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Problem CC

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Let X and Y be topological spaces, and $R \subseteq X \times Y$ a relation. A **selector** for R (Fremlin 87, 10A) is a function $f \subseteq R$ such that $\operatorname{dom} f = R^{-1}[Y]$. Assuming the axiom of choice, every relation has a selector; but we can look for conditions on R which will ensure that f has interesting topological properties. The most important questions concern metrizable X and Y, though we shall find that some of the results can easily be extended to more general cases.

1 σ-isolated refinements

1A Definitions I will use the following ideas.

(a) If $\mathcal{E}$ is a family of sets, then a family $\langle A_i \rangle_{i \in I}$ of sets is **$\mathcal{E}$-additive** if $\bigcup_{i \in J} A_i \in \mathcal{E}$ for every $J \subseteq I$.

(b) Let X be a topological space. A family $\langle A_i \rangle_{i \in I}$ of subsets of X is **isolated** if it is disjoint and every A_i is relatively open in $\bigcup_{i \in I} A_i$. $\langle A_i \rangle_{i \in I}$ is **σ-isolated** if I is expressible as $\bigcup_{n \in \mathbb{N}} I_n$ where $\langle A_i \rangle_{i \in I_n}$ is isolated for every $n \in \mathbb{N}$. $\langle A_i \rangle_{i \in I}$ is **σ-isolatedly refinable** if there is a family $\langle C_{in} \rangle_{i \in I, n \in \mathbb{N}}$ such that (α) $C_{in} \subseteq A_i$ whenever $i \in I$ and $n \in \mathbb{N}$ (β) $\langle C_{in} \rangle_{i \in I}$ is isolated for each $n \in \mathbb{N}$ (γ) $\bigcup_{i \in I, n \in \mathbb{N}} C_{in} = \bigcup_{i \in I} A_i$.

(c) For any topological space X, its **augmented Lindelöf number** $L^+(X)$ is the least cardinal κ such that whenever $\mathcal{G}$ is an open cover of X then there is a $\mathcal{G}' \in [\mathcal{G}]^{<\kappa}$ covering X.

(d) A topological space X is **hereditarily weakly θ-refinable** (see Fremlin 03, §438) if every family of open sets in X is σ-isolatedly refinable.

(e) Let X be a topological space. Following Holický & Spurný 03 I define the (additive) **Borel classes** $\mathcal{B}_\xi(X)$, for ordinals $\xi \geq 1$, by saying that

$\mathcal{B}_0(X)$ is the algebra generated by the open subsets of X,

if $\xi > 0$, then

$$\mathcal{B}_\xi(X) = \{\bigcup_{n \in \mathbb{N}} X \setminus E_n : E_n \in \bigcup_{\eta < \xi} \mathcal{B}_\eta(X) \text{ for every } n \in \mathbb{N}\}.$$

$\mathcal{B}(X) = \mathcal{B}_{\omega_1}(X) = \mathcal{B}_{\omega_1+1}(X) = \bigcup_{\xi < \omega_1} \mathcal{B}_\xi(X)$ is the Borel σ-algebra $\mathcal{B}(X)$ of X. Note that, for $\xi \geq 1$, $\mathcal{B}_\xi(X)$ is closed under countable unions and finite intersections, and that $\langle \mathcal{B}_\xi(X) \rangle_{\xi < \omega_1}$ is non-decreasing. (See also Fremlin n05.)

It will be convenient to have a corresponding notation for multiplicative Baire classes; set

$$\mathcal{B}'_\xi(X) = \{X \setminus E : E \in \mathcal{B}_\xi(X)\}$$

for each ξ. Observe that for $\xi \geq 1$ we shall have

$$\mathcal{B}_\xi(X) = \{\bigcup_{n \in \mathbb{N}} E_n : E_n \in \bigcup_{\eta < \xi} \mathcal{B}'_\eta(X) \text{ for every } n \in \mathbb{N}\},$$

$$\mathcal{B}'_\xi(X) = \{\bigcap_{n \in \mathbb{N}} E_n : E_n \in \bigcup_{\eta < \xi} \mathcal{B}_\eta(X) \text{ for every } n \in \mathbb{N}\}.$$

These do not exactly match the standard labelling of Borel classes in metrizable spaces. However, if X is metrizable, then every set in $\mathcal{B}_0(X)$ is both F_σ and G_δ, so $\mathcal{B}_1(X)$ is the family of all F_σ sets and $\mathcal{B}'_1(X)$ the family of all G_δ sets.

1B Lemma Let X be a topological space and $\langle A_i \rangle_{i \in I}$ a family of subsets of X. Let $\mathcal{I}$ be the family of those sets $E \subseteq X$ such that $\langle A_i \cap E \rangle_{i \in I}$ is σ-isolatedly refinable.

