Problem EZ

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1. The problem Set $E_n = \{a : n \in a \subseteq \mathbb{N}\}$ for each $n \in \mathbb{N}$. Given a Borel set $F \subseteq \mathcal{P}\mathbb{N}$, can we always find a Radon probability measure μ on $\mathcal{P}\mathbb{N}$ such that either $\lim_{n\to\infty} \mu E_n = 0 < \mu F$ or $\lim_{n\to\infty} \mu E_n = 1 > \mu F$?

2. First reduction Let A be the set of those Radon probability measures μ on $\mathcal{P}\mathbb{N}$ such that $\lim_{n\to\infty} \mu E_n = 0$, and $\mathcal{I}$ the σ-ideal of those sets $F \subseteq \mathcal{P}\mathbb{N}$ such that $\mu F = 0$ for every $\mu \in A$. (Warning! The Radon measures here are completed, as in Fremlin 03. It is not clear that $\mathcal{I}$ is generated by the Borel sets it contains, and in Theorem 4 below it seems that we need to look at analytic non-Borel sets in $\mathcal{I}$.) For $a \subseteq \mathbb{N}$ set $\pi a = \mathbb{N} \setminus a$, so that π is a continuous involution, and $\pi^{-1}[E_n] = \mathcal{P}\mathbb{N} \setminus E_n$ for every n. Accordingly, for a Radon probability measure μ on A,

$$\lim_{n\to\infty} \mu E_n = 1 \iff \lim_{n\to\infty} \mu\pi^{-1}[E_n] = 1 \iff \mu\pi^{-1} \in A,$$

writing $\mu\pi^{-1}$ for the image measure (Fremlin 03, §418). Now the question in §1 can be rephrased (in negative form) as

> Is there a Borel set $F \subseteq \mathcal{P}\mathbb{N}$ such that $\mu F = 0$ for every $\mu \in A$ and $\mu F = 1$ whenever $\mu\pi^{-1} \in A$?

or (since $\mu = (\mu\pi^{-1})\pi^{-1}$ for every μ)

> Is there a Borel set $F \subseteq \mathcal{P}\mathbb{N}$ such that $\mu F = \mu\pi^{-1}[\mathcal{P}\mathbb{N} \setminus F] = 0$ for every $\mu \in A$?

that is,

> Is there a Borel set $F \subseteq \mathcal{P}\mathbb{N}$ such that F and $\pi^{-1}[\mathcal{P}\mathbb{N} \setminus F]$ both belong to $\mathcal{I}$?

or (thinking of F' as $F \cup \pi^{-1}[\mathcal{P}\mathbb{N} \setminus F]$)

> Is there a Borel set $F' \in \mathcal{I}$ such that, for every $a \subseteq \mathbb{N}$, at least one of a, $\mathbb{N}\setminus a$ belongs to F'?

3. On A and $\mathcal{I}$ (a) Observe that if $d \in [\mathbb{N}]^{<\omega}$ and we write $\phi_d(a) = a \triangle d$ for $a \subseteq \mathbb{N}$, then $\phi_d^{-1}[E_n] = E_n$ for all $n \in \mathbb{N}\setminus d$, so that $\mu\phi_d^{-1} \in A$ for every $\mu \in A$ and $\phi_d^{-1}[F] \in \mathcal{I}$ for every $F \in \mathcal{I}$. Accordingly, for any $F \in \mathcal{I}$, $F^* = \bigcup\{\phi_d^{-1}[F] : d \in [\mathbb{N}]^{<\omega}\} = \{a \triangle d : a \in F, d \in [\mathbb{N}]^{<\omega}\}$ belongs to $\mathcal{I}$.

We had better identify the point-supported measures in A; they are exactly those Radon probability measures μ on $\mathcal{P}\mathbb{N}$ such that $\mu[\mathbb{N}]^{<}\omega = 1$.

(b) If $h : \mathbb{N} \to \mathbb{N}$ is any bijection, and we set $\psi_h(a) = h^{-1}[a]$ for $a \subseteq \mathbb{N}$, then $\psi_h^{-1}[E_n] = E_{h(n)}$ for every n, so $\mu\psi_h^{-1} \in A$ for every $\mu \in A$ and $\psi_h^{-1}[F] \in \mathcal{I}$ for every $F \in \mathcal{I}$.

