Problem FH

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This is a variation of a problem W.Weiss (Toronto) brought to the Workshop on Set-Theoretic Topology, Budapest, July 1999.

1. The original question Let us say that a **star** in the plane $\mathbb{R}^2$ with center x is the union of a family of rays of finite length starting at x. How many stars do we need to cover the plane? Two stars won't do, for the trivial reason that there's no way to cover the line joining their centers; there are also deeper obstacles. Komjáth 01 shows that if the continuum hypothesis is true, we can cover the plane with three stars. I begin by showing that if $\mathfrak{c} \le \omega_r$ then we can cover the plane with at most $r+2$ stars, using the following fact.

2. Proposition Suppose that X is a set and that $\mathcal{I} \subseteq \mathcal{P}X$ is such that

(*) $\{I : J \subseteq I \in \mathcal{I}\}$ is finite whenever $\#(J) \ge 2$

(ii) for some $r \in \mathbb{N}$, $\#(X) \le \omega_r$ and $\#(I) \ge r+2$ for every $I \in \mathcal{I}$.

Then there is a finite-to-one choice function $c : \mathcal{I} \to X$.

proof Induce on r. If X is finite, so is $\mathcal{I}$, and any choice function will do. If $\#(X) = \omega$, enumerate X as $\langle x_n \rangle_{n \in \mathbb{N}}$, and set $c(I) = x_n$ whenever $x_n \in I$ and there is just one $i < n$ such that $x_i \in I$.

For the inductive step to $r+1$, enumerate X as $\langle x_\xi \rangle_{\xi < \kappa}$, where $\kappa \le \omega_{r+1}$. For each $I \in \mathcal{I}$ let $\tilde{I}$ be the set consisting of the first $r+3$ members of I in the ordering defined by the enumeration $\langle x_\xi \rangle_{\xi < \kappa}$. For $\xi \le \kappa$ let X_ξ be the smallest subset of X, including $\{x_\eta : \eta < \xi\}$, such that $\tilde{I} \subseteq X_\xi$ whenever $I \in \mathcal{I}$ and $\#(I \cap X_\xi) \ge 2$. By condition (*), $\#(X_\xi) \le \max(\omega, \#(\xi)) \le \omega_r$ for every $\xi < \kappa$. Let $\mathcal{I}_\xi$ be $\{I : I \in \mathcal{I},\, \tilde{I} \subseteq X_{\xi+1},\, \tilde{I} \not\subseteq X_\xi\}$.

For each $\xi < \kappa$, consider

$$\mathcal{J}_\xi = \{\tilde{I} \setminus X_\xi : I \in \mathcal{I}_\xi\}.$$

Then each member of $\mathcal{J}_\xi$ has at least $r+2$ members and is included in $X_{\xi+1} \setminus X_\xi$; moreover, if $\#(K) = 2$, then

$$\#(\{J : J \in \mathcal{J}_\xi,\, K \subseteq J\}) \le \#(\{I : J \in \mathcal{I},\, K \subseteq J\})$$

is finite. Since $\#(X_{\xi+1} \setminus X_\xi) \le \omega_r$, the inductive hypothesis tells us that there is a finite-to-one choice function c_ξ for $\mathcal{J}_\xi$. Since the map $I \mapsto \tilde{I}$ is also finite-to-one, the map $I \mapsto c_\xi(\tilde{I})$ is also finite-to-one. Set $c(I) = c_\xi(\tilde{I})$ for every $I \in \mathcal{I}_\xi$.

3. Proposition If $\mathfrak{c} \le \omega_r$ then we can cover the plane with $r+2$ stars.

proof Let D be the set of 'directions' of rays in the plane; if you like, we can identify D with the unit circle S^1; in any case, $\#(D) = \mathfrak{c} \le \omega_r$. Fix any set $A \subseteq \mathbb{R}^2$ with $r+2$ points, not all collinear. Let B be the union of all the lines passing through two points of A. For $x \in \mathbb{R}^2 \setminus B$, let $I_x \subseteq D$ be the set of directions of rays from points of A to x, and set $\mathcal{I} = \{I_x : x \in \mathbb{R}^2 \setminus B\}$. Observe that if $J \in [X]^2$ then there are only finitely many points $x \in \mathbb{R}^2 \setminus B$ such that both directions in J are directions from points of A to x, so that $\{I : J \subseteq I \in \mathcal{I}\}$ is finite. By Proposition 2, there is a finite-to-one choice function $c : \mathcal{I} \to D$. For each $d \in D$, let K_d be the finite set $\{x : x \in \mathbb{R}^2 \setminus B,\ c(I_x) = d\}$ and let $g(d) = \sup(\{0\} \cup \{\|x - a\| : x \in K_d, a \in A\}$.

