}$ is the ideal of negligibles of $([0,1],\mathcal{B},\mathcal{M})\ltimes([0,1],\mathcal{B},\mathcal{N})$, where $\mathcal{B}$ is the Borel σ-algebra of $[0,1]$, $\mathcal{M}$ is the ideal of meager sets, and $\mathcal{N}$ the ideal of Lebesgue negligible sets. We have $\operatorname{add}\mathcal{I}=\omega_1$ and $\operatorname{cf}\mathcal{I}=\mathfrak{c}$ (Fremlin 06?, §526, or Fremlin 91, Theorem 3H). Let Σ be the σ-algebra of subsets of $[0,1]^2$ generated by $(\mathcal{I}\cap\mathcal{B}_2)\cup\{[0,1]\times E:E\in\mathcal{B}\}$. Then we have a measure μ with domain Σ defined by saying that $\mu H=0$ if $H\in\mathcal{I}\cap\mathcal{B}_2$ and $\mu([0,1]\times E)=\mu_L E$ if $E\in\mathcal{B}$, where μ_L is Lebesgue measure on $\mathbb{R}$. Note that $\mathcal{I}$ is the ideal of μ-negligible sets. By Proposition 3, μ is weakly α-favourable.

5. Subproblem of Problem FN In ZFC, can we prove that μ is countably compact? (Compare Theorem 2 of Plebanek 01.)

Subproblem of Problem FO In ZFC, can we prove that μ is α-favourable?

6. Lemma (a) There is a μ-conegligible set H_1 such that if $K \in \Sigma$ is included in H_1 and has compact vertical sections then $\mu K = 0$.

(b) There is a μ-conegligible set H_2 such that if $K \in \Sigma$ is included in H_1 and has compact horizontal sections then $\mu K = 0$.

proof Let $E_1 \subseteq [0,1]$ be a meager F_σ set with Lebesgue measure 1. Set $H_1 = \{(x,y) : x, y \in [0,1], x+y \in F$ or $x+y-1 \in F\}$. Then every vertical section of H_1 has Lebesgue measure 1 so H_1 is μ-conegligible. If $K \in \Sigma$ has compact vertical sections and is included in H_1, let $E \in \mathcal{B}$ be such that $K \triangle ([0,1] \times E) \in \mathcal{I}$. Then $C = \{x : K[\{x\}] \triangle E$ is Lebesgue negligible$\}$ is comeager. Set $F = \bigcap_{x \in C} K[\{x\}]$; then $F \triangle E$ is Lebesgue negligible and $C \times F \subseteq H_1$. But as every horizontal section of H_1 is meager, this means that $F = \emptyset$ and $\mu K = \mu_L E = 0$.

(b) Set $H_2 = ([0,1] \setminus E_1) \times [0,1]$. If $K \in \Sigma$, $K \subseteq H_2$ has compact horizontal sections, they are all nowhere dense. Let $E \in \mathcal{B}$ be such that $K \triangle ([0,1] \times E) \in \mathcal{I}$. Then $C = \{x : K[\{x\}] \triangle E$ is Lebesgue negligible$\}$ is comeager. **?** If $\mu_L E > 0$, then there is a non-empty open set U such that $E_1 = \{y : y \in E, U \cap K^{-1}[\{y\}] = \emptyset\}$ is not Lebesgue negligible. Take $x \in U \cap C$; then $K[\{x\}]$ does not meet E_1 so $K[\{x\}] \triangle E$ is not negligible. **X**

7. Lemma Suppose that $\mathcal{K} \subseteq \Sigma$ is a countably compact class, closed under finite unions and countable intersections, such that μ is inner regular with respect to $\mathcal{K}$.

(a) Let $\mathcal{K}'$ be the set

$$\{K : K \in \mathcal{K}, \exists\, E \in \mathcal{B}, K \subseteq [0,1] \times E, \mu K = \mu_L E\}.$$

Then $\mathcal{K}'$ is closed under finite unions and countable intersections, and μ is inner regular with respect to $\mathcal{K}'$.

(b) Let $\mathcal{L}$ be the set of unions $\bigcup_{n\in\mathbb{N}} K_n$ of sequences $\langle K_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ such that $\mu(\bigcup_{n\in\mathbb{N}} K_n) = 1$ and there are disjoint $E_n \in \mathcal{B}$ such that $K_n \subseteq [0,1] \times E_n$ for every n. Then every μ-conegligible set includes a member of $\mathcal{L}$.

