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Problem ET

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1. Vector-valued gauge integrals Let X be a set, $\mathcal{C}$ a family of subsets of X, $T \subseteq [X \times \mathcal{C}]^{<\omega}$ a non-empty set of 'tagged partitions', $\mathcal{F}$ a filter on T and U a normed space. For $\boldsymbol{t} \in T$, set $W_{\boldsymbol{t}} = \bigcup\{C : (x, C) \in \boldsymbol{t}\}$. If $\delta : X \to \mathcal{P}X$ is a function, say that $\boldsymbol{t} \in T$ is **δ-fine** if $C \subseteq \delta(x)$ whenever $(x, C) \in \boldsymbol{t}$. If $\mathcal{R} \subseteq \mathcal{P}X$, say that $\boldsymbol{t}$ is **$\mathcal{R}$-filling** if $X \setminus W_{\boldsymbol{t}} \in \mathcal{R}$.

For $f : X \to U$, $\nu : \mathcal{C} \to \mathbb{R}$ and $\boldsymbol{t} \in T$, set $S_{\boldsymbol{t}} = \sum_{(x,C)\in\boldsymbol{t}} \nu C \cdot f(x)$. Now set

$$I_{\nu,\mathcal{F}}(f) = \lim_{\boldsymbol{t}\to\mathcal{F}} S_{\boldsymbol{t}}(f, \nu)$$

if this is defined in X for the norm topology. Then $I_{\nu,\mathcal{F}}$ is a linear functional defined on a linear subspace of U^X, a **gauge integral** for U-valued functions.

For the elementary theory of gauge integrals, see Fremlin 03, §§481-482.

2. The McShane integral (Fremlin 03, 481N, or Fremlin 95) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space, and take $\Sigma^f = \{E : E \in \Sigma,\ \mu E < \infty\}$. Let $T = T(X, \Sigma^f)$ be the family of finite subsets $\boldsymbol{t}$ of $X \times \Sigma^f$ such that $C \cap C'$ is empty whenever (x, C), (x', C') are distinct members of $\boldsymbol{t}$. Let $\Delta = \Delta(X, \mathfrak{T})$ be the set of all **neighbourhood gauges** on X, that is, functions $\delta : X \to \mathfrak{T}$ such that $x \in \delta(x)$ for every $x \in X$. For any $E \in \Sigma^f$ and $\eta > 0$, let $\mathcal{R}_{E\eta}$ be the set $\{F : F \in \Sigma,\ \mu(F \cap E) \le \eta\}$. Let $\mathcal{F}$ be the filter on T generated by sets of the form

$$\{\boldsymbol{t} : \boldsymbol{t} \in T \text{ is } \delta\text{-fine and } \mathcal{R}_{E\eta}\text{-filling}\}$$

where δ runs over Δ, E over Σ^f and η over $]0, \infty[$. (The check that these sets do indeed generate a filter is in Fremlin 03, 481N.) I will call the corresponding functional $I_{\mu,\mathcal{F}}$ the **McShane integral** for vector-valued functions defined on X, and write $\mathrm{MS}\!\!\int f d\mu = I_{\mu,\mathcal{F}}(f)$ for the McShane integral of a function f.

3 The definition here is designed to match the structures developed in Fremlin 03, chap. 48, and differs from the formulation in Fremlin 95. I should therefore explain why they are the same in the context considered in the earlier paper.

Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a non-empty σ-finite quasi-Radon measure space in which μ is outer regular with respect to the open sets, U a Banach space, $f : X \to U$ a function and $w \in U$. Then the following are equiveridical:

(i) w is the McShane integral of f as defined in §2;

(ii) for every $\epsilon > 0$ there is a $\delta \in \Delta = \Delta(X, \mathfrak{T})$ such that

$$\limsup_{n\to\infty} \|w - \sum_{i\le n} \mu E_i \cdot f(x_i)\| \le \epsilon$$

whenever $\langle E_i\rangle_{i\in\mathbb{N}}$ is a disjoint sequence in Σ^f with conegligible union, and $x_i \in X$, $E_i \subseteq \delta(x_i)$ for each i.

proof (i)⇒(ii) If w is the McShane integral of f, and $\epsilon > 0$, let $\delta \in \Delta$, $E \in \Sigma^f$ and $\eta > 0$ be such that $\|S_{\boldsymbol{t}}(f, \mu) - w\| \le \epsilon$ whenever $\boldsymbol{t} \in T = T(X, \Sigma^f)$ is δ-fine and $\mu(E \setminus W_{\boldsymbol{t}}) \le \eta$. Let $\langle E_i\rangle_{i\in\mathbb{N}}$ be a disjoint sequence in Σ^f with conegligible union, and suppose that $x_i \in X$, $E_i \subseteq \delta(x_i)$ for each i. Then $\boldsymbol{t}_n = \{(x_i, E_i) : i \le n\} \in T$ for every n, and $W_{\boldsymbol{t}_n} = \bigcup_{i\le n} E_i$, so there is an n_0 such that $\mu(E \setminus W_{\boldsymbol{t}_n}) \le \eta$ for every $n \ge n_0$. Now

$$\limsup_{n\to\infty} \|w - \sum_{i\le n} \mu E_i \cdot f(x_i)\| \le \sup_{n\ge n_0} \|w - S_{\boldsymbol{t}_n}(f, \mu)\| \le \epsilon.$$

As ϵ is arbitrary, (ii) is true.

(ii)⇒(i) Assume (ii). Because μ is σ-finite and outer regular with respect to the open sets, there is a non-decreasing sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of open sets of finite measure covering X.

(α) For any $\epsilon > 0$ there are $\delta \in \Delta$, $E \in \Sigma^f$ and $\eta > 0$ such that $\|S_{\boldsymbol{t}}(f, \mu)\| \le 5\epsilon$ whenever $\boldsymbol{t} \in T$ is δ-fine and $\mu(E \cap W_{\boldsymbol{t}}) \le \eta$. **P?** Otherwise, let $\delta \in \Delta$ be such that

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$$\limsup_{n\to\infty} \|w - \sum_{i\le n} \mu E_i \cdot f(x_i)\| \le \epsilon$$

whenever $\langle E_i\rangle_{i\in\mathbb{N}}$ is a disjoint sequence in Σ^f with conegligible union and $x_i \in X$, $E_i \subseteq \delta(x_i)$ for each i. As μ is locally finite, we can suppose that $\mu(\delta(x)) < \infty$ for every x. Choose $\langle \boldsymbol{t}_n\rangle_{n\in\mathbb{N}}$ in T inductively, as follows. Given $\langle \boldsymbol{t}_i\rangle_{i<n}$, set $F_n = \bigcup_{i<n} W_{\boldsymbol{t}_i}$, and let $\eta_n > 0$ be such that $\eta_n\|f(x)\| \le 2^{-n}\epsilon$ whenever $(x,E) \in \bigcup_{i<n} \boldsymbol{t}_i$; let $\boldsymbol{t}_n \in T$ be such that $\|S_{\boldsymbol{t}_n}(f,\mu)\| > 5\epsilon$ and $\mu(F_n \cap W_{\boldsymbol{t}_n}) \le \eta_n$. Continue.

Set

$$\boldsymbol{t}'_n = \{(x, E\setminus \bigcup_{m>n} W_{\boldsymbol{t}_m}) : (x,E) \in \boldsymbol{t}_n\} \in T$$

for each n, so that $\langle W_{\boldsymbol{t}'_n}\rangle_{n\in\mathbb{N}}$ is disjoint. Also

$$\begin{aligned}\|S_{\boldsymbol{t}_n}(f,\mu) - S_{\boldsymbol{t}'_n}(f,\mu)\| &\le \sum_{(x,E)\in\boldsymbol{t}_n} \|f(x)\|\mu(E\cap \bigcup_{m>n} W_{\boldsymbol{t}_m})\\ &\le \sup_{(x,E)\in\boldsymbol{t}_n} \|f(x)\|\cdot\mu(W_{\boldsymbol{t}_n}\cap \bigcup_{m>n} W_{\boldsymbol{t}_m})\\ &\le \sup_{(x,E)\in\boldsymbol{t}_n} \|f(x)\|\cdot \sum_{m=n+1}^{\infty}\eta_m \le \sum_{m=n+1}^{\infty} 2^{-m}\epsilon \le \epsilon,\end{aligned}$$

so $\|S_{\boldsymbol{t}'_n}(f,\mu)\| \ge 4\epsilon$.