(c) If we give the set P of Radon probability measures on $\mathcal{P}\mathbb{N}$ its narrow topology (Fremlin 03, §437), the functionals $\mu \mapsto \mu E_n$ are all continuous, so A is $F_{\sigma\delta}$ ($= \mathbf{\Pi}^0_3$), and in particular is Borel.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

(d) At a deeper level we have the following fact. Take any $I \subseteq \mathbb{N}$. Write A_I for the set of Radon probability measures μ on $\mathcal{P}I$ such that $\lim_{n\in I, n\to\infty} \mu\{a : n \in a \subseteq I\} = 0$ (if I is finite, A_I is to be the set of all Radon probability measures on $\mathcal{P}I$), and $\mathcal{I}_I$ for the family of sets $F \subseteq \mathcal{P}I$ such that $\mu F = 0$ for every $\mu \in A_I$. Of course the embedding $\mathcal{P}I \subseteq \mathcal{P}\mathbb{N}$ identifies A_I with $\{\mu : \mu \in A,\ \mu(\mathcal{P}I) = 1\}$ and $\mathcal{I}_I$ with $\mathcal{I} \cap \mathcal{P}I$.

4. Theorem Suppose that $I \subseteq \mathbb{N}$. Set $J = \mathbb{N} \setminus I$ and for $E \subseteq \mathcal{P}\mathbb{N}$, $b \subseteq I$ write $E[\{b\}] = \{c : c \subseteq J,\ b \cup c \in F\}$. If $F \in \mathcal{I}$ is a Borel set then $\{b : b \subseteq I,\ F[\{b\}] \notin \mathcal{I}_J\} \in \mathcal{I}_I$.

proof (a) First note that if we give the set P_J of Radon probability measures on J the narrow topology, then $(b, \nu) \mapsto \nu F[\{b\}] : \mathcal{P}I \times P_J \to [0, 1]$ is Borel measurable; this is elementary if F is open-and-closed, and the general case follows by the Monotone Class Theoremn (FREMLIN 00, §136). Next, setting $H = \{b : b \subseteq I,\ F[\{b\}] \notin \mathcal{I}_J\} = \{b : b \subseteq I$, there is a $\nu \in A_J$ such that $\nu F[\{b\}] > 0\}$, H is analytic in $\mathcal{P}I$.

(b) ? If $H \notin \mathcal{I}_I$, then there is a $\mu \in A_I$ such that $\mu H > 0$. Let $b \mapsto \nu_b : H \to A_J$ be a universally measurable function such that $\nu_b(F[\{b\}]) > 0$ for every $b \in H$ (FREMLIN 03, 423M-423N). For $b \in \mathcal{P}I \setminus K$ take $\nu_b \in A_J$ to be the point mass at $\emptyset$. Now $b \mapsto \nu_b : \mathcal{P}I \to A_J$ is universally measurable, so we have a Radon probability measure λ on $\mathcal{P}\mathbb{N}$ defined by setting

$$\lambda E = \int \nu_b(F[\{b\}])\mu(db)$$

for every Borel set $E \subseteq \mathcal{P}\mathbb{N}$. Now $\lambda \in A$. **P** If $n \in I$ then $(E_n)[\{b\}] = \mathcal{P}J$ if $n \in b$, $\emptyset$ otherwise, so

$$\lim_{n\in I, n\to\infty} \lambda E_n = \lim_{n\in I, n\to\infty} \mu(E_n \cap \mathcal{P}I) = 0.$$