(c) For $x \in [0,1]$ set $\mathcal{K}_x = \{F : F \times \{x\} \in \mathcal{K}\}$. Set $\mathcal{K}'' = \{K : K \in \mathcal{K}', K^{-1}[\{x\}] \in \mathcal{K}_x$ for every $x \in [0,1]\}$. Then μ is inner regular with respect to $\mathcal{K}''$.

(d) Let $\mathcal{L}'$ be $\{H : H \in \Sigma, H \cap ([0,1] \times \{x\}) \in \mathcal{K}$ for every $x \in [0,1]\}$. Then every μ-conegligible set includes a conegligible member of $\mathcal{L}'$.

(e) There is a function $\theta : \mathcal{I} \to \mathcal{I}$ such that $\theta H \supseteq H$ for every $H \in \mathcal{I}$ and whenever $\langle H_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{I}$ such that $\theta H_n \subseteq \theta H_{n+1}$ for every n, then $\pi_2[[0,1]^2 \setminus \bigcup_{n\in\mathbb{N}} H_n] = \bigcap_{n\in\mathbb{N}} \pi_2[[0,1]^2 \setminus H_n]$, where $\pi_2(x,y) = y$ for $x, y \in [0,1]$.

proof (a) Let $H \in \Sigma$, $0 \le \gamma < \mu H$. Choose $\langle K_n \rangle_{n\in\mathbb{N}}$, $\langle E_n \rangle_{n\in\mathbb{N}}$ inductively as follows. $K_0 \in \mathcal{K}$, $K_0 \subseteq H$, $\mu K_0 > \gamma$. Given $K_n \in \mathcal{K}$, choose $E_n \in \mathcal{B}$ such that $K_n \triangle ([0,1] \times E_n) \in \mathcal{I}$, and now take $K_{n+1} \in \mathcal{K}$ such that $K_{n+1} \subseteq K_n \cap ([0,1] \times E_n)$ and $\mu K_{n+1} > \gamma$. Set $K = \bigcap_{n\in\mathbb{N}} K_n$, $E = \bigcap_{n\in\mathbb{N}} E_n$; then $\mu_L E = \mu K \ge \gamma$ and $K \subseteq [0,1] \times E$, so $K \in \mathcal{K}'$, while $K \subseteq H$.

(b) Let $H \in \Sigma$ be μ-conegligible. Choose $\langle K_n \rangle_{n\in\mathbb{N}}$, $\langle E_n \rangle_{n\in\mathbb{N}}$ inductively such that $K_n \subseteq H \setminus ([0,1] \times \bigcup_{i<n} E_i)$, $\mu K_n \ge \mu(H \setminus ([0,1] \times \bigcup_{i<n} E_i)) - 2^{-n}$, $K_n \in \mathcal{K}'$, $E_n \in \mathcal{B}$, $E_n \cap \bigcup_{i<n} E_i$, $K_n \subseteq [0,1] \times E_n$, $\mu K_n = \mu_L E_n$. Then $L = \bigcup_{n\in\mathbb{N}} K_n \in \mathcal{L}$, $\mu L = 1$ and $L \subseteq H$.

(c) Follow the method of (b) to build $K_\sigma \in \mathcal{K}'$, $E_\sigma \in \mathcal{B}$, for $\sigma \in \bigcup_{n\ge1} \mathbb{N}^n$, such that $K_\sigma \subseteq [0,1] \times E_\sigma$ and $\mu K_\sigma = \mu_L E_\sigma$ for every σ, while $\langle E_{\sigma^\frown i} \rangle_{i\in\mathbb{N}}$ is a disjoint family of subsets of E_σ and $\sum_{i=0}^\infty \mu K_{\sigma^\frown i} = \mu K_\sigma$, and $\sum_{i=0}^\infty \mu K_{(i)} = 1$; also ensuring that $\operatorname{diam} E_\sigma \le 2^{-\#(\sigma)}$ for every σ. Now look at $L = \bigcap_{n\ge1} \bigcup_{\sigma\in\mathbb{N}^n} K_\sigma$. This is μ-conegligible. If $x \in [0,1]$, then either $L^{-1}[\{x\}]$ is empty and certainly belongs to $\mathcal{K}_x$, or there is a unique $\phi \in \mathbb{N}^\mathbb{N}$ such that $x \in E_{\phi\restriction n}$ for every $n \ge 1$, in which case $L^{-1}[\{x\}] \times \{x\} = \bigcap_{n\ge1} K_{\phi\restriction n} \in \mathcal{K}'$ and $L^{-1}[\{x\}] \in \mathcal{K}_x$.

