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Problem CE

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1. The problem Is it consistent to suppose that there is a cardinal κ such that (i) $\mathcal{P}\kappa$ is not countably generated as σ-algebra (ii) whenever J is a set and $\mathcal{A} \subseteq \mathcal{P}J$ has cardinal κ, there is a countable $\mathcal{B} \subseteq \mathcal{P}J$ such that $\mathcal{A}$ is included in the σ-algebra of subsets of J generated by $\mathcal{B}$? The answer would fill a gap left by Theorem 12 of Fremlin Hansell & Junnila 83.

A.W.Miller and S.Shelah have pointed out that only cardinals κ with cofinality ω_1 are relevant. Here I set out a version of Shelah's argument (October 2002).

2. Lemma Let κ be any cardinal. Then the following are equiveridical:

(i) $\mathcal{P}\kappa$ is countably generated as σ-algebra;

(ii) there is a set $D \subseteq \mathcal{P}\mathbb{N}$, of cardinal κ, such that every subset of D is relatively Borel.

proof (i)⇒(ii) If $\mathcal{P}\kappa$ is σ-generated by $\langle A_n\rangle_{n\in\mathbb{N}}$, define $\phi : \kappa \to \mathcal{P}\mathbb{N}$ by setting $\phi(\xi) = \{n : n \in \mathbb{N},\ \xi \in A_n\}$ for every $\xi < \kappa$. Now $\{\phi^{-1}[E] : E \subseteq \mathcal{P}\mathbb{N}$ is Borel$\}$ is a σ-algebra of subsets of κ containing every A_n, and is therefore $\mathcal{P}\kappa$. This shows that ϕ is injective, so that $D = \phi[\kappa]$ has cardinal κ, and also that every subset of D is relatively Borel.

(ii)⇒(i) If (ii) is true, then $\mathcal{P}D$ is countably σ-generated, so $\mathcal{P}\kappa$ also is.

3. Lemma Let κ be any cardinal. Then the following are equiveridical:

(i) whenever J is a set and $\mathcal{A} \subseteq \mathcal{P}J$ has cardinal κ, there is a countable $\mathcal{B} \subseteq \mathcal{P}J$ such that $\mathcal{A}$ is included in the σ-algebra of subsets of J generated by $\mathcal{B}$;

(ii) setting $A_\xi = \{a : \xi \in a \subseteq \kappa\}$ for each $\xi < \kappa$, there is a countable $\mathcal{B} \subseteq \mathcal{P}(\mathcal{P}\kappa)$ such that every A_ξ belongs to the σ-algebra of subsets of $\mathcal{P}\kappa$ generated by $\mathcal{B}$;

(iii) there is a family $\langle E_\xi\rangle_{\xi<\kappa}$ of Borel subsets of $\mathcal{P}\mathbb{N}$ such that for every $a \subseteq \kappa$ there is an $x \in \mathcal{P}\mathbb{N}$ such that $a = \{\xi : \xi < \kappa,\ x \in E_\xi\}$.

proof (i)⇒(ii) is trivial.

(ii)⇒(iii) Let $\langle B_n\rangle_{n\in\mathbb{N}}$ run over $\mathcal{B} \cup \{\emptyset\}$. Define $\phi : \mathcal{P}\kappa \to \mathcal{P}\mathbb{N}$ by setting $\phi(a) = \{n : n \in \mathbb{N},\ a \in B_n\}$ for $a \subseteq \kappa$. Then $\{\phi^{-1}[E] : E \subseteq \mathcal{P}\mathbb{N}$ is Borel$\}$ contains every B_n so contains every A_ξ, and we can find for each $\xi < \kappa$ a Borel set $E_\xi \subseteq \mathcal{P}\mathbb{N}$ such that $\phi^{-1}[E_\xi] = A_\xi$. Turning this round, we now find that if $a \subseteq \kappa$ and $x = \phi(a) \in \mathcal{P}\mathbb{N}$, then $\{\xi : x \in E_\xi\} = \{\xi : a \in A_\xi\}$. So $\langle E_\xi\rangle_{\xi<\kappa}$ satisfies (ii).