Set $H = \bigcup_{n\in\mathbb{N}} W_{\boldsymbol{t}'_n} = \bigcup_{n\in\mathbb{N}} W_{\boldsymbol{t}_n}$. There is a sequence $\langle y_n\rangle_{n\in\mathbb{N}}$ in X such that $H\setminus\bigcup_{n\in\mathbb{N}}\delta(y_n)$ is negligible; set $H_n = H\cap\delta(y_n)\setminus\bigcup_{i<n}\delta(y_i)$ for each n. Now define $\langle(x_n,E_n)\rangle_{n\in\mathbb{N}}$ in T as

$$\mathbf{z}_0{}^\frown(y_0,H_0)^\frown\mathbf{z}_1{}^\frown(y_1,H_1)^\frown\mathbf{z}_2\ldots,$$

where $\mathbf{z}_n$ is an enumeration of $\boldsymbol{t}'_n$ for each n. Then $\langle E_n\rangle_{n\in\mathbb{N}}$ is disjoint, $E_n\subseteq\delta(x_n)$ for every n and $\bigcup_{n\in\mathbb{N}} E_n$ is conegligible. By the choice of δ, we ought to have $\|w - \sum_{i=0}^n \mu E_i\cdot f(x_i)\| < 2\epsilon$ for all sufficiently large n. But this will mean that $\|\sum_{i=m}^n \mu E_i\cdot f(x_i)\| < 4\epsilon$ for all sufficiently large $m < n$, and, in particular, $\|S_{\boldsymbol{t}'_n}(f,\mu)\| < 4\epsilon$ for all sufficiently large n; which is not the case. **XQ**

(β) Let $\epsilon > 0$. Let $\delta_0 \in \Delta$ be such that

$$\limsup_{n\to\infty} \|w - \sum_{i\le n} \mu E_i \cdot f(x_i)\| \le \epsilon$$

whenever $\langle E_i\rangle_{i\in\mathbb{N}}$ is a disjoint sequence in Σ^f with conegligible union, and $x_i\in X$, $E_i\subseteq\delta_0(x_i)$ for each i. Let $F\in\Sigma^f$, $\eta>0$, $\delta_1\in\Delta$ be such that $\|S_{\boldsymbol{s}}(f,\mu)\|\le\epsilon$ whenever $\boldsymbol{s}\in T$ is δ_1-fine and $\mu(F\cap W_{\boldsymbol{s}})\le\eta$.

Suppose that $\boldsymbol{t}\in T$ is δ_0-fine and that $\mu(F\setminus W_{\boldsymbol{t}})\le\eta$. Enumerate $\boldsymbol{t}$ as $\langle(x_n,E_n)\rangle_{n<m}$ and extend to a sequence $\langle(x_n,E_n)\rangle_{n\in\mathbb{N}}$ such that $\langle E_n\rangle_{n\in\mathbb{N}}$ is disjoint and has conegligible union and $E_n\subseteq\delta_0(x_n)\cap\delta_1(x_n)$ for every $n\ge m$. Then there must be some $n\ge m$ such that $\|w-\sum_{i=0}^n\mu E_i\cdot f(x_i)\|\le 2\epsilon$. Setting $\boldsymbol{s} = \{(x_i,E_i) : m\le i\le n\}$ we see that $\boldsymbol{s}\in T$ is δ_1-fine and that $W_{\boldsymbol{s}}$ is disjoint from $W_{\boldsymbol{t}}$, so that

$$\|w - S_{\boldsymbol{t}}(f,\mu)\| \le \|w-\sum_{i=0}^n \mu E_i\cdot f(x_i)\| + \|S_{\boldsymbol{s}}(f,\mu)\| \le 2\epsilon.$$

As ϵ is arbitrary, w is the McShane integral of f, and (i) is true.

4 We can therefore use the results of Fremlin 95, in particular, the following.

Proposition (a) If $(X,\mathfrak{T},\Sigma,\mu)$ is a compact Radon measure space, then the McShane integral can also be defined by the filter $\mathcal{F}'$ on $T = T(X,\Sigma^f)$ generated by sets of the form

$$\{\boldsymbol{t} : \boldsymbol{t}\in T \text{ is } \delta\text{-fine and } W_{\boldsymbol{t}} = X\}$$

as δ runs over $\Delta = \Delta(X,\mathfrak{T})$.

(b) If $(X,\mathfrak{T},\Sigma,\mu)$ is a σ-finite quasi-Radon measure space in which μ is outer regular with respect to the open sets, U is a Banach space, and $f : X\to U$ is McShane integrable, then $\chi E\cdot f : X\to U$ is McShane integrable for any $E\in\Sigma$.

(c) If $(X,\mathfrak{T},\Sigma,\mu)$ is a σ-finite quasi-Radon measure space in which μ is outer regular with respect to the open sets, then any McShane integrable function on X is Pettis integrable, with the same integral.

(d) If $(X,\mathfrak{T},\Sigma,\mu)$ is a σ-finite quasi-Radon measure space in which μ is outer regular with respect to the open sets, U is a Banach space, and $f : X\to U$ is McShane integrable, then for any $A\subseteq X$ the restriction $f\restriction A$ is McShane integrable if we give A the subspace topology and measure.

proof Fremlin 95, 1E, 2F, 1Q and 1N.

5 Lemma Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite quasi-Radon measure space in which μ is outer regular with respect to the open sets, U a Banach space, and $f : X \to U$ a McShane integrable function.

(a) There are $M \geq 0$ and $\delta \in \Delta = \Delta(X, \mathfrak{T})$ such that $\|S_{\boldsymbol{t}}(f, \mu)\| \leq M$ whenever $\boldsymbol{t} \in T = T(X, \Sigma^f)$ is δ-fine.

(b) If $h : X \to \mathbb{R}$ is a bounded measurable function, then $h \cdot f : X \to U$ is McShane integrable.

proof (a) Set $M = 1 + \sup_{E \in \Sigma} \| \mathrm{P}\!\!\!\!\int_E f d\mu\|$, where $\mathrm{P}\!\!\!\!\int$ is the Pettis integral. By Fremlin 95, 2B, there is a $\delta \in \Delta$ such that $\|S_{\boldsymbol{t}}(f, \mu) - \mathrm{P}\!\!\!\!\int_{W_{\boldsymbol{t}}} f\| \leq 1$ whenever $\boldsymbol{t} \in T$ is δ-fine; so $\|S_{\boldsymbol{t}}(f, \mu)\| \leq M$ whenever $\boldsymbol{t} \in T$ is δ-fine.