If $K \in \mathcal{K}'$ and $\gamma < \mu K$, then there is some $\psi \in \mathbb{N}^\mathbb{N}$ such that $\mu(K \cap L_\psi) \ge \gamma$, where $L_\psi = \bigcap_{n\ge1} \bigcup_{\sigma\in\mathbb{N}^n, \sigma \le \psi\restriction n}$; so that $K \cap L_\psi \cap ([0,1] \times \{x\})$ is either empty or $K \cap L \cap ([0,1] \times \{x\})$ and in either case belongs to $\mathcal{K}_1$; so that $K \cap L_\psi \in \mathcal{K}''$.

(d) If $H \in \mathcal{L}$ then $H \cap L \in \mathcal{L}'$.

(e) For $H \in \mathcal{I}$ take $\theta H \supseteq H$ such that $[0,1]^2 \setminus \theta H \in \mathcal{L}'$. Then all the horizontal sections of $[0,1]^2 \setminus \theta H$ are in the countably compact class $\mathcal{K}'$.

8. Proposition Let (X, Σ, μ) be a weakly α-favourable probability space with measure algebra $\mathfrak{A}$. Set $\kappa = \min(\operatorname{add}\mathcal{N}_\mu, \operatorname{wdistr}(\mathfrak{A}))$. Suppose that there are sets $\mathcal{V}$, $\mathcal{A}$ such that

$\mathcal{V} \subseteq \Sigma$,

μ is inner regular with respect to the σ-algebra generated by $\mathcal{V}$,

$\mathcal{A} \subseteq \mathcal{N}_\mu$,

if $\mathcal{A}' \subseteq \mathcal{A}$ and $\#(\mathcal{A}') \geq \kappa$ then $\bigcup \mathcal{A}'$ has full outer measure,

$\#(\mathcal{V}) \leq \#(\mathcal{A})$.

Then (X, Σ, μ) is α-favourable.

proof (a) Let $\mathcal{W}$ be the subalgebra of $\mathcal{P}X$ generated by $\mathcal{V}$. Let $Q \subseteq \bigcup_{n\geq 1}(\Sigma \setminus \mathcal{N}_\mu)^n$ be a family witnessing that (X, Σ, μ) is weakly α-favourable, as in FREMLIN 00, Lemma 1D, and for $s \in \bigcup_{n\geq 1} \mathcal{W}^n$ choose countable subsets $Q_s \subseteq Q$, $\mathcal{H}_s \subseteq \Sigma$ such that

whenever $n \geq 1$ and $s \in \mathcal{W}^n$ then $Q_s \subseteq (\Sigma \setminus \mathcal{N}_\mu)^n$;

if s extends t then every member of Q_s extends a member of Q_t;

$\mathcal{H}_s = \{K_n : (K_0, \dots, K_n) \in Q_s$ whenever $s \in \mathcal{W}^{n+1}$;

$\mathcal{H}_s$ is a disjoint family with conegligible union;

if $s \in \mathcal{W}^{n+1}$ and $K \in \mathcal{H}_s$, then either $K \subseteq s(n)$ or $K \cap s(n) = \emptyset$.

(See FREMLIN 00, Lemma 3A.)

Fix an injective function $\phi : \mathcal{W} \to \mathcal{A}$.

(b) Let $\mathcal{M}$ be the family of sets $M \subseteq \mathcal{W}$ such that

$X \in M$;

whenever $E \in M \setminus \mathcal{N}_\mu$, $W \in \mathcal{W}$ and $\phi(W) \cap E = \emptyset$ then $W \in M$;

$\#(M) < \kappa$.