(iii)⇒(i) Assume (iii). If J is a set and $\langle A_\xi\rangle_{\xi<\kappa}$ is a family of subsets of J, choose $\phi : J \to \mathcal{P}\mathbb{N}$ such that $\{\xi : \phi(j) \in E_\xi\} = \{\xi : j \in A_\xi\}$ for every $j \in J$. Set $B_n = \{j : n \in \phi(j)\}$ for each $n \in \mathbb{N}$. If Σ is the σ-algebra of subsets of J generated by $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$, then $\mathrm{T} = \{E : E \subseteq \mathcal{P}\mathbb{N},\ \phi^{-1}[E] \in \Sigma\}$ is a σ-algebra of subsets of $\mathcal{P}\mathbb{N}$ containing all the sets $\{x : n \in x \subseteq \mathbb{N}\}$, so contains every Borel set. In particular, $E_\xi \in \mathrm{T}$ and $A_\xi \in \Sigma$ for every $\xi < \kappa$. So the countably σ-generated algebra Σ contains every A_ξ.

4. Proposition If $\mathrm{cf}\,\kappa \neq \omega_1$ and κ satisfies the conditions of Lemma 3, it satisfies the conditions of Lemma 2.

proof The case of finite κ is trivial; suppose that κ is infinite. Let $\langle E_\xi \rangle_{\xi<\kappa}$ be a family of Borel subsets of $\mathcal{P}\mathbb{N}$ such that for every $a \subseteq \kappa$ there is an $x \subseteq \mathbb{N}$ such that $a = \{\xi : \xi < \kappa,$ $x \in E_\xi\}$. Let $\langle \mathbf{\Sigma}^0_\alpha \rangle_{\alpha<\omega_1}$ be the additive Borel classes of $\mathcal{P}\mathbb{N}$ (Kechris 95, 11.B). Because $\mathrm{cf}\,\kappa \neq \omega_1$, there is an $\alpha < \omega_1$ such that $C = \{\xi : \xi < \kappa,\ E_\xi \in \mathbf{\Sigma}^0_\alpha\}$ has cardinal κ. Next, there is a Borel set $W \subseteq \mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{N}$ such that $\mathbf{\Sigma}^0_\alpha \subseteq \{W[\{z\}] : z \in \mathcal{P}\mathbb{N}\}$ (Kechris 95, Theorem 22.3). For each $\xi \in C$, let $z_\xi \subseteq \mathbb{N}$ be such that $E_\xi = W[\{z_\xi\}]$. If $a \subseteq C$, there is an $x \subseteq \mathbb{N}$ such that $a = \{\xi : \xi < \kappa,\ x \in E_\xi\}$. Set $F = W^{-1}[\{x\}]$, so that $F \subseteq \mathcal{P}\mathbb{N}$ is a Borel set. Then

$$\{\xi : \xi \in C,\ z_\xi \in F\} = \{\xi : \xi \in C,\ (z_\xi, x) \in W\} = \{\xi : \xi \in C,\ x \in E_\xi\} = C \cap a = a.$$

But this means that $D = \{z_\xi : \xi \in C\}$ has cardinal $\#(C) = \kappa$ and every subset of D is relatively Borel, so D witnesses that condition (ii) of Lemma 2 is satisfied.

5. The problem revised Problem CE asked: is it consistent to suppose that there is a cardinal κ satisfying condition (i) of Lemma 3 but not satisfying condition (i) of Lemma 2. In view of §§2-4 here, we can rephrase this as

> is it consistent to suppose that there is a cardinal κ, of cofinality ω_1, such that there is a completely independent family $\langle E_\xi \rangle_{\xi<\kappa}$ of Borel subsets of $\mathcal{P}\mathbb{N}$, but there is no subset of $\mathcal{P}\mathbb{N}$, of cardinal κ, of which every subset is relatively Borel?