(b) Given $\epsilon > 0$, let $M > 0$ and $\delta_0 \in \Delta$ be such that $\|S_{\boldsymbol{t}}(f, \mu)\| \leq M$ whenever $\boldsymbol{t} \in T$ is δ_0-fine. Then $\|S_{\boldsymbol{t}}(g \cdot f, \mu)\| \leq M$ whenever $\boldsymbol{t} \in T$ is δ_0-fine and $g : X \to [0, 1]$ is a function. $\mathbf{P}$ $\|S_{\boldsymbol{t}}(g \cdot f, \mu) = \sum_{(x,E) \in \boldsymbol{t}} g(x) \mu E \cdot f(x)\|$ belongs to the convex hull of $\{\|S_{\boldsymbol{s}}(f, \mu) : \boldsymbol{s} \subseteq \boldsymbol{t}\}$. $\mathbf{Q}$

Let $h_0 : X \to \mathbb{R}$ be a measurable function taking only finitely many values such that $h_0 \leq h \leq h_0 + \frac{\epsilon}{M} \chi X$. Then $h_0 \cdot f$ is McShane integrable, by Proposition 4b. So there are $w_0 \in U$, $\delta \in \Delta$, $F \in \Sigma^f$ and $\eta > 0$ such that $\|S_{\boldsymbol{t}}(h_0 \cdot f, \mu) - w_0\| \leq \epsilon$ whenever $\boldsymbol{t} \in T$ is δ-fine and $\mu(F \setminus W_{\boldsymbol{t}}) \leq \eta$; and we may suppose that $\delta(x) \subseteq \delta_0(x)$ for every $x \in X$. In this case,

$$\begin{aligned}\|S_{\boldsymbol{t}}(h \cdot f, \mu) - w_0\| &\leq \|S_{\boldsymbol{t}}(h_0 \cdot f, \mu) - w_0\| + \|S_{\boldsymbol{t}}((h - h_0) \cdot f, \mu)\| \\ &\leq \epsilon + M\|h - h_0\|_\infty \leq 2\epsilon\end{aligned}$$

whenever $\boldsymbol{t} \in T$ is δ-fine and $\mu(F \setminus W_{\boldsymbol{t}}) \leq \eta$. So if $\boldsymbol{s}$ and $\boldsymbol{t}$ are δ-fine members of T such that $\mu(F \setminus W_{\boldsymbol{s}}) \leq \eta$ and $\mu(F \setminus W_{\boldsymbol{t}}) \leq \eta$, $\|S_{\boldsymbol{s}}(h \cdot f, \mu) - S_{\boldsymbol{t}}(h \cdot f, \mu)\| \leq 4\epsilon$. As U is supposed to be complete, this is enough to ensure that $h \cdot f$ is McShane integrable.

6 Lemma Let (X, Σ, μ) be an atomless totally finite measure space, and $G_0, \ldots, G_n \in \Sigma$. Suppose that $\alpha_0, \ldots, \alpha_n \geq 0$ are such that $\sum_{i \in I} \alpha_i \leq \mu(\bigcup_{i \in I} G_i)$ for every $I \subseteq n + 1$. Then there are disjoint $E_0, \ldots, E_n \in \Sigma$ such that $E_i \subseteq G_i$ for every i and $\mu E_i = \alpha_i$ for every i.

proof Bollobás & Varopoulos 75.

7 Lemma Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an atomless totally finite quasi-Radon measure space, U a Banach space, $f : X \to U$ a McShane integrable function with $\mathrm{MS}\!\!\!\!\!\int f d\mu = w$ and $\epsilon > 0$. Then there are $\delta \in \Delta = \Delta(X, \mathfrak{T})$ and $\eta > 0$ such that $\|w - \sum_{i<n} \alpha_i f(x_i)\| \leq \epsilon$ whenever $x_i \in X$, $\alpha_i \geq 0$ are such that $\sum_{i \in I} \alpha_i \leq \mu(\bigcup_{i \in I} \delta(x_i))$ for every $I \subseteq n$ and $\sum_{i<n} \alpha_i \geq \mu X - \eta$.

proof Let $\delta \in \Delta$, $\eta > 0$ be such that $\|w - S_{\boldsymbol{t}}(f, \mu)\| \leq \epsilon$ whenever $\boldsymbol{t} \in T$ is δ-fine and $\mu W_{\boldsymbol{t}} \geq \mu X - \eta$. Suppose that $x_i \in X$, $\alpha_i \geq 0$ are such that $\sum_{i \in I} \alpha_i \leq \mu(\bigcup_{i \in I} \delta(x_i))$ for every $I \subseteq n$ and $\sum_{i<n} \alpha_i \geq \mu X - \eta$. By Lemma 6, there are disjoint measurable $F_i \subseteq \delta(x_i)$, for $i < n$, such that $\mu F_i = \alpha_i$ for each i. So $\boldsymbol{t} = \{(x_i, F_i) : i < n\}$ belongs to $T(X, \Sigma)$ and is δ-fine, and $\mu W_{\boldsymbol{t}} = \sum_{i<n} \alpha_i \geq \mu X - \eta$. Accordingly

$$\|w - \textstyle\sum_{i<n} \alpha_i f(x_i)\| = \|w - S_{\boldsymbol{t}}(f, \mu)\| \leq \epsilon.$$

As x_i and α_i are arbitrary, we have appropriate δ and η.

8 Proposition Suppose that $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ are σ-finite quasi-Radon measure spaces in which the measures are outer regular with respect to the open sets, and $\phi : X \to Y$ an almost continuous inverse-measure-preserving function. If U is a Banach space and $f : Y \to U$ is McShane integrable, then $f\phi : X \to Y$ is McShane integrable, with the same integral.

proof (a) Suppose first that μ and ν are atomless and $\mu X = \nu Y$ is finite. Write w for $\mathrm{MS}\!\!\!\!\!\int f\, d\nu$. Let $\epsilon > 0$. By Lemma 7, there are $\delta \in \Delta(Y, \mathfrak{S})$ and $\eta > 0$ such that $\|w - \sum_{i<n} \alpha_i f(y_i)\| \leq \epsilon$ whenever $y_i \in Y$, $\alpha_i \geq 0$ are such that $\sum_{i \in I} \alpha_i \leq \nu(\bigcup_{i \in I} \delta(y_i))$ for every $I \subseteq n$ and $\sum_{i<n} \alpha_i \geq \nu Y - 3\eta$.

For each $n \in \mathbb{N}$, let $K_n \subseteq X$ be a closed set such that $\phi \restriction K_n$ is continuous and $\mu(X \setminus K_n) \leq 2^{-n} \min(\eta, \epsilon)$. For $x \in X$, let $n_x \in \mathbb{N}$ be such that $n_x \leq \|f(\phi(x))\| < n_x + 1$ and take $\delta'(x) \in \mathfrak{T}$ such that $x \in \delta'(x)$ and $K_{n_x} \cap \delta'(x) \subseteq \phi^{-1}[\delta(\phi(x))]$. Let $\boldsymbol{s} \in T(X, \Sigma)$ be δ'-fine and such that $\mu W_{\boldsymbol{s}} \geq \mu X - \eta$. Set

$$\boldsymbol{s}' = \{(x, E\cap K_{n_x}) : (x,E)\in\boldsymbol{s}\}.$$

so that $\boldsymbol{s}' \in T(X,\Sigma)$ is δ'-fine. Then

$$\begin{aligned}
\|S_{\boldsymbol{s}}(f\phi,\mu) - S_{\boldsymbol{s}'}(f\phi,\mu)\| &\le \sum_{(x,E)\in\boldsymbol{s}} \|f(x)\|\mu(E\setminus K_{n_x}) \\
&= \sum_{n=0}^{\infty} \sum_{(x,E)\in\boldsymbol{s}, n_x=n} \|f(x)\|\mu(E\setminus K_{n_x}) \\
&\le \sum_{n=0}^{\infty}(n+1)\mu(X\setminus K_n) \le \sum_{n=0}^{\infty} 2^{-n}(n+1)\epsilon = 4\epsilon, \\
\mu(W_{\boldsymbol{s}}\setminus W_{\boldsymbol{s}'}) &\le \sum_{(x,E)\in\boldsymbol{s}} \mu(E\setminus K_{n_x}) \\
&= \sum_{n=0}^{\infty} \sum_{(x,E)\in\boldsymbol{s}, n_x=n} \mu(E\setminus K_{n_x}) \\
&\le \sum_{n=0}^{\infty}\mu(X\setminus K_n) \le \sum_{n=0}^{\infty} 2^{-n}\eta = 2\eta.
\end{aligned}$$

Enumerate $\boldsymbol{s}'$ as $\langle (x_i,E_i)\rangle_{i<n}$. Then for any $I\subseteq n$,

$$\textstyle\sum_{i\in I}\mu E_i \le \mu(\bigcup_{i\in I}\phi^{-1}[\delta(\phi(x_i))]) = \nu(\bigcup_{i\in I}\delta(\phi(x_i))),$$

while

$$\textstyle\sum_{i<n}\mu E_i = \mu W_{\boldsymbol{s}'} \ge \mu W_{\boldsymbol{s}} - 2\eta \ge \mu X - 3\eta = \nu Y - 3\eta.$$

So

$$\textstyle\|w - S_{\boldsymbol{s}'}(f\phi,\mu)\| = \|w - \sum_{i<n}\mu E_i\cdot f(\phi(x_i))\| \le \epsilon,$$

and $\|w - S_{\boldsymbol{s}}(f\phi,\mu)\| \le 5\epsilon$. As ϵ is arbitrary, $\mathrm{McS}\!\int f\phi\,d\mu = w$, as required.