Because $\operatorname{cf}\kappa > \omega$ there are lots of sets in $\mathcal{M}$. For $E \in \Sigma \setminus \mathcal{N}_\mu$ choose $M_k(E)$, $F_k(E)$ inductively, for $k \in \mathbb{N}$, so that $F_0(E) \subseteq E$ and, for every $k \in \mathbb{N}$,

$F_k(E) \in M_k(E) \in \mathcal{M}$;

$E \in M_k(E)$;

$\mu F_k(E) \geq (\frac{1}{2} + 2^{-k-2})\mu E$;

$F_k(E)$ is expressible as the intersection of a sequence in $M_k(E)$;

for any $s \in \bigcup_{n\geq 1} M_k(E)^n$ there is a finite subset of $\mathcal{H}_s$ covering $F_{k+1}(E)$;

$F_{k+1}(E) \subseteq F_k(E)$;

$F_{k+1}(E) \cap \phi(W) = \emptyset$ for every $W \in M_k(E)$.

(This is possible by Lemma 3C of FREMLIN 00.) Set $\tau(E) = \bigcap_{k\in\mathbb{N}} F_k(E)$. Note that $\mu(\tau(E)) \geq \frac{1}{2}\mu E > 0$, so $\tau(E) \in \Sigma \setminus \mathcal{N}_\mu$, while $\tau(E) \subseteq F_0(E) \subseteq E$; thus τ is a tactic in the game on $\Sigma \setminus \mathcal{N}_\mu$.

(c) Now suppose that $\langle E_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\Sigma \setminus \mathcal{N}_\mu$ such that $E_{n+1} \subseteq \tau(E_n)$ for every n. Let $\mathcal{F}$ be an ultrafilter on X containing every E_n. For each $n \in \mathbb{N}$ there is a countable set $\mathcal{E}_n \subseteq \bigcup_{k\in\mathbb{N}} M_k(E_n)$ such that $E_n = \bigcap \mathcal{E}_n$. Let $\langle V_n \rangle_{n\in\mathbb{N}}$ run over $\{X\} \cup \bigcup_{n\in\mathbb{N}} \mathcal{E}_n$, with $V_0 = X$. Set $s_n = (V_0, \dots, V_n)$ for each n.

For each $n \in \mathbb{N}$ there is a $K_n \in \mathcal{F} \cap \mathcal{H}_{s_n}$. **P** Let $m \in \mathbb{N}$ be such that $V_i \in \bigcup_{j\leq m} \mathcal{E}_j$ for every $i \leq n$. Then for every $i \leq n$ there are $j \leq m$, $k \in \mathbb{N}$ such that $V_i \in M_k(E_j)$, so that $E_{m+1} \subseteq E_{j+1} \subseteq \tau(E_j) \subseteq F_{k+1}(E_j)$ is disjoint from $\phi(V_i)$ and $V_i \in M_0(E_{m+1})$. This means that $s_n \in M_0(E_{m+1})^{n+1}$ and $F_1(E_{m+1})$ is covered by a finite subset of $\mathcal{H}_{s_n}$. As $F_1(E_{m+1}) \supseteq \tau(E_{m+1}) \supseteq E_{m+2} \in \mathcal{F}$, there must be a set in $\mathcal{F} \cap \mathcal{H}_{s_n}$. **Q**

Because $K_{n+1} \cap K_n \neq \emptyset$ for every n, we now see that $(K_0, \dots, K_n) \in Q_{s_n} \subseteq Q$ for every n, so that there is a point $x \in \bigcap_{n\in\mathbb{N}} K_n$. Also, for each n, we must have $K_n \subseteq V_n$ or $K_n \subseteq X \setminus V_n$; but as both K_n and V_n belong to $\mathcal{F}$, $K_n \subseteq V_n$. This means that $x \in V_n$ for every $n \in \mathbb{N}$ so $x \in \bigcap_{n\in\mathbb{N}} V_n = \bigcap_{n\in\mathbb{N}} \bigcap \mathcal{E}_n = \bigcap_{n\in\mathbb{N}} E_n$.

As $\langle E_n \rangle_{n\in\mathbb{N}}$ is arbitrary, τ is a winning tactic and (X, Σ, μ) is α-favourable.