Of course any uncountable Polish space can be used in place of $\mathcal{P}\mathbb{N}$ at any point in the arguments above. I note that the results of Fremlin Hansell and Junnila 83 are already enough to show that any κ satisfying the conditions of Lemma 2 also satisfies the conditions of Lemma 3. Note also that any κ satisfying the conditions of Lemma 3 must have $2^\kappa \leq \mathfrak{c}$, and any κ not satisfying the conditions of Lemma 2 must have $\kappa \geq \mathfrak{p}$ (Fremlin 84, 23B); so we need consider only $\mathfrak{p} \leq \kappa < \mathfrak{c}$, and under Martin's axiom, for instance, we have finished.

6. The same ideas lead us to the following result.

Proposition For any cardinal κ, the following are equiveridical:

(i) there is a countable subset $\mathcal{B}$ of $\mathcal{P}\kappa$ such that $\mathcal{P}\kappa$ is the family $\mathcal{S}(\mathcal{B})$ of sets obtainable by Souslin's operation from $\mathcal{B}$;

(ii) there is a set $D \subseteq \mathcal{P}\mathbb{N}$, of cardinal κ, such that every subset of D is Souslin-F in D;

(iii) whenever J is a set and $\mathcal{A} \subseteq \mathcal{P}J$ has cardinal κ, there is a countable $\mathcal{B} \subseteq \mathcal{P}J$ such that $\mathcal{A} \subseteq \mathcal{S}(\mathcal{B})$;

(iv) setting $A_\xi = \{a : \xi \in a \subseteq \kappa\}$ for each $\xi < \kappa$, there is a countable $\mathcal{B} \subseteq \mathcal{P}(\mathcal{P}\kappa)$ such that every A_ξ belongs to $\mathcal{S}(\mathcal{B})$;

(v) there is a family $\langle E_\xi \rangle_{\xi<\kappa}$ of Souslin subsets of $\mathcal{P}\mathbb{N}$ such that for every $a \subseteq \kappa$ there is an $x \in \mathcal{P}\mathbb{N}$ such that $a = \{\xi : \xi < \kappa,\ x \in E_\xi\}$.

proof (i) $\iff$ (ii) As in Lemma 2. We need to know that if $\phi : X \to Y$ is a function and $\mathcal{B} \subseteq \mathcal{P}Y$, then $\mathcal{S}(\mathcal{B}) = \{E : E \subseteq Y,\ \phi^{-1}[E] \in \mathcal{S}(\mathcal{F})\}$, where $\mathcal{F} = \{\phi^{-1}[B] : B \in \mathcal{B}\}$ (Fremlin 03, 421C), and that if D is a separable metric space with at least two points then the Souslin-F sets are obtainable by Souslin's operation from any base for the topology.

(ii)⇒(v) Assume (ii). Let $W \subseteq \mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{N}$ be a Souslin set such that every Souslin subset of $\mathcal{P}\mathbb{N}$ is of the form $W[\{x\}]$ for some $x \subseteq \mathbb{N}$. Enumerate D as $\langle z_\xi \rangle_{\xi<\kappa}$ and set $E_\xi = W^{-1}[\{z_\xi\}]$ for each ξ. If $a \subseteq \kappa$, $D_a = \{z_\xi : \xi \in a\}$ is Souslin-F in D, so there is a Souslin set $A \subseteq \mathcal{P}\mathbb{N}$ such that $D_a = D \cap A$. Let $x \subseteq \mathbb{N}$ be such that $A = W[\{x\}]$; then $a = \{\xi : x \in E_\xi\}$.

(v)⇒(ii) Similarly, as in Proposition 4.

(v)⇒(iii)⇒(iv) As in Lemma 3. This time we need to know that if $\phi : X \to Y$ is a function and $\mathcal{F} \subseteq \mathcal{P}X$ is closed under Souslin's operation, so is $\{E : E \subseteq Y, \phi^{-1}[E] \in \mathcal{F}\}$.

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