(b) If Y is an atom for ν, then there is a closed conegligible set $F\subseteq Y$ such that the subspace topology on F is the indiscrete topology and the induced σ-algebra is $\{\emptyset, F\}$ (Fremlin 03, 415Xa). In this case, since f is Pettis integrable, gf is Σ-measurable and $gf\restriction F$ is constant for every $g\in U^*$; accordingly $f\restriction F$ is constant. Let u be its constant value. Since of course $\phi^{-1}[F]$ is also conegligible,

$$\mathrm{McS}\!\int f\phi\,d\mu = \mu\phi^{-1}[F]\cdot u = \nu F\cdot u = \mathrm{McS}\!\int f\,d\nu$$

(using Fremlin 95, 1I and 2G).

(c) For the general case, let $\langle F_n\rangle_{n\in\mathbb{N}}$ be a partition of Y into measurable sets such that each is either atomless and of finite measure or an atom for ν. Set $E_n = \phi^{-1}[F_n]$. By (a)-(b), applied to the subspace topologies and measures on E_n and F_n and the functions $\phi\restriction E_n$, $f\restriction F_n$, we have $\mathrm{McS}\!\int_{E_n} f\phi\,d\mu = \mathrm{McS}\!\int_{F_n} f\,d\nu$ for every n. Next, because f is Pettis integrable and ϕ is inverse-measure-preserving (and ν is σ-finite), $f\phi$ is Pettis integrable, with $\mathrm{P}\!\int f\phi\,d\mu = \mathrm{P}\!\int f\,d\nu$. But this means that, setting $\psi_n = \chi(\bigcup_{i\le n}E_i)\cdot f\phi$ for each n, $f(\phi(x)) = \lim_{n\to\infty}\psi_n(x)$ for every $x\in X$, ψ_n is McShane integrable for every n, and

$$\lim_{n\to\infty}\mathrm{McS}\!\int_E \psi_n = \lim_{n\to\infty}\mathrm{P}\!\int_{E\cap\bigcup_{i\le n}} f\phi\,d\mu = \mathrm{P}\!\int_E f\phi\,d\mu$$

for the weak topology of U, for every $E\in\Sigma$. By Fremlin 95, 4A, $f\phi$ is McShane integrable with integral $\mathrm{P}\!\int f\phi\,d\mu = \mathrm{McS}\!\int f\,d\nu$. So we have the general result.

9 Problem Suppose that $(X,\mathfrak{T},\Sigma,\mu)$ is a quasi-Radon measure space and $\mathfrak{S}$ is another topology on X for which μ is still quasi-Radon. Do $(X,\mathfrak{T},\Sigma,\mu)$ and $(X,\mathfrak{S},\Sigma,\mu)$ have the same McShane integrable vector-valued functions?

In Fremlin n92 I show that if $(X,\mathfrak{T},\Sigma,\mu)$ is a quasi-Radon measure space in which X is covered by a sequence of open sets of finite measure, a function $f: X\to\ell^\infty$ is Henstock integrable iff it is Birkhoff integrable. So if $\mathfrak{S}$ is another topology on X with the same properties with respect to μ, and U is a normed space with a weak*-separable dual unit ball, then the U-valued McShane integrals will be the same for both topologies.

10 Problem ET Let μ_L be Lebesgue measure on $[0,1]$. Consider the statement

($\dagger$) Whenever $\langle G_x\rangle_{x\in[0,1]}$ is a family of open subsets of $\mathbb{R}$ such that $\mu_L(A\setminus\bigcup_{x\in A}G_x)=0$ for every $A\subseteq[0,1]$, and $\epsilon>0$, there is a family $\langle H_x\rangle_{x\in[0,1]}$ of open sets such that $x\in H_x$ for every $x\in[0,1]$ and $\mu_L(\bigcup_{x\in A}H_x\setminus\bigcup_{x\in A}G_x)\le\epsilon$ for every $A\subseteq[0,1]$.

If this is true, we have the following result.

11 Proposition Let $\mathfrak{T}$ be the usual topology on $[0,1]$, and $\mathfrak{S}$ another topology on $[0,1]$ with respect to which Lebesgue measure μ_L on $[0,1]$ is quasi-Radon. Let U be a Banach space, and $I_{\mathfrak{T}}$, $I_{\mathfrak{S}}$ the U-valued McShane integrals. Then $I_{\mathfrak{S}}$ extends $I_{\mathfrak{T}}$, and if ($\dagger$) in §10 is true, then $I_{\mathfrak{S}}=I_{\mathfrak{T}}$.

proof (a) The identity map from $([0,1],\mathfrak{S})$ to $([0,1],\mathfrak{T})$ is measurable, therefore almost continuous (Fremlin 03, 418J), so every $I_{\mathfrak{T}}$-integrable function is $I_{\mathfrak{S}}$-integrable with the right integral (Proposition 8).

(b) Now suppose that ($\dagger$) is true, and that U is a Banach space and that $f:[0,1]\to U$ is such that $w=I_{\mathfrak{S}}(f)$ is defined. Let $\epsilon>0$. Then there are a $\delta\in\Delta([0,1],\mathfrak{S})$ and an $\eta>0$ such that $\|w-\sum_{i<n}\alpha_i f(x_i)\|\le\epsilon$ whenever $x_i\in[0,1]$, $\alpha_i\ge0$ are such that $\sum_{i\in I}\alpha_i\le\mu_L(\bigcup_{i\in I}\delta(x_i))$ for every $I\subseteq n$ and $\sum_{i<n}\alpha_i\ge1-5\eta$.

Set $A_n=\{x:x\in[0,1],\,n\le\|f(x)\|<n+1\}$ for $n\in\mathbb{N}$. Let $\mathcal{I}$ be the family of all open intervals in $[0,1]$ of the form $]2^{-n}k,2^{-n}(k+1)[$ for $n\in\mathbb{N}$, $k<2^n$. For $x\in A_n$, set

$$G_x=\bigcup\{I:I\in\mathcal{I},\,\mu_L(I\setminus\delta(x))\le2^{-n}\eta\mu_LI\},$$

so that $G_x\in\mathfrak{T}$ and $\mu_L(\delta(x)\setminus G_x)=0$ (Fremlin 01, 261Yb). If $A\subseteq[0,1]$, $\{\delta(x):x\in A\}\subseteq\mathfrak{S}$; now μ_L is τ-additive for $\mathfrak{S}$, so there is a countable set $A'\subseteq A$ such that $\bigcup_{x\in A}\delta(x)\setminus\bigcup_{x\in A'}\delta(x)$ is negligible. But now

$$A\setminus\bigcup_{x\in A}G_x\subseteq(\bigcup_{x\in A}\delta(x)\setminus\bigcup_{x\in A'}\delta(x))\cup\bigcup_{x\in A'}\delta(x)\setminus G_x$$

is negligible. So for each $n\in\mathbb{N}$ we have a $\delta'_n\in\Delta([0,1],\mathfrak{T})$ such that $\mu_L(\bigcup_{x\in A}\delta'_n(x)\setminus\bigcup_{x\in A}G_x)\le2^{-n}\eta$ for every $A\subseteq[0,1]$. Set $\delta'(x)=\delta'_n(x)$ for $n\in\mathbb{N}$, $x\in A_n$, so that $\delta'\in\Delta([0,1],\mathfrak{T})$.