9. Corollary If *either* there is a Sierpiński subset of $\mathbb{R}$ of cardinal $\mathfrak{c}$ *or* there is a Lusin subset of $\mathbb{R}$ of cardinal $\mathfrak{c}$, then the space of Example 4 is α-favourable.

proof If $B \in [\mathbb{R}]^{\mathfrak{c}}$ is a Sierpiński set, then $\mathcal{A} = \{[0,1] \times ((t + \mathbb{Q}) \cap [0,1]) : t \in B\}$, $\mathcal{V} = \mathcal{B}_2$ will serve in Proposition 8. if $B \in [\mathbb{R}]^{\mathfrak{c}}$ is a Lusin set, use $\mathcal{A} = \{((t + \mathbb{Q}) \cap [0,1]) \times [0,1] : t \in B\}$.

10. Another σ-algebra Let $\Sigma_{\mathbb{Q}}$ be the σ-algebra $\{E : E \in \mathcal{B}(\mathbb{R}),\ E + \mathbb{Q} = E\}$. On $\Sigma_{\mathbb{Q}}$ we can identify the following types of totally finite measure.

(a) Measures extendible to $\mathcal{B}(\mathbb{R})$. By Pachl's theorem (FREMLIN 03, 452L), these must all be countably compact.

(b) Simple category measures. If, for instance, $C \subseteq [0,1]$ is the Cantor set, then $E \cap C$ is either meager or comeager in C for every $E \in \Sigma_{\mathbb{Q}}$, so we have a corresponding zero-one measure.

(c) Mixed category measures. Build a compact set $K \subseteq \mathbb{R}$ as follows. K is to be $\bigcap_{n\in\mathbb{N}} K_n$ where each K_n consists of 2^n closed intervals of length 8^{-n} and K_{n+1} is formed by taking the end sections of each interval in K_n. Note that $K_n - K_n$ is covered by 4^n intervals of length $2 \cdot 8^{-n}$, so $K - K$ is Lebesgue negligible, therefore nowhere dense. The intervals of K_n meet K in congruent sets obtained by rational translations.

If $F \subseteq \mathbb{R}$ is closed and $F \cap K$ is not meager in K, then $K \subseteq F + \mathbb{Q}$, because F covers $K \cap I$ for one of the intervals I of one of the K_n, and now K is covered by finitely many rational translates of $K \cap I$. If $E \in \Sigma_{\mathbb{Q}}$ and $E \cap K$ is not meager in K, let I be one of the closed intervals of one of the K_n such that $E \cap K \cap I$ is comeager in $K \cap I$. Then finitely many rational translates of $E \cap K \cap I$ cover a comeager subset of K, so $E \cap K$ is comeager in K.

For $x \in \mathbb{R}$ define a zero-one measure μ_x with domain $\Sigma_{\mathbb{Q}}$ by saying that $\mu_x E = 1$ iff $(E - x) \cap K$ is comeager in K. For any $E \in \Sigma_{\mathbb{Q}}$, $x \mapsto \mu_x E$ is Borel measurable. **P** $\{x : \mu_x E = 0\} = \{x : K \cap (E - x)$ is meager in $K\}$. Now consider the set $W = \{(x, y) : x \in \mathbb{R},\ y \in K \cap (E - x)\}$. This is a Borel subset of $\mathbb{R} \times K$. By FREMLIN 03, 4A3Sa, $\{x : \mu_x E = 0\} = \{x : W[\{x\}]$ is meager in $K\}$ is a Borel subset of $\mathbb{R}$. **Q**

Now $(K - K) + \mathbb{Q}$ is meager, so there is a non-empty compact set $Z \subseteq \mathbb{R}$ without isolated points such that $z - z' \notin (K - K) + \mathbb{Q}$ for any distinct z, $z' \in Z$. Let ν be an atomless Radon probability measure on Z, and set $\mu E = \int_Z \mu_z(E)\nu(dz)$ for $E \in \Sigma_{\mathbb{Q}}$.

μ is atomless. **P** If x, $x' \in K$, z, $z' \in Z$, q, $q' \in \mathbb{Q}$ and $x+z+q = x'+z'+q'$ then $z-z' = (x'-x)+(q'-q) \in (K - K) + \mathbb{Q}$, so $z = z'$. What this means is that if $F \subseteq Z$ is closed and $E = K + F + \mathbb{Q} \in \Sigma_{\mathbb{Q}}$, we shall have

— if $z \in F$ then $E - z \supseteq K$ and $\mu_z E = 1$,

— if $z \in Z \setminus F$ then $K \cap (K + \mathbb{Q} + z' - z) = \emptyset$ for every $z' \in F$, $K \cap (E - z)$ is empty and $\mu_z E = 0$.

