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Half Cohen reals

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The problem Is there a partially ordered set $\mathbb{P}$ such that

(a) there is a $\mathbb{P}$-name $\ulcorner h \urcorner$ for a function from $\mathbb{N}$ to $\mathbb{N}$ such that for every $f \in \mathbb{N}^{\mathbb{N}}$ we have

$$\Vdash_{\mathbb{P}} \exists\, n \in \mathbb{N},\ \ulcorner h \urcorner(n) = f(n),$$

(b) for every $\mathbb{P}$-name $\ulcorner g \urcorner$ for a function from $\mathbb{N}$ to $\mathbb{N}$ there is a sequence $\langle \sigma_n \rangle_{n\in\mathbb{N}}$ in $S^* = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ such that every member of S^* is an initial segment of some σ_n and

$$\Vdash_{\mathbb{P}} \forall\, n \in \mathbb{N},\ \sigma_n \not\subseteq \ulcorner g \urcorner?$$

Definition Let $\mathcal{T}$ be the set of all $T \subseteq S^*$ such that

$\sigma \in T \Rightarrow \sigma{\restriction}k \in T$ for every $k \le \#(\sigma)$,

for every $\sigma \in T$ and $f \in \mathbb{N}^{\mathbb{N}}$ there is a $\tau \in T$ such that $\sigma \subseteq \tau$ and $\tau \cap f \ne \emptyset$.

Give $\mathcal{T}$ the ordering $\subseteq$; we think of $\mathcal{T}$ as active downwards. Note that if $T \in \mathcal{T}$ then (i) for every $n \in \mathbb{N}$ there is a $\sigma \in T \cap \mathbb{N}^n$ (ii) for every $\sigma \in T$, $T_\sigma = \{\tau : \tau \in T$, either $\tau \subseteq \sigma$ or $\sigma \subseteq \tau\}$ belongs to $\mathcal{T}$. We therefore have a $\mathcal{T}$-name $\ulcorner h \urcorner$ for a function from $\mathbb{N}$ to itself defined by saying

$$\text{if } \sigma(n) = i \text{ whenever } \sigma \in T \text{ and } \#(\sigma) > n \text{ then } T \Vdash_{\mathcal{T}} \ulcorner h \urcorner(n) = i;$$

and now for every $f \in \mathbb{N}^{\mathbb{N}}$ we have

$$\Vdash_{\mathbb{P}} \exists\, n \in \mathbb{N},\ \ulcorner h \urcorner(n) = f(n).$$

Thus $\mathcal{T}$ satisfies (a) of the problem.

Lemma If $\sigma \in T \in \mathcal{T}$ there is an $n \in \mathbb{N}$ such that for every $i \in \mathbb{N}$ there is a $\tau \in T$, extending σ, with $\tau(n) = i$.

proof ? Otherwise, for each $n \in \mathbb{N}$, we can choose $f(n)$ such that $(n, f(n)) \notin \tau$ whenever $\sigma \subseteq \tau \in T$. **X**

Proposition There are a $T^* \in \mathcal{T}$ and a $\mathcal{T}$-name $\ulcorner g \urcorner$ for a function from $\mathbb{N}$ to $\mathbb{N}$ such that whenever $\langle \sigma_n \rangle_{n\in\mathbb{N}}$ is a sequence in S^* such that every member of S^* is an initial segment of some σ_n then

$$T^* \Vdash_{\mathbb{P}} \exists\, n \in \mathbb{N},\ \sigma_n \subseteq \ulcorner g \urcorner.$$

proof (a) Let $q : S^* \to \mathbb{N}$ be a bijection such that $q(\sigma) \ge \#(\sigma)$ for every $\sigma \in S^*$. We must have $q(\emptyset) = 0$. For $\sigma \in S^*$, $i \in \mathbb{N}$ define $\tau_{\sigma i} \in \mathbb{N}^{q(\sigma)+1}$ by saying

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$$\begin{aligned}\tau_{\sigma i}(n) &= \sigma(n) \text{ if } n < \#(\sigma),\\ &= 0 \text{ if } \#(\sigma) \le n < q(\sigma),\\ &= i \text{ if } n = q(\sigma).\end{aligned}$$

Let $R \subseteq S^*$ be the smallest set such that $\emptyset \in R$ and $\tau_{\sigma i} \in R$ whenever $\sigma \in R$ and $i \in \mathbb{N}$, and set $T^* = \{\sigma\restriction k : \sigma \in R,\ k \le \#(\sigma)\}$. Then $T^* \in \mathcal{T}$.

For $\sigma \in R$ define $\upsilon(\sigma) \in S^*$ by saying that

$\upsilon(\emptyset) = \emptyset$,

for every $\sigma \in R$ and $i \in \mathbb{N}$, $\upsilon(\tau_{\sigma i}) = \upsilon(\sigma)^\frown q^{-1}(i)$.