Now, for each $a \in A$ and each $d \in D$, let $R(a, d)$ be the half-line from a in direction d, and $J(a, d)$ be the finite set

$$\{x : x \in R(a,d) \setminus B,\ c(I_x) = d\}$$
$$\cup \{x : x \in R(a,d) \text{ lies on a line through two points of } A \text{ which does not contain } a\}.$$

If we take $S(a)$ to be the union of the line segments $[a, x]$ for $x \in \bigcup_{d \in D} J(a, d)$, then it is easy to check that $\bigcup_{a \in A} S(a) = \mathbb{R}^2$, so that $\mathbb{R}^2$ is the union of $r + 2$ stars.

4. The problem FH Similarly,

> if $\mathfrak{c} \leq \omega_n$, then there is a non-decreasing sequence $\langle D_m \rangle_{m \in \mathbb{N}}$ of sets covering $\mathbb{R}$ such that no D_m includes any $n+2$-term arithmetic sequence with diameter 2^{-m} or less.

Komjáth p01 shows (i) that if $\mathfrak{m} = \mathfrak{c}$, then there is a non-decreasing sequence $\langle D_m \rangle_{m \in \mathbb{N}}$ covering $\mathbb{R}$ such that no D_m includes any 3-term arithmetic sequence with diameter 2^{-m} or less (ii) that if we add ω_4 Cohen reals to a model of GCH, then we obtain a model in which there is no such sequence $\langle D_m \rangle_{m \in \mathbb{N}}$ and the plane cannot be covered by three stars.

6. Example To see that in Proposition 2 we cannot find a finite-to-one choice function if $\#(X) = \omega_2$, $\#(I) = 3$ for every I, consider the following example. Take a disjoint family $\langle X_\xi \rangle_{\xi < \omega_2}$ of sets of cardinal ω_1. For $0 < \xi < \omega_2$ choose an injection $\phi_\xi : [X_\xi]^2 \to \bigcup_{\eta < \xi} X_\eta$. Set

$$\mathcal{I} = \{J \cup \{\phi_\xi(J)\} : 0 < \xi < \omega_2,\ J \in [X_\xi]^2\}.$$

It is easy to check that $\mathcal{I}$ satisfies (*) of Proposition 2, and that in fact there is at most one member of $\mathcal{I}$ including any given doubleton set.

? Suppose, however, that c were a finite-to-one choice function for $\mathcal{I}$. Take ξ such that $0 < \xi < \omega_2$ and $I \subseteq \bigcup_{\eta < \xi} X_\eta$ whenever $I \in \mathcal{I}$ and $c(I) \in \bigcup_{\eta < \xi} X_\eta$. Look at

$$c'(J) = c(J \cup \{\phi_\xi(J)\})$$

for $J \in [X_\xi]^2$. Then $c'(J) \in J$ for every J, so c' is a finite-to-one choice function for $[X_\xi]^2$, where X_ξ is uncountable. But this is impossible. (Take a countable non-empty $Y \subseteq X_\xi$ such that $J \subseteq Y$ whenever $c'(J) \in Y$, and any $z \in X_\xi \setminus Y$; then $c'(\{y, z\}) = z$ for every $y \in Y$, so c' is not finite-to-one.) **X**

7. Example The original problem (involving stars in the plane) and also Problem FH (on arithmetic sequences) are not committed to finite-to-one choice functions of the sort produced by Proposition 2. What they ask is much closer to the following question:

> given a family $\mathcal{I} \subseteq \mathcal{P}X$ satisfying (*) of Proposition 2, and a function $h : \mathcal{I} \to \mathbb{N}$, when can we find a function $f : X \to \mathbb{N}$ such that $h(I) \leq \max f[I]$ for every $I \in \mathcal{I}$?