If $A\subseteq A_n$, then $\mu_L(\bigcup_{x\in A}\delta'(x)\setminus\bigcup_{x\in A}\delta(x))\le2^{-n+1}\eta$. **P** Set

$$\mathcal{J}=\{I:I\in\mathcal{I},\,\mu_L(I\setminus\delta(x))\le2^{-n}\eta\mu_LI\text{ for some }x\in A\}.$$

Then $\bigcup_{x\in A}G_x\subseteq\bigcup\mathcal{J}$. Let $\mathcal{J}_0$ be the family of maximal elements of $\mathcal{J}$; then $\bigcup\mathcal{J}_0=\bigcup\mathcal{J}$ and $\mathcal{J}_0$ is disjoint. So

$$\begin{aligned}\mu_L(\bigcup_{x\in A}G_x\setminus\bigcup_{x\in A}\delta(x))&\le\sum_{J\in\mathcal{J}_0}\mu_L(J_0\setminus\bigcup_{x\in A}\delta(x))\\&\le\sum_{J\in\mathcal{J}_0}2^{-n}\eta\mu_LJ\le2^{-n}\eta.\end{aligned}$$

On the other hand,

$$\mu_L(\bigcup_{x\in A}\delta'(x)\setminus\bigcup_{x\in A}G_x)=\mu_L(\bigcup_{x\in A}\delta'_n(x)\setminus\bigcup_{x\in A}G_x)\le2^{-n}\eta$$

by the choice of δ'_n; putting these together we have the result. **Q**

Let $\boldsymbol{t}\in T$ be δ'-fine and such that $\mu_LW_{\boldsymbol{t}}\ge1-\eta$. For each $n\in\mathbb{N}$, set $H_n=\bigcup_{(x,E)\in\boldsymbol{t},x\in A_n}\delta'(x)\setminus\bigcup_{(x,E)\in\boldsymbol{t}_n}\delta(x)$, so that $\mu_LH_n\le2^{-n+1}\eta$. Set $\alpha_{(x,E)}=\mu_L(E\setminus H_n)$ when $(x,E)\in\boldsymbol{t}$ and $x\in A_n$. Then for any $\boldsymbol{s}\subseteq\boldsymbol{t}$,

$$\begin{aligned}
\sum_{(x,E)\in\boldsymbol{s}} \alpha_{(x,E)} &= \sum_{n=0}^{\infty} \sum_{\substack{(x,E)\in\boldsymbol{s}\\ x\in A_n}} \mu_L(E\setminus H_n)\\
&= \mu_L(\bigcup_{n\in\mathbb{N}}(\bigcup_{\substack{(x,E)\in\boldsymbol{s}\\ x\in A_n}} E\setminus H_n))\\
&\le \mu_L(\bigcup_{n\in\mathbb{N}}(\bigcup_{\substack{(x,E)\in\boldsymbol{s}\\ x\in A_n}} \delta'(x)\setminus H_n))\\
&\le \mu_L(\bigcup_{n\in\mathbb{N}} \bigcup_{\substack{(x,E)\in\boldsymbol{s}\\ x\in A_n}} \delta(x)) = \mu_L(\bigcup_{(x,E)\in\boldsymbol{s}} \delta(x)).
\end{aligned}$$

Also

$$\begin{aligned}
\sum_{(x,E)\in\boldsymbol{t}} (\mu_L E - \alpha_{(x,E)}) &= \sum_{n=0}^{\infty} \sum_{\substack{(x,E)\in\boldsymbol{t}\\ x\in A_n}} \mu_L(E\cap H_n)\\
&\le \sum_{n=0}^{\infty} \mu_L H_n \le 4\eta,
\end{aligned}$$

so $\sum_{(x,E)\in\boldsymbol{t}} \alpha_{(x,E)} \ge 1-5\eta$. By the choice of δ and η, $\|w - \sum_{(x,E)\in\boldsymbol{t}} \alpha_{(x,E)} f(x)\| \le \epsilon$. On the other hand

$$\begin{aligned}
\|S_{\boldsymbol{t}}(f,\mu_L) - \sum_{(x,E)\in\boldsymbol{t}} \alpha_{(x,E)} f(x)\| &\le \sum_{n=0}^{\infty} \sum_{\substack{(x,E)\in\boldsymbol{t}\\ x\in A_n}} (n+1)(\mu_L E - \alpha_{(x,E)})\\
&\le \sum_{n=0}^{\infty} \sum_{\substack{(x,E)\in\boldsymbol{t}\\ x\in A_n}} (n+1)(\mu_L E - \alpha_{(x,E)})\\
&= \sum_{n=0}^{\infty} (n+1) \sum_{\substack{(x,E)\in\boldsymbol{t}\\ x\in A_n}} \mu_L(E\cap H_n)\\
&\le \sum_{n=0}^{\infty} (n+1)\mu_L H_n\\
&\le \sum_{n=0}^{\infty} 2^{-n+1}(n+1)\eta = 8\eta \le 8\epsilon.
\end{aligned}$$

So $\|w - S_{\boldsymbol{t}}(f,\mu_L)\| \le 9\epsilon$. As ϵ is arbitrary, $I_{\mathfrak{T}}(f) = w$; as f is arbitrary, $I_{\mathfrak{S}} = I_{\mathfrak{T}}$.

12 Remarks (a) It is easy to show that if $\langle G_x\rangle_{x\in[0,1]}$ is any family of relatively open subsets of $[0,1]$ such that $A\setminus\bigcup_{x\in A} G_x$ is negligible for every $A\subseteq[0,1]$, and $\mathfrak{S}$ is the topology on $[0,1]$ generated by the usual topology and $\{\{x\}\cup G_x : x\in[0,1]\}$, then Lebesgue measure is τ-additive for $\mathfrak{S}$, therefore quasi-Radon. So families of this kind can be taken as representative of neighbourhood gauges for topologies on $[0,1]$ which are compatible in this sense with Lebesgue measure.

(b) Suppose that $\langle G_x\rangle_{x\in[0,1]}$ is a family of relatively open subsets of $[0,1]$ such that $\limsup_{\eta\downarrow 0} \frac{1}{2\eta}\mu_L(G_x\cap[x-\eta,x+\eta]) > 0$ for every $x\in[0,1]$. Then $A' = A\setminus\bigcup_{x\in A} G_x$ is negligible for every $A\subseteq[0,1]$. **P?** If not, then there is an $x\in A'$ such that $\lim_{\eta\downarrow 0}\frac{1}{2\eta}\mu_L^*(A'\cap[x-\eta,x+\eta]) = 1$ (Fremlin 01, 261D). But as $A'\cap G_x = \emptyset$ this is impossible. **XQ** (Compare Fremlin 03, 453Xk.)

Even for families of this kind I have not been able to determine whether (†) is true; but see §16 below.