Let $\ulcorner g \urcorner$ be a $\mathcal{T}$-name for a function from $\mathbb{N}$ to itself such that

$$T^*_\sigma \Vdash_{\mathcal{T}} \ulcorner g \urcorner \supseteq \upsilon(\sigma)$$

for every $\sigma \in R$.

(b) The point is that if $T \in \mathcal{T}$ and $\sigma \in T \subseteq T^*$ then there is a $\sigma' \in R$ such that $\sigma \subseteq \sigma'$ and $\tau_{\sigma' i} \in T$ for every $i \ge 1$. **P** There must be an $n \ge \#(\sigma)$ such that for every $i \in \mathbb{N}$ there is a $\tau_i \in T$ such that $\sigma \subseteq \tau_i$ and $\tau_i(n) = 1$. Now there are $\tau'_i \in R$ such that $\tau'_i \supseteq \tau_i$, so $\tau'_i(n) = 1$. But this means that (at least for $i \ne 0$) there is a $\sigma'_i \in R$ such that $n = q(\sigma'_i)$ and $\tau'_i \supseteq \tau_{\sigma'_i, i}$. Because q is bijective, all the σ'_i, for $i \ge 1$, are the same; call the common value σ'; we now have $\tau_{\sigma' i} \subseteq \tau_i \in T$ for every $i \ge 1$. **Q**

(c) Now suppose that $\langle \sigma_n \rangle_{n\in\mathbb{N}}$ is a sequence in S^* such that $S^* = \{\sigma_n\restriction k : n \in \mathbb{N}, k \le \#(\sigma_n)\}$. **?** If

$$T^* \not\Vdash_{\mathbb{P}} \exists\, n \in \mathbb{N},\ \sigma_n \subseteq \ulcorner g \urcorner,$$

then for some $T \in \mathcal{T}$ such that $T \subseteq T^*$ we have

$$T \Vdash_{\mathbb{P}} \forall\, n \in \mathbb{N},\ \sigma_n \not\subseteq \ulcorner g \urcorner.$$

Applying (b) with $\sigma = \emptyset$, we see that there is a $\sigma' \in R$ such that $\tau_{\sigma' i} \in T$ for every $i \ge 1$. Let $n \in \mathbb{N}$ be such that σ_n properly extends $\upsilon(\sigma')$. Let $i \in \mathbb{N}$ be such that $\sigma_n = \upsilon(\sigma')^\frown q(i)$; because $q(\emptyset) = 0$, $i \ge 1$, so $\tau_{\sigma' i} \in T$. But in this case $T_{\tau_{\sigma' i}} \subseteq T$ and

$$T_{\tau_{\sigma' i}} \Vdash_{\mathbb{P}} \sigma_n = \upsilon(\tau_{\sigma' i}) \subseteq \ulcorner g \urcorner,$$

which is supposed to be impossible. **X**

So T^* and $\ulcorner g \urcorner$ have the required properties.

Definition Let $\mathcal{T}^* \subseteq \mathcal{T}$ be the set of those $T \in \mathcal{T}$ such that

for every $\sigma \in T$ there is an $n \in \mathbb{N}$ such that whenever $m \ge n$ and $i \in \mathbb{N}$ there is a $\tau \in T$ such that $\tau \supseteq \sigma$ and $\tau(m) = i$.

Note that if $T \in \mathcal{T}^*$ then $T_\sigma \in \mathcal{T}^*$ for every $\sigma \in T$.

Proposition Suppose that $\langle \mathcal{Q}_k \rangle_{k\in\mathbb{N}}$ is a sequence of down-open coinitial subsets of $\mathcal{T}^*$. Then for every $T \in \mathcal{T}^*$ there are a $T' \in \mathcal{T}^*$ and a sequence $\langle n_k \rangle_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $T' \subseteq T$ and $T'_\sigma \in \mathcal{Q}_k$ whenever $\sigma \in T'$ and $\#(\sigma) = n_{k+1}$.

proof (a) Fix a well-ordering $\preccurlyeq$ of S^* of order type ω. Choose $\langle n_k \rangle_{k\in\mathbb{N}}$, $\langle T^{(k)} \rangle_{k\in\mathbb{N}}$, $\langle S_k \rangle_{k\in\mathbb{N}}$, $\langle m_\sigma \rangle_{\sigma\in S_k}$ inductively, as follows. The inductive hypothesis will be

$T^{(k)} \in \mathcal{T}^*$,
$n_k \in \mathbb{N}$,
$S_k \subseteq T^{(k)}$ is finite,
$m_\sigma \le n_k$ for every $\sigma \in S_k$,
if $\sigma \in S_k$, $m \ge m_\sigma$, $i \in \mathbb{N}$ then there is a $\tau \in T^{(k)}$ such that $\sigma \subseteq \tau$ and $\tau(m) = i$,

and the induction will proceed in such a way that

$n_{k+1} > n_k$,
$T^{(k+1)} \subseteq T^{(k)}$,
$\sigma \in T^{(k+1)}$ whenever $\sigma \in T^{(k)}$ and $\#(\sigma) \le n_{k+1}$,
$T^{(k+1)}_\sigma \in \mathcal{Q}_k$ whenever $\sigma \in T^{(k+1)}$ and $\#(\sigma) = n_{k+1}$,
$S_{k+1} \supseteq S_k$.