Of course we shall always be able to find such an f if there is a finite-to-one choice function $c:\mathcal{I}\to X$ (set $f(x)=\sup\{h(I):c(I)=x\}$); but there are many cases in which f can be built in other ways. One limitation, however, is given by the following example. Set $\kappa=2^{\omega_1}$; let X be a set of size κ, and $\langle X_\xi\rangle_{\xi<\kappa}$ a partition of X into sets of size ω_1; let $\theta:\kappa\setminus\{0\}\to[X_0]^{\omega_1}$ be a surjection; for each $\xi\in\kappa\setminus\{0\}$ let $\phi_\xi:[X_\xi]^2\to\theta(\xi)$ be an injection. As in Example 6, set

$$\mathcal{I}=\{J\cup\{\phi_\xi(J)\}:0<\xi<\kappa,\ J\in[X_\xi]^2\}.$$

Next, for each $\xi<\kappa$, let $h_\xi:[X_\xi]^2\to\mathbb{N}$ be such that for every $g:X_\xi\to\mathbb{N}$ there is a $J\in[X_\xi]^2$ such that $h_\xi(J)<\max g[J]$. (To construct such a function, well-order X_ξ with order type ω_1, and arrange that $h_\xi(\{x,y\})\neq h_\xi(\{x',y\})$ whenever x, x' are distinct elements of X_ξ preceding y.) Set $h(J\cup\{\phi_\xi(J)\})=h_\xi(J)$ whenever $0<\xi<\kappa$ and $J\in[X_\xi]^2$.

Now let $f:X\to\mathbb{N}$ be any function. Let $k\in\mathbb{N}$ be such that $K=\{x:x\in X_0,\ f(x)=k\}$ is uncountable, and $\xi\in\kappa\setminus\{0\}$ such that $\theta(\xi)=K$. Set $g(x)=\max(k,f(x))$ for $x\in X_\xi$; then there is a $J\in[X_\xi]^2$ such that $h_\xi(J)>\max g[J]$, in which case, setting $I=J\cup\{\phi_\xi(J)\}$, $h(I)>\max f[I]$ (since $\phi_\xi(J)\in\theta(\xi)=K$).

Thus we have a counter-example with $\#(I)=3$ for every $I\in\mathcal{I}$ and $\#(X)=2^{\omega_1}$.

8. Problem Can we improve '2^{ω_1}', in Example 7, to 'ω_2', without using special axioms?

Note that Example 7 really depends on the following fact: there is a $\lambda\le\kappa$ and a family $\langle A_\xi\rangle_{\xi<\kappa}$ in $[\lambda]^{\omega_1}$ such that whenever $f:\lambda\to\mathbb{N}$ is a function there is some A_ξ on which f is bounded. So an appropriate version of $\clubsuit$ on ω_2 would also be sufficient.

9. Why the marriage lemma doesn't work Returning to the original geometric problems, the whole discussion would be short-circuited if we could use the marriage lemma in the following form:

> If L and Y are sets, and $R\subseteq Y\times L$ is a relation with finite vertical sections such that $m\#(R[K])\ge\#(K)$ for every finite set $K\subseteq Y$, then there is a $\le m$-to-one function $c:Y\to L$ covered by R.

The intended application would be to a version of Proposition 3 in which L is the set of rays from the centers of the stars, Y is the set $\mathbb{R}^2\setminus B$ of points in the plane not on any of the lines defined by these centers, and $R=\{(y,\ell):y$ lies on the ray $\ell\}$. The relation 'the point y lies on the ray ℓ' certainly has finite vertical sections. To see that the other condition of the marriage lemma is not satisfied, however, take $(1,0)$, $(0,1)$ and $(0,-1)$ for the centers. For α, $\beta\in\mathbb{R}$ such that $\alpha+\beta+1\neq0$, set

$$x(\alpha,\beta)=\left(\tfrac{1}{\alpha+\beta+1},\tfrac{\alpha-\beta}{\alpha+\beta+1}\right).$$

Then

(i) the slope of the line from $(1,0)$ to $x(\alpha,\beta)$ is $-\frac{\alpha-\beta}{\alpha+\beta}$, depending only on the ratio $\alpha:\beta$,

(ii) the slope of the line from $(0,1)$ to $x(\alpha,\beta)$ is $-2\beta-1$, depending only on β,

(iii) the slope of the line from $(0,-1)$ to $x(\alpha,\beta)$ is $2\alpha+1$, depending only on α.

So if we set $y_{ij}=x(-\frac12\gamma^i,-\frac12\gamma^j)$ for $0\le i,j<n$, where $0<\gamma<1$, we shall have n^2 points and only $4n+1$ rays to cover them with; some rays will have to deal with many points.