(c) If X is any topological group with a Haar measure μ, $G\subseteq X$ is an open set such that the identity e belongs to the closure of G, and $A\subseteq X$ is not negligible, then $A\cap AG\neq\emptyset$. **P** Setting $F=\overline{A}$, $F^{-1}F$

includes a neighbourhood of e, so meets G; as $A^{-1}A$ is dense in $F^{-1}F$, it meets G. **Q** So if we set $G_x = xG$ for every $x \in X$, then $A \setminus \bigcup_{x\in A} G_x$ will be negligible for every $A \subseteq X$. **P** Set $A' = A \setminus \bigcup_{x\in A} G_x = A \setminus AG$. Then $A' \cap A'G$ is empty, so A' is negligible. **Q**

13 For families as described in 12c, I can show that (†) is true, as follows.

Proposition Let $G \subseteq \mathbb{R}$ be an open set such that $0 \in \overline{G}$, and set $G_x = x + G$ for $x \in [0,1]$. Then for any $\epsilon > 0$ there is a $\delta > 0$ such that $\mu(\bigcup_{x\in A}]x-\delta, x+\delta[\setminus \bigcup_{x\in A} G_x) \le \epsilon$ for every $A \subseteq [0,1]$, where μ is Lebesgue measure.

proof ? Otherwise, then for each $n \in \mathbb{N}$ we have a set $A_n \subseteq [0,1]$ such that

$$E_n = \bigcup_{x\in A_n}]x-2^{-n}, x+2^{-n}[\setminus \bigcup_{x\in A_n} G_x$$

has measure at least ϵ. Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$, and set $F = \bigcap_{I\in\mathcal{F}} \overline{\bigcup_{n\in I} E_n}$; then F is the intersection of a downwards-directed family of closed sets of measure at least ϵ, so $\mu F \ge \epsilon$. It follows that $F - F$ includes a neighbourhood of 0 and meets G; say x, $y \in F$ are such that $x - y \in G$. Let $n \in \mathbb{N}$ be so large that $x' - y' \in G$ whenever $|x - x'| \le 2^{-n+1}$ and $|y - y'| \le 2^{-n+1}$. The set $\{i : i \in \mathbb{N}, \rho(x, E_i) \ge 2^{-n}\}$ cannot belong to $\mathcal{F}$, so its complement belongs to $\mathcal{F}$; similarly, $\{i : \rho(y, E_i) < 2^{-n}\} \in \mathcal{F}$ and there is an $i \ge n$ such that $\rho(x, E_i)$ and $\rho(y, E_i)$ are both less than 2^{-n}. Let x', $y' \in E_i$ be such that $|x-x'|$ and $|y-y'|$ are both less than 2^{-n}. Let $z \in A_i$ be such that $|y' - z| < 2^{-n}$. Then $x' - z \in G$ so $x' \in G_z$ and $x' \notin E_i$, which is impossible. **X**

14 D.Preiss suggested the following variation on 12b-12c. As I do not have a reference to hand I give a proof of a standard fact.

Proposition Let $r \ge 1$ be an integer, and $\langle z_n\rangle_{n\in\mathbb{N}}$, $\langle \delta_n\rangle_{n\in\mathbb{N}}$ sequences in $\mathbb{R}^r$ and $]0,\infty[$ respectively such that $\lim_{n\to\infty} \delta_n = 0$ and $\gamma = \sup_{m,n\in\mathbb{N},m>n} \frac{\delta_m + \|z_m\|}{\delta_n}$ is finite. Let μ be Lebesgue measure on $\mathbb{R}^r$. Then, for any $A \subseteq \mathbb{R}^r$,

$$\lim_{n\to\infty} \frac{\mu^*(A\cap(x+B(z_n,\delta_n)))}{\mu B(z_n,\delta_n)} = 1$$

for almost every $x \in A$.

proof (a) Take $M \ge 0$ and $\epsilon \in]0,1[$ and set

$$A' = \{x : \|x\| \le M, \liminf_{n\to\infty} \frac{\mu^*(A\cap(x+B(z_n,\delta_n)))}{\mu B(z_n,\delta_n)} < 1 - \epsilon\}.$$

Let $G \supseteq A'$ be an open set such that $\mu G \le \mu^* A' + \epsilon^2$. Set

$$\mathcal{W}_n = \{W : W = x + B(z_n,\delta_n) \text{ for some } x \in A',\ W \subseteq G,\ \frac{\mu^*(A\cap W)}{\mu W} \le 1 - \epsilon\}$$

for each n. For $W \in \mathcal{W}_n$, let y_W be its centre and set $\phi_n(W) = B(y_W - z_n, 2\delta_n) \cup B(y_W, (\gamma+1)\delta_n)$; note that $\mu(\phi_n(W)) \le K\mu W$ where $K = 2^r + (\gamma+1)^r$. Choose x_k, n_k and W_k inductively as follows. Given $\langle W_i\rangle_{i<k}$, set

$$\mathcal{V}_k = \{W : W \in \bigcup_{n\in\mathbb{N}} \mathcal{W}_n,\ W \cap W_i = \emptyset \text{ for every } i < k\}.$$

If $\mathcal{V}_k$ is empty, stop; otherwise set $n_k = \min\{n : \mathcal{V}_k \cap \mathcal{W}_n \ne \emptyset\}$ and choose $x_k \in A'$ such that $W_k = x_k + B(z_{n_k}, \delta_{n_k}) \in \mathcal{V}_k$.

case 1 If the procedure stops with $\mathcal{V}_k = \emptyset$, then $A' \subseteq \bigcup_{i<k} W_i$. **P?** If $x \in A' \setminus \bigcup_{i<k} W_i$, then, because G is open and $\bigcup_{i<k} W_i$ is closed and $\lim_{n\to\infty} \delta_n + \|z_n\| \le \lim_{n\to\infty} \gamma\delta_{n-1} = 0$ and $\{n : \mu^*(A\cap(x+B(z_n,\delta_n))) \le (1-\epsilon)\mu B(z_n,\delta_n)\}$ is infinite, there is an n such that $x+B(z_n,\delta_n)$ belongs to $\mathcal{W}_n$ and is disjoint from $\bigcup_{i<k} W_i$. **XQ** So in this case

$$\mu^* A' \le \textstyle\sum_{i<k} \mu^*(A \cap W_i) \le (1-\epsilon)\sum_{i<k} \mu W_i \le (1-\epsilon)\mu G \le (1-\epsilon)(\epsilon^2 + \mu^* A')$$

and $\mu^* A' \le \epsilon$.

case 2 If the procedure continues indefinitely, then $A' \subseteq \bigcup_{i<k} W_i \cup \bigcup_{i\ge k} \phi_{n_i}(W_i)$ for every $k \in \mathbb{N}$. **P?** If $x \in A' \setminus (\bigcup_{i<k} W_i \cup \bigcup_{i\ge k} \phi_{n_i}(W_i))$, then, as in case 1 above, there is a least n such that $x + B(z_n,\delta_n)$ belongs

to $\mathcal{W}_n$ and is disjoint from $\bigcup_{i<k} W_i$. On the other hand, because $\langle W_i\rangle_{i\in\mathbb{N}}$ is disjoint, $\sum_{i=0}^{\infty}\mu W_i \le \mu G$ is finite, so $\lim_{i\to\infty} n_i = \infty$ (this is where we really need to know that every δ_i is strictly positive). In particular, $n_i > n$ for all large i, so there must be a first j such that $x + B(z_n, \delta_n) \notin \mathcal{V}_{j+1}$. In this case, $W_j \cap (x + B(z_n, \delta_n))$ is non-empty, and $j \ge k$; also $n_j \le n$, since $x + B(z_n, \delta_n) \in \mathcal{V}_j$.