On the other hand, if $n \in J$, then $(E_n)[\{b\}] = \{c : n \in c \subseteq J\}$ for every $b \subseteq I$, so

$$\lim_{n\in J, n\to\infty} \lambda E_n = \lim_{n\in J, n\to\infty} \int \nu_b(E_n \cap \mathcal{P}J)\mu(db) = 0\}$$

by Lebesgue's Dominated Convergence Theorem (FREMLIN 00, §123). **Q**

However

$$\lambda F = \int \nu_b(F[\{b\}])\mu(db) > 0,$$

so this is impossible. **X**

5. Theorem If $F \in \mathcal{I}$ has the Baire property, it is meager.

proof (a) It will help if I begin by setting out an elementary fact. For $d \subseteq n \in \mathbb{N}$ set $G(n, d) = \{a : a \subseteq \mathbb{N},\ a \cap n = d\}$, so that $G(n, d)$ is a non-empty open-and-closed subset of $\mathcal{P}\mathbb{N}$. Now if $n \in \mathbb{N}$ and $M \subseteq \mathcal{P}\mathbb{N}$ is nowhere dense and $\langle d_i \rangle_{i\le k}$ is any finite sequence of subsets of n, there are an $m \ge n$ and a family $\langle e_i \rangle_{i\le k}$ of subsets of m such that (i) for each $i \le k$, $e_i \cap n = d_i$ and $G(m, e_i) \cap M = \emptyset$ (ii) if $i < j \le k$ then $e_i \cap e_j \subseteq n$. **P** Choose e_i one by one; all we need to know is that given d_j and $\langle e_i \rangle_{i<j}$ and $m_j \ge n$ such that $e_i \subseteq m_j$ for $i < j$, there are an $m_{j+1} \ge m_j$ and an $e_j \subseteq m_{j+1} \setminus m_j$ such that $G(d_j \cup e_j, m_{j+1}) \cap M$ is empty. At the end, set $m = m_{k+1}$. **Q**

(b) Suppose that $E \subseteq \mathcal{P}\mathbb{N}$ is a non-meager set with the Baire property. Then $E \notin \mathcal{I}$. **P** Start by choosing $n_0 \in \mathbb{N}$ and $d_\emptyset \subseteq n_0$ such that $E \cap G(n_0, d_\emptyset)$ is comeager in $G(n_0, d_\emptyset)$;

let $\langle M_k\rangle_{k\in\mathbb{N}}$ be a sequence of nowhere dense sets covering $G(n_0,d_\emptyset)\setminus E$. Given n_k and $d_\sigma\subseteq n_k$ for $\sigma\in\{0,1\}^k$, choose $n_{k+1}\ge n_k$ and d_τ, for $\tau\in\{0,1\}^{k+1}$, such that

$$d_\tau\cap n_k=d_{\tau\restriction k}\text{ for every }\tau\in\{0,1\}^{k+1},$$

$$G(n_{k+1},d_\tau)\cap M_k=\emptyset\text{ for every }\tau\in\{0,1\}^{k+1},$$

$$\text{if }\tau,\ \tau'\in\{0,1\}^{k+1}\text{ are distinct, then }d_\tau\cap d_{\tau'}\subseteq n_k.$$

Continue.

At the end of the induction, define $f:\{0,1\}^{\mathbb{N}}\to\mathcal{P}\mathbb{N}$ by setting $f(x)=\bigcup_{k\in\mathbb{N}}d_{x\restriction k}$. Then f is continuous, and $f(x)\in G(n_k,d_{x\restriction k})\subseteq G(n_0,d_\emptyset)$ for every $x\in\{0,1\}^{\mathbb{N}}$, $k\in\mathbb{N}$. So $f(x)\in E$ for every $x\in\{0,1\}^{\mathbb{N}}$. Let λ be the usual measure on $\{0,1\}^{\mathbb{N}}$ and set $\mu=\lambda f^{-1}$. Then $\mu E=1$. If $k\in\mathbb{N}$ and $n_k\le n<n_{k+1}$, then

$$\mu E_n=\lambda\{x:n\in f(x)\}=\lambda\{x:n\in d_{x\restriction k+1}\}\le 2^{-k-1}$$

because there can be at most one $\tau\in\{0,1\}^{k+1}$ such that $n\in d_\tau$. But this shows that $\lim_{n\to\infty}\mu E_n=0$. (The construction above allows $\langle n_k\rangle_{k\in\mathbb{N}}$ to be bounded; but if $n\ge n_k$ for every $k\in\mathbb{N}$, then $n\notin d_\tau$ for every τ, so $n\notin f(x)$ for every x and $\mu E_n=0$.) So μ witnesses that $E\notin\mathcal{I}$.

(c) Turning (b) round, we have the result.