10. Long arithmetic progressions P.Komjáth has pointed out that there is always a non-decreasing sequence $\langle A_m\rangle_{m\in\mathbb{N}}$, with union $\mathbb{R}$, such that no A_m includes an $m+1$-term arithmetic progression. To see this, start from a Hamel basis B for $\mathbb{R}$ over $\mathbb{Q}$. Let D be the set of all triples $(n,\boldsymbol{q},\boldsymbol{r})$ where $n\in\mathbb{N}$, $\boldsymbol{q}=(q_0,\dots,q_n)\in(\mathbb{Q}\setminus\{0\})^{n+1}$ and $\boldsymbol{r}=(r_0,\dots,r_{n-1})\in\mathbb{Q}^n$ is strictly increasing. (Allow $\boldsymbol{r}$ to be the empty sequence if $n=0$.) For $(n,\boldsymbol{q},\boldsymbol{r})\in D$, set

$$B_{n\boldsymbol{qr}}=\{\sum_{i=0}^{n}q_ib_i:b_0,\dots,b_n\in B,\ b_i<r_i\le b_{i+1}\text{ if }i<n\}.$$

Then $B_{n\boldsymbol{qr}}$ is linearly independent over $\mathbb{Q}$, and $\mathbb{R}=\{0\}\cup\bigcup_{(n,\boldsymbol{q},\boldsymbol{r})\in D}B_{n\boldsymbol{qr}}$. Enumerate $\{\{0\}\}\cup\{B_{n\boldsymbol{qr}}:(n,\boldsymbol{q},\boldsymbol{r})\in D\}$ as $\langle C_n\rangle_{n\in\mathbb{N}}$, so that no C_n contains any arithmetic progression of length 3. Set $C'_n=\bigcup_{k<n}C_k$; then there is for each $n\in\mathbb{N}$ an $f(n)\in\mathbb{N}$ such that C'_n does not contain any arithmetic progression of length greater than $f(n)$, by van der Waerden's theorem. Finally, define $A_m=\bigcup\{C'_n:n\in\mathbb{N},\ f(n)\le m\}$ for $m\in\mathbb{N}$.**13.**

11. Of course in §§4 and 10 we cannot expect our sets A_m or D_m to be regular in any simple way. For instance, if $E\subseteq\mathbb{R}$ is either a non-negligible Lebesgue measurable set or a non-meager set with the Baire property, then E includes arbitrarily long arithmetic progressions.

If we have a projective well-ordering of $\mathbb{R}$ (for instance, if V=L), then we can make the A_m, D_m projective.

I do not know, however, whether there can be a non-decreasing sequence $\langle B_m\rangle_{m\in\mathbb{N}}$ of PCA ($=\boldsymbol{\Sigma}^1_2$) sets, covering $\mathbb{R}$, such that no B_m includes any arithmetic progression of length greater than m.

12. A similar fact: Proposition Assume the continuum hypothesis. Then there is a non-decreasing sequence $\langle A_n\rangle_{m\in\mathbb{N}}$, with union $\mathbb{R}^2$, such that no circle of diameter less than or equal to 2^{-n} meets A_n in more than three points.

proof Express $\mathbb{R}^2$ as a union $\bigcup_{\xi<\omega_1}X_\xi$ where $\langle X_\xi\rangle_{\xi<\omega_1}$ is a non-decreasing family of countable sets, $X_\xi=\bigcup_{\eta<\xi}X_\eta$ for every limit ordinal $\xi<\omega_1$, and whenever C_1, C_2 are distinct circles meeting X_ξ in three or more points then $C_1\cap C_2\subseteq X_\xi$. Choose inductively functions $f_\xi:X_\xi\to\mathbb{N}$ in such a way that if $n\in\mathbb{N}$ and C is a circle of diameter less than or equal to 2^{-n} then $\#(\{x:x\in C,\ f_\xi(x)\le n\})\le 3$. Only the step to a successor ordinal $\xi+1$ is non-trivial. Given f_ξ, enumerate $X_{\xi+1}\setminus X_\xi$ as $\langle x_i\rangle_{i<N}$ where $N\le\omega$. Now any x_i can belong to at most one circle meeting X_ξ in three or more points. So we can choose $f_{\xi+1}(x_i)=m$ where m is so large that

– if $x_i\in C$ where C is a circle meeting X_ξ in three or more points, then $\mathrm{diam}(C)>2^{-m}$,

– if $j<i$ then $\|x_j-x_i\|>2^{-m}$.

This works.

13. Further questions Note that all the arguments above are based entirely on combinatorial ideas. So, for instance, we could look at regular polygons in $\mathbb{R}^2$ in place of arithmetic progressions, and expect the same conclusions.

I do not know whether Martin's axiom is enough to ensure that the plane is the union of three stars.

Reference

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