Set $x' = y_{W_j} - z_{n_j}$, so that $B(x', 2\delta_{n_j}) \subseteq \phi_{n_j}(W_j)$ does not contain x. As $2\delta_{n_j} < \|x - x'\| \le \|z_n - z_{n_j}\| + \delta_n + \delta_{n_j}$, $n \ne n_j$ and $n > n_j$. But this means that $\|x - y_{W_j}\| \le \|z_n\| + \delta_n + \delta_{n_j} \le (\gamma+1)\delta_{n_j}$ and $x \in B(y_{W_j}, (\gamma+1)\delta_{n_j}) \subseteq \phi_{n_j}(W_j)$, which is supposed to be impossible. **XQ** So in this case we have

$$\begin{aligned}\mu^* A' &\le \sum_{i=0}^{k-1}\mu(W_i \cap A') + \sum_{i=k}^{\infty}\mu(\phi_{n_i}(W_i))\\ &\le (1-\epsilon)\sum_{i=0}^{k-1}\mu W_i + \sum_{i=k}^{\infty} K\mu W_i\\ &\le (1-\epsilon)(\epsilon^2 + \mu^* A') + K\sum_{i=k}^{\infty}\mu W_i\end{aligned}$$

for every k; as $\sum_{i=0}^{\infty}\mu W_i$ is finite, we again have $\mu^* A' \le (1-\epsilon)(\epsilon^2 + \mu^* A')$ and $\mu^* A' \le \epsilon$.

(b) Now this is supposed to be true for every $\epsilon > 0$, so we see that

$$\{x : \|x\| \le M,\ \liminf_{n\to\infty} \tfrac{\mu^*(A\cap(x+B(z_n,\delta_n)))}{\mu B(z_n,\delta_n)} < 1\}$$

must be negligible. As M is arbitrary, we have the result.

15 Example Let $r \ge 1$ be an integer, and $\langle z_n\rangle_{n\in\mathbb{N}}$, $\langle\delta_n\rangle_{n\in\mathbb{N}}$ sequences in $\mathbb{R}^r$ and $]0,\infty[$ respectively such that $\lim_{n\to\infty}\delta_n = 0$ and $\gamma = \sup_{m,n\in\mathbb{N}, m>n} \frac{\delta_m + \|z_m\|}{\delta_n}$ is finite. Let μ be Lebesgue measure on $\mathbb{R}^r$. For each $x \in \mathbb{R}^r$, let $G_x \subseteq \mathbb{R}^r$ be an open set such that

$$\limsup_{n\to\infty} \tfrac{\mu^*(G_x\cap(x+B(z_n,\delta_n)))}{\mu B(z_n,\delta_n)} > 0.$$

Then, for any $A \subseteq \mathbb{R}^r$, $A' = A \setminus \bigcup_{x\in A} G_x$ is negligible. **P** If $x \in A'$, then

$$\liminf_{n\to\infty} \tfrac{\mu^*(A'\cap(x+B(z_n,\delta_n)))}{\mu B(z_n,\delta_n)} \le 1 - \limsup_{n\to\infty} \tfrac{\mu^*(G_x\cap(x+B(z_n,\delta_n)))}{\mu B(z_n,\delta_n)} < 1;$$

by Prop. 14, A' must be negligible. **Q**

So, taking $r = 1$, we have another family of examples in which to look for a counter-example to (†).

16 M.Csörnyei and D.Preiss have shown that for adequately definable families of the type in §12b we have a positive result.

Proposition Let $r \ge 1$ be an integer, and $\langle G_x\rangle_{x\in\mathbb{R}^r}$ a family of open subsets of $\mathbb{R}^r$ such that (i) $\limsup_{\delta\downarrow 0} \frac{\mu(G_x\cap B(x,\delta))}{\mu B(x,\delta)} > 0$ for every $x \in \mathbb{R}^r$, where μ is Lebesgue measure on $\mathbb{R}^r$ (ii) $\{(x,y) : x \in \mathbb{R}^r,\ y \in G_x\}$ is measurable in $\mathbb{R}^r \times \mathbb{R}^r$ (for the product measure, which may be identified with Lebesgue measure on $\mathbb{R}^{2r}$). Then for any $\epsilon > 0$ there is a measurable function $\delta : \mathbb{R}^r \to]0,\infty[$ such that $\mu((\bigcup_{x\in A} U(x,\delta)) \setminus (\bigcup_{x\in A} G_x)) \le \epsilon$ for every $A \subseteq \mathbb{R}^r$.

Remark I write $B(x,\delta) = \{y : \|y - x\| \le \delta\}$, where $\|\,\|$ is the Euclidean norm on $\mathbb{R}^r$, and $U(x,\delta)$ for $\{y : \|y - x\| < \delta\}$. For $A \subseteq \mathbb{R}^r$ and $\delta > 0$, I will write $U(A,\delta) = \bigcup_{x\in A} U(x,\delta)$.

proof (a) Note first that for any measurable set $E \subseteq \mathbb{R}^r$ of finite measure, the function $x \mapsto \mu(G_x \cap E)$ is measurable, by Fubini's theorem; hence, or otherwise, the function $x \mapsto \mu(G_x \cap B(x,\delta))$ is measurable for every δ, so that if we write $q(x)$ for the upper density $\limsup_{\delta\downarrow 0} \frac{\mu(G_x\cap B(x,\delta))}{\mu B(x,\delta)}$ of G_x at x, $q : \mathbb{R}^r \to [0,1]$ is measurable. For $\alpha > 0$, set $\mathcal{U}_\alpha = \{U(y,\alpha) : y \in \mathbb{Q}^r\}$. For $x \in \mathbb{R}^r$, $\alpha > 0$ set $\tilde G_{x\alpha} = \bigcup\{U : U \in \mathcal{U}_\alpha,\ \mu(U \setminus G_x) = 0\}$. Note that $\tilde G_{x\alpha} \subseteq \tilde G_{x\beta}$ whenever $0 < \beta \le \alpha$, and that $\{(x,y) : x \in \mathbb{R}^r,\ y \in \tilde G_{x\alpha}\}$ is measurable. For $x \in \mathbb{R}^r$ set $\tilde{G}_x = \bigcup_{\alpha>0} \tilde{G}_{x\alpha}$; then $\tilde{G}_x \supseteq G_x$ and $\tilde{G}_x \setminus G_x$ is negligible. Let $\langle E_n \rangle_{n\in\mathbb{N}}$ be a partition of $\mathbb{R}^r$ into non-empty bounded measurable sets of finite measure such that $\gamma_n = \frac{1}{2}\inf\{q(x) : x \in E_n\}$ is non-zero for each n; for each $n \in \mathbb{N}$ set $\tilde{\gamma}_n = \mu U(E_n, 1)$. Let $\langle \epsilon_n \rangle_{n\in\mathbb{N}}$ be a sequence of strictly positive numbers with sum at most $\frac{1}{3}\epsilon$.

(b) Readers unfamiliar with standard maximal inequalities (like myself) may find the following reminders helpful.

(i) If $A \subseteq \mathbb{R}^r$ is bounded and α, $\eta > 0$, then $\mu(U(A,(1+\eta)\alpha) \setminus U(A,\alpha)) \le 5^r((1+\eta)^r - 1)\mu(A,\alpha)$. **P** Set $F = U(A,(1+\eta)\alpha) \setminus U(A,\alpha)$ and $\mathcal{I} = \{B(x,(1+\eta)\alpha) : x \in F\}$. By Besicovitch's Covering Lemma (Fremlin 03, 472B), there is a family $\langle \mathcal{I}_k \rangle_{k<5^r}$ of countable subsets of $\mathcal{I}$ such that each $\mathcal{I}_k$ is disjoint and $\bigcup_{k<5^r} \mathcal{I}_k$ covers F. Now, for each $k < 5^r$, let $F_k \subseteq F$ be the set of centres of members of $\mathcal{I}_k$. Then F_k is finite (because A is bounded) and

$$\begin{aligned}\mu(F \cap \bigcup \mathcal{I}_k) &\le \mu(\bigcup_{x\in F_k} U(x,(1+\eta)\alpha) \setminus U(x,\alpha)) \\ &= ((1+\eta)^r - 1)\alpha^r \#(F_k)\mu B(\mathbf{0},1) \le ((1+\eta)^r - 1)\mu(A,\alpha).\end{aligned}$$

So $\mu F \le 5^r((1+\eta)^r - 1)\mu(A,\alpha)$. **Q**

(ii) If $E \subseteq \mathbb{R}^r$ is a bounded measurable set and $\alpha > 0$, then $F = \{x : \sup_{\delta>0} \frac{\mu(E\cap B(x,\delta))}{\mu B(x,\delta)} > \alpha\}$ has measure at most $\frac{5^r}{\alpha}\mu E$. **P** Apply Besicovitch's lemma to the family $\mathcal{I}$ of non-trivial closed balls B such that $\mu(B \cap E) \ge \alpha\mu B$; cf. Fremlin 03, 472B. **Q**

(c) For the time being, fix $n \in \mathbb{N}$. Let $m \ge 1$ be such that $10^{-r}\gamma_n\epsilon_n m \ge \tilde{\gamma}_n$. Let $\langle \epsilon'_k \rangle_{k<m}$ be a non-increasing sequence in $]0,1]$ such that $5^r\tilde{\gamma}_n \sum_{k=0}^{m-1}((1+2\epsilon'_k)^r - 1) \le \epsilon_n$.