(a)(i) $\mathcal{I}$ is a σ-ideal of subsets of X.

(ii) If $\langle E_j \rangle_{j \in J}$ is a σ-isolated family in $\mathcal{I}$ then $\bigcup_{j \in J} E_j \in \mathcal{I}$.

(iii) In particular, every isolated subset of X belongs to $\mathcal{I}$.

(b) If X is hereditarily weakly θ-refinable, there is an open set $G \subseteq X$ such that $\langle A_i \cap G\rangle_{i\in I}$ is σ-isolatedly refinable but $\langle A_i \cap H \setminus G\rangle_{i\in I}$ is not σ-isolatedly refinable for any open set H not included in G.

proof (a)(i) If $\langle E_m\rangle_{m\in\mathbb{N}}$ is a sequence in $\mathcal{I}$ and $E \subseteq \bigcup_{m\in\mathbb{N}} E_m$, then for each $m \in \mathbb{N}$ let $\langle C_{inm}\rangle_{i\in I,n\in\mathbb{N}}$ be a witness that $\langle A_i \cap E_m\rangle_{i\in I}$ is σ-isolatedly refinable; then $\langle C_{inm} \cap E\rangle_{i\in I,n,m\in\mathbb{N}}$ witnesses that $\langle A_i \cap E\rangle_{i\in I}$ is σ-isolatedly refinable and $E \in \mathcal{I}$.

(ii) Because $\mathcal{I}$ is a σ-ideal, it is enough to consider an isolated family $\langle E_j\rangle_{j\in J}$ in $\mathcal{I}$ with union E. For each $j \in J$ let $\langle C_{jin}\rangle_{i\in I,n\in\mathbb{N}}$ witness that $\langle A_i \cap E_j\rangle_{i\in I}$ is σ-isolatedly refinable. Set $C_{in} = \bigcup_{j\in J} C_{jin}$ for $i \in I$, $n \in \mathbb{N}$. Then $C_{in} \subseteq A_i \cap E$ for all i and n, and $\bigcup_{i\in I,n\in\mathbb{N}} C_{in} = \bigcup_{i\in I} A_i \cap E$. If $n \in \mathbb{N}$, $i \in I$ and $x \in C_{in}$, let $j \in J$ be such that $x \in C_{jin}$. There is a neighbourhood U of x not meeting $E_{j'}$ for any $j' \in J \setminus \{j\}$, and not meeting $C_{ji'n}$ for any $i' \in I \setminus \{i\}$; so that $U \cap C_{i'n}$ is empty for every $i' \in I \setminus \{i\}$. Thus $\langle C_{in}\rangle_{i\in I}$ is isolated for each n, and $\langle C_{in}\rangle_{i\in I,n\in\mathbb{N}}$ witnesses that $\langle A_i \cap E\rangle_{i\in I}$ is σ-isolatedly refinable.

(iii) We have only to observe that singleton sets belong to $\mathcal{I}$ and use (ii).

(b) Let $\mathcal{H}$ be the family of open sets belonging to $\mathcal{I}$, and $G = \bigcup \mathcal{H}$. Because $\mathcal{H}$ (or, if you prefer, $\langle H\rangle_{H\in\mathcal{H}}$) is σ-isolatedly refinable, (a-ii) tells us that $G \in \mathcal{I}$. If $H \subseteq X$ is open and $H \not\subseteq G$, then $H \notin \mathcal{H}$ so $H \notin \mathcal{I}$ and $H \setminus G \notin \mathcal{I}$.

1C Lemma Suppose that X is a topological space and $1 \le \xi < \omega_1$.
(a) Let $\langle E_i\rangle_{i\in I}$ be a σ-isolatedly refinable family in $\mathcal{B}_\xi(X)$. Then $\bigcup_{i\in I} E_i \in \mathcal{B}_\xi(X)$.
(b) Let $\langle E_i\rangle_{i\in I}$ be an isolated family in $\mathcal{B}'_\xi(X)$. Then $\bigcup_{i\in I} E_i \in \mathcal{B}'_\xi(X)$.