So if we look at the sequentially order-continuous Boolean homomorphism $E \mapsto \psi E = \{z : \mu_z E = 1\} : \Sigma_{\mathbb{Q}} \to \mathcal{B}(Z)$, the set of values of this homomorphism, which is a σ-subalgebra of $\mathcal{B}(Z)$ (FREMLIN 02, 314F(b-i)), contains every closed subset of Z, and is therefore the whole of $\mathcal{B}(Z)$. Also $\mu E = \nu\psi E$ for every $E \in \Sigma_{\mathbb{Q}}$. If $\epsilon > 0$, Z can be covered by finitely many Borel sets of measure at most ϵ (for ν), so $\mathbb{R}$ can be covered by finitely many members of $\Sigma_{\mathbb{Q}}$ of measure at most ϵ (for μ). **Q**

We need to know that if $F \subseteq K$ is meager in K then $(F + x) \cap K$ is meager in K for every $x \in \mathbb{R}$. **P** It is enough to consider the case in which F is closed. **?** If $F' = (F + x) \cap K$ is not meager in K, then it includes $I \cap K$ for some interval I of some K_n. Now $(I \cap K) - x \subseteq F \subseteq K$. But as distinct intervals of K_n are separated by gaps of length at least $8^{-n+1} - 2 \cdot 8^{-n} > 8^{-n}$, and the endpoints of $(I \cap K) - x$ are distance 8^{-n} apart, $(I \cap K) - x$ must be included in a single interval J of K_n, and the endpoints of I must match the endpoints of J. Because $K \cap I$ and $K \cap J$ are congruent, $F \supseteq K \cap J$ is not meager in K. **X** Thus $(F + x) \cap K$ is nowhere dense. **Q**

It follows that if $F \subseteq K$ is meager in K then $\mu(F + \mathbb{Q} + Z) = 0$. **P** If $z \in Z$ then $K \cap (F + \mathbb{Q} + Z - z) = K \cap (F + \mathbb{Q})$, because $F \subseteq K$, so $F + \mathbb{Q} + z' - z$ can meet K only if $z' = z$. But $K \cap (F + \mathbb{Q})$ is meager, so $\mu_z(F + \mathbb{Q} + Z) = 0$. **Q**

μ does not extend to a Borel measure on $\mathbb{R}$. **P** Note that $\mu_z(K + \mathbb{Q} + Z) = 1$ for every $z \in Z$, so $\mu(K + \mathbb{Q} + Z) = 1$. **?** If $\tilde{\mu}$ were a Borel measure on $\mathbb{R}$ extending μ, consider the product $(K + \mathbb{Q}) \times Z$ and the map $(x, z) \mapsto x + z : (K + \mathbb{Q}) \times Z \to \mathbb{R}$. This is Borel measurable and injective. So its inverse is Borel measurable and we have a Borel measurable function $f : K + \mathbb{Q} + Z \to \mathbb{R}$ such that $f(x + z) = x$ whenever $x \in K + \mathbb{Q}$ and $z \in Z$. Set $\lambda = \tilde{\mu}f^{-1}$, so that λ is a topological probability measure on $\mathbb{R}$ and

$$\lambda(K + \mathbb{Q}) = \tilde{\mu}(K + \mathbb{Q} + Z) = \mu(K + \mathbb{Q} + Z) = 1.$$

Let $q \in \mathbb{Q}$ be such that $\lambda(K + q) > 0$, and set $\lambda' E = \lambda(E - q)$ for $E \in \mathcal{B}(K)$, so that $\lambda' K > 0$. If $F \subseteq K$ is closed and nowhere dense in K, then

$$\begin{aligned}\lambda' F = \lambda(F - q) \le \lambda(F + \mathbb{Q}) &= \tilde{\mu}f^{-1}[F + \mathbb{Q}] \\ &\le \tilde{\mu}(F + \mathbb{Q} + Z) = \mu(F + \mathbb{Q} + Z) = 0.\end{aligned}$$

Thus we have a non-zero totally finite Borel measure on K for which all nowhere dense sets are negligible, which is impossible. **XQ**

References

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