6. Corollary The question in §1 has the answer ‘yes’ and the questions in §2 all have the answer ‘no’.

proof If $F\in\mathcal{I}$ is a Borel set it is meager, so $\{a:\mathbb{N}\setminus a\in F\}$ is also meager and there must be an $a\subseteq\mathbb{N}$ such that neither a nor $\mathbb{N}\setminus a$ belongs to F.

7. Another question The following was asked by B.de Pagter and W.Ritter (private communication, October 2002). Let A^* be the set of those Radon probability measures μ on $\mathcal{P}\mathbb{N}$ such that $\lim_{m,n\to\infty}\mu(E_m\triangle E_n)=0$. Is there a Borel set $F\subseteq\mathbb{N}$ such that $\mu F=\lim_{n\to\infty}\mu E_n$ for every $\mu\in A^*$? But as A and $\{\mu:\mu\pi^{-1}\in A\}=\{\mu:\lim_{n\to\infty}\mu E_n=1\}$ are both included in A^*, such a set would have to provide a positive answer to the questions in §2 and therefore cannot exist.

8. A question in Fremlin 02**.** In 323Z of Fremlin 02 (2002 edition) I issued the invitation:

> Find a localizable measure algebra $(\mathfrak{A},\bar\mu)$, a σ-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ and a sequence $\langle b_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{B}$ which converges, for the measure-algebra topology, to a member of $\mathfrak{A}\setminus\mathfrak{B}$.

I can now offer a construction. Write $\mathcal{B}$ for the set of Borel subsets of $\mathcal{P}\mathbb{N}$. For each $F\in\mathcal{B}$ let ν_F be a Radon measure on $\{0,1\}^{\mathbb{N}}$ such that either $\lim_{n\to\infty}\nu_FE_n=0<\nu_FF$ or $\lim_{n\to\infty}\nu_FE_n=1>\nu_FF$. Let (X,μ) be the direct sum of $\langle\mathcal{P}\mathbb{N},\nu_F\rangle_{F\in\mathcal{B}}$, $(\mathfrak{A},\bar\mu)$ its measure algebra. For $n\in\mathbb{N}$ set $B_n=\{(F,a):F\in\mathcal{B},\ a\in E_n\}\subseteq X$, and $b_n=B_n^\bullet\in\mathfrak{A}$. Set

$$B=\{(F,a):F\in\mathcal{B},\ \lim_{n\to\infty}\nu_FE_n=1,\ a\in\mathcal{P}\mathbb{N}\}\subseteq X,\qquad b=B^\bullet\in\mathfrak{A}.$$

Then $\langle b_n\rangle_{n\in\mathbb{N}}\to b$ for the measure-algebra topology, since for any $F\in\mathcal{B}$

— if $\lim_{n\to\infty}\nu_F E_n = 1$, then $\{a : (F,a) \in B_n\}\triangle\{a : (F,a) \in B\} = E_n\triangle\mathcal{P}\mathbb{N}$ and $\lim_{n\to\infty}\nu_F(\{a : (F,a) \in B_n\}\triangle\{a : (F,a) \in B\}) = 0$

— if $\lim_{n\to\infty}\nu_F E_n = 0$, then $\{a : (F,a) \in B_n\}\triangle\{a : (F,a) \in B\} = \emptyset$ and $\lim_{n\to\infty}\nu_F(\{a : (F,a) \in B_n\}\triangle\{a : (F,a) \in B\}) = 0$.

If we set

$$B_H = \{(F,x) : F \in \mathcal{B},\ x \in H\}$$

for $H \in \mathcal{B}$, then the σ-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by $\{b_n : n \in \mathbb{N}\}$ is just $\{B_H^\bullet : H \in \mathcal{B}\}$, because $H \mapsto B_H$ is a sequentially order-continuous Boolean homomorphism, and $b_n = B_{E_n}^\bullet$ for each n. **?** If $b \in \mathfrak{B}$, say $b = B_H^\bullet$. Then $\lim_{n\to\infty}\nu_H(H\triangle E_n) = 0$, contrary to the choice of ν_H. **X**

Thus we have an example for 323Z; a localizable measure algebra with a σ-subalgebra which is not sequentially closed for the measure-algebra topology.

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References

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