proof (a)(i) Suppose first that $F \subseteq X$ is closed, $\langle G_i\rangle_{i\in I}$ is a family of open sets such that $\langle G_i \cap F\rangle_{i\in I}$ is disjoint, and $\langle E_i\rangle_{i\in I}$ is a family in $\mathcal{B}_\xi(X)$ such that $E_i \subseteq G_i \cap F$ for every i. Then $\bigcup_{i\in I} E_i \in \mathcal{B}_\xi(X)$. **P** Induce on ξ. The case $\xi = 1$ seems to require special attention. In this case, each E_i is expressible as $\bigcup_{n\in\mathbb{N}} G_{in} \cap E_{in}$ where every G_{in} is open and every E_{in} is closed; of course we may suppose that $G_{in} \subseteq G_i$ and $E_{in} \subseteq F$ for all i and n. Now, for each n, set $H_n = \bigcup_{i\in I} G_{in} \cap E_{in}$ and $E_n = \overline{H}_n \subseteq F$. If $i \in I$, then G_{in} is disjoint from $G_{jn} \cap E_{jn}$ for every $j \ne i$, so $G_{in} \cap E_n \subseteq E_{in}$. Accordingly $H_n = \bigcup_{i\in I} G_{in} \cap E_n \in \mathcal{B}_0(X)$ and $\bigcup_{i\in I} E_i = \bigcup_{n\in\mathbb{N}} H_n$ belongs to $\mathcal{B}_1(X)$.

For the inductive step to a countable ordinal $\xi > 1$, we can express each E_i as $\bigcup_{n\in\mathbb{N}} X \setminus E_{in}$ where $E_{in} \in \bigcup_{\eta<\xi} \mathcal{B}_\eta(X)$ for $n \in \mathbb{N}$. For $n \in \mathbb{N}$ and $1 \le \eta < \xi$, set $I_{n\eta} = \{i : i \in I,\ E_{in} \in \mathcal{B}_\eta(X)\}$. For $i \in I_{n\eta}$, set $E'_{in} = G_i \cap F \cap E_{in}$. Then the inductive hypothesis tells us that $H_{n\eta} = \bigcup_{i\in I_{n\eta}} E'_{in}$ belongs to $\mathcal{B}_\eta(X)$. So

$$\bigcup_{i\in I} E_i = \bigcup_{i\in I} G_i \cap F \cap E_i = \bigcup_{i\in I,n\in\mathbb{N}} G_i \cap F \setminus E_{in} = \bigcup_{i\in I,n\in\mathbb{N}} G_i \cap F \setminus E'_{in}$$
$$= \bigcup_{n\in\mathbb{N},\eta<\xi} \bigcup_{i\in I_{n\eta}} G_i \cap F \setminus E'_{in} = \bigcup_{n\in\mathbb{N},\eta<\xi} \bigcup_{i\in I_{n\eta}} G_i \cap F \setminus H_{n\eta}$$

(because $\langle G_i \cap F\rangle_{i\in I}$ is disjoint and $G_i \cap F \cap H_{n\eta} = G_i \cap F \cap E'_{in}$ if $i \in I_{n\eta}$)

$$= \bigcup_{n\in\mathbb{N},\eta<\xi} \Big(\bigcup_{i\in I_{n\eta}} G_i\Big) \cap F \setminus H_{n\eta}$$

is a countable union of members of $\mathcal{B}_\xi(X)$ and belongs to $\mathcal{B}_\xi(X)$. Thus the induction proceeds. **Q**

(ii) Now suppose that $\langle E_i\rangle_{i\in I}$ is a σ-isolatedly refinable family in $\mathcal{B}_\xi(X)$, where $1 \le \xi < \omega_1$. Let $\langle C_{in}\rangle_{i\in I,n\in\mathbb{N}}$ witness that $\langle E_i\rangle_{i\in I}$ is σ-isolately refinable. For $i \in I$, $n \in \mathbb{N}$ set $G_{in} = X \setminus \overline{\bigcup_{j\in I,j\ne i} C_{jn}}$; for $n \in \mathbb{N}$ set $G_n = \bigcup_{i,j\in I,i\ne j} G_{in} \cap G_{jn}$; now set $E_{in} = E_i \cap G_{in} \setminus G_n$. Then $E_{in} \in \mathcal{B}_\xi(X)$ for every i and n because $\mathcal{B}_\xi(X)$ contains all open sets and all closed sets and is closed under finite intersections. But $\langle G_{in} \setminus G_n\rangle_{i\in I}$ is disjoint, so (a) tells us that $\bigcup_{i\in I} E_{in} \in \mathcal{B}_\xi(X)$; as this is true for every n,

$$\textstyle\bigcup_{i\in I} E_i = \bigcup_{n\in\mathbb{N}} \bigcup_{i\in I} C_{in} = \bigcup_{n\in\mathbb{N}} \bigcup_{i\in I} E_{in}$$

also belongs to $\mathcal{B}_\xi(X)$.

(b) If $\langle E_i\rangle_{i\in I}$ is an isolated family in $\mathcal{B}'_\xi(X)$, then, as in (a-ii) above, we can find a family $\langle G_i\rangle_{i\in I}$ of open sets and an open set G such that $\langle G_i \setminus G\rangle_{i\in