The induction starts with $n_0 = 0$, $T^{(0)} = T$, $S_0 = \emptyset$.

(b) *Inductive step* First add the $\preccurlyeq$-first member of $T^{(k)} \setminus S_k$ to S_k to form S_{k+1}. Let $\tilde{S}_{k+1}$ be the set of maximal elements of S_{k+1}, and for each $\sigma \in \tilde{S}_{k+1}$ choose $U^{k,\sigma} \in \mathcal{Q}_k$ such that $U^{k,\sigma} \subseteq T^{(k)}_\sigma$. Let $n_{k+1} > n_k$ be so large that for every $\sigma \in \tilde{S}_{k+1}$, $m \ge n_{k+1}$ and $i \in \mathbb{N}$ there is a $\tau \in U^{k,\sigma}$ such that $\tau(m) = i$. For each $\tau \in T^{(k)}$ such that $\#(\tau) = n_{k+1}$, choose $V^{k,\tau} \in \mathcal{Q}_k$ such that $V^{k,\tau} \subseteq T^{(k)}_\tau$; do this in such a way that if $\sigma \in \tilde{S}_{k+1}$ and $\tau \in U^{k,\sigma}$ then $V^{k,\tau} = U^{k,\sigma}_\tau$. (There is no problem with this because the different members of $\tilde{S}_{k+1}$ are incompatible, so no τ of length n_{k+1} can belong to more than on $U^{k,\sigma}$.) Set $T^{(k+1)} = \bigcup\{V^{k,\tau} : \tau \in T^{(k)}, \#(\tau) = n_{k+1}\}$. If σ is the member of $S_{k+1} \setminus S_k$ set $m_\sigma = n_{k+1}$.

If $\sigma \in T^{(k+1)}$ there is a $\tau \in T^{(k)}$ such that $\#(\tau) = n_{k+1}$ and $\sigma \in V^{(k,\tau)} \in \mathcal{T}^*$. Now there is an $n \in \mathbb{N}$ such that for every $m \ge n$, $i \in \mathbb{N}$ there is a $\tau' \in V^{(k,\tau)}$, extending σ, with $\tau'(m) = i$; in which case $\sigma \subseteq \tau' \in T^{(k+1)}$. As σ is arbitrary, $T^{(k+1)} \in \mathcal{T}^*$. Of course $T^{(k+1)}$ includes $\{\sigma : \sigma \in T^{(k)}, \#(\sigma) = n_{k+1}\}$ and therefore includes $\{\sigma : \sigma \in T^{(k)}, \#(\sigma) \le n_{k+1}\}$.

If $\sigma \in S_{k+1}$, $m_\sigma \le m$ and $i \in \mathbb{N}$, then if $m < n_{k+1}$ we have $\sigma \in S_k$ so there is a $\tau \in T^{(k)}$ such that $\sigma \subseteq \tau$ and $\tau(m) = i$. We may suppose that $\#(\tau) = n_{k+1}$ and now $\tau \in V^{k,\tau} \subseteq T^{(k+1)}$. If $m \ge n_{k+1}$ then there is a $\tilde{\sigma} \in \tilde{S}_{k+1}$ such that $\sigma \subseteq \tilde{\sigma}$ and there is a $\tau \in U^{k,\tilde{\sigma}}$ such that $\tau(m) = i$; now

$$\tau \in U^{k,\tilde{\sigma}}_{\tau'} = V^{k,\tau'} \subseteq T^{(k+1)}$$

where $\tau' = \tau \restriction n_{k+1}$.

If $\sigma \in T^{(k+1)}$ and $\#(\sigma) = n_{k+1}$ then $T^{(k+1)}_\sigma = V^{k,\sigma} \in \mathcal{Q}_k$.

(c) At the end of the induction set $T' = \bigcap_{k \in \mathbb{N}} T^{(k)}$. If $\sigma \in T'$, then there is some $k \in \mathbb{N}$ such that $\sigma \in S_k$. If now $m \ge m_\sigma$ and $i \in \mathbb{N}$, let $l \ge k$ be such that $m < n_l$. As $\sigma \in S_l$, there is a $\tau \in T^{(l)}$ such that $\sigma \subseteq \tau$ and $\tau(m) = i$. We may suppose that $\#(\tau) \le n_l$, in which case $\tau \in T'$. This shows that $T' \in \mathcal{T}^*$.

If $k \in \mathbb{N}$, $\sigma \in T'$ and $\#(\sigma) = n_{k+1}$, then $\sigma \in T^{(k+1)}$ and $T'_\sigma \subseteq T^{(k+1)}_\sigma \in \mathcal{Q}_k$ so $T'_\sigma \in \mathcal{Q}_k$.

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