For rational η, $\alpha > 0$ set $W(\eta,\alpha) = \{x : \mu(\tilde{G}_{x\alpha} \cap B(x,\eta)) \ge \gamma_n \mu B(x,\eta)\}$. For every $x \in E_n$ and $\alpha' > 0$ we know that there is a rational $\eta \in]0,\alpha']$ such that $\mu(G_x \cap B(x,\eta)) > \gamma_n\mu B(x,\eta)$, so that $x \in W(\eta,\alpha)$ for all sufficiently small α.

Let $m \ge 1$ be such that $10^r\gamma_n\epsilon_n m \ge \tilde{\gamma}_n$. Choose strictly positive rational numbers $\alpha_0, \dots, \alpha_m$ and measurable sets $F_0, \dots, F_m$ as follows. $\alpha_0 = 1$, $F_0 = E_n$. Given that $\alpha_k > 0$, $F_k \subseteq E_n$ and $\mu(E_n \setminus F_k) < \epsilon_n$, where $k < m$, take $\alpha_{k+1} \in]0, \epsilon'_k\alpha_k]$ such that $\mu(E_n \setminus F_{k+1}) < \epsilon_n$, where

$$F_{k+1} = F_k \cap \bigcup\{W(\eta,\alpha_{k+1}) : \eta \in [\alpha_{k+1}, \epsilon'_k\alpha_k] \text{ is rational}\}.$$

Now, for any $A \subseteq F_m$,

$$\mu(U(A,\alpha_m) \setminus \textstyle\bigcup_{x\in A} \tilde{G}_x) \le 2\epsilon_n.$$

P For $k < m$ set $V_k = \bigcup_{x\in A} \tilde{G}_{x,\alpha_k}$. Observe that V_k is of the form $U(C,\alpha_k)$ for some C. So (b-i) tells us that

$$\begin{aligned}\mu(U(V_k, 2\epsilon'_k\alpha_k) \setminus V_k) &= \mu(U(C,(1+2\epsilon'_k)\alpha_k) \setminus U(C,\alpha_k)) \\ &\le 5^r((1+2\epsilon'_k)^r - 1)\mu U(C,\alpha_k) \le 5^r\tilde{\gamma}_n((1+2\epsilon'_k)^r - 1).\end{aligned}$$

Next, there must be a j such that $1 \le j \le m$ and $\mu V'_j \le 10^{-r}\gamma_n\epsilon_n$, where $V'_j = V_j \setminus \bigcup_{k<j} V_k$. If $y \in U(A,\alpha_m) \setminus \bigcup_{k<m} U(V_k, 2\epsilon'_k\alpha_k)$, let $x \in A$ be such that $y \in U(x,\alpha_m)$. As $\alpha_m \le \epsilon'_k\alpha_k$, $x \notin U(V_k, \epsilon'_k\alpha_k)$ for every $k < m$; at the same time, $x \in F_j$, so there must be an $\eta \in [\alpha_j, \epsilon'_{j-1}\alpha_{j-1}]$ such that $x \in W(\eta,\alpha_j)$, that is, $\mu(\tilde{G}_{x,\alpha_j} \cap B(x,\eta)) \ge \gamma_n\mu B(x,\eta)$. For $k < j$, $x \notin U(V_k, \epsilon'_k\alpha_k)$ and $\eta \le \epsilon'_k\alpha_k$, so $B(x,\eta)$ does not meet V_k; accordingly $\mu(B(x,\eta) \cap V'_j) \ge \gamma_n\mu B(x,\eta)$. As $\eta \ge \alpha_m$, $\mu(B(y,2\eta) \cap V'_j) \ge 2^{-r}\gamma_n\mu B(y,2\eta)$ But from (b-ii) we know that $\{y : \mu(B(y,\zeta) \cap V'_j) \ge 2^{-r}\gamma_n\mu B(y,\zeta)$ for some $\zeta > 0\}$ has measure at most $\frac{10^r}{\gamma_n}\mu V'_j \le \epsilon_n$. So $\mu(U(A,\alpha_m) \setminus \bigcup_{k\le m} U(V_k, \epsilon'_k\alpha_k)) \le \epsilon_n$.

Putting these together,

$$\mu(U(A,\alpha_m)\setminus\bigcup_{x\in A}\tilde G_x)\le\sum_{k=0}^{m-1}\mu(U(V_k,2\epsilon'_k\alpha_k)\setminus V_k)+\mu(U(A,\alpha_m)\setminus\bigcup_{k<m}U(V_k,\epsilon'_k\alpha_k))$$
$$\le 5^r\tilde\gamma_n\sum_{k=0}^{m-1}((1+2\epsilon'_k)^r-1)+\epsilon_n\le 2\epsilon_n.\ \mathbf{Q}$$

Note that in fact

$$\begin{aligned}\mu(U(A,\alpha_m)\setminus\bigcup_{x\in A}G_x)&=\sup_{A'\subseteq A\text{ is finite}}\mu(U(A',\alpha_m)\setminus\bigcup_{x\in A}G_x)\\&\le\sup_{A'\subseteq A\text{ is finite}}\mu(U(A',\alpha_m)\setminus\bigcup_{x\in A'}G_x)\\&=\sup_{A'\subseteq A\text{ is finite}}\mu(U(A',\alpha_m)\setminus\bigcup_{x\in A'}\tilde G_x)\le 2\epsilon_n.\end{aligned}$$

We also know that $\mu(E_n\setminus F_m)<\epsilon_n$; let H be an open set of measure at most ϵ_n including $E_n\setminus F_m$, and define $\delta:E_n\to\,]0,\infty[$ by setting

$$\begin{aligned}\delta(x)&=\alpha_m\text{ if }x\in F_m,\\&=\rho(x,\mathbb{R}^r\setminus H)\text{ if }x\in E_n\setminus F_m.\end{aligned}$$

Then if $A\subseteq E_n$,

$$\textstyle\bigcup_{x\in A}U(x,\delta(x))\setminus\bigcup_{x\in A}G_x\subseteq H\cup(\bigcup_{x\in A\cap F_m}U(x,\alpha_m)\setminus\bigcup_{x\in A}G_x)$$

has measure at most $3\epsilon_n$.

(d) Putting these together as n varies, we have a measurable function $\delta:\mathbb{R}^r\to\,]0,\infty[$ such that, for any $A\subseteq\mathbb{R}^r$,

$$\textstyle\bigcup_{x\in A}U(x,\delta(x))\setminus\bigcup_{x\in A}G_x\subseteq\bigcup_{n\in\mathbb{N}}\bigl(\bigcup_{x\in A\cap E_n}U(x,\delta(x))\setminus\bigcup_{x\in A\cap E_n}G_x\bigr)$$

has measure at most $3\sum_{n=0}^{\infty}\epsilon_n\le\epsilon$. As ϵ is arbitrary, (†) is true for the given family $\langle G_x\rangle_{x\in\mathbb{R}^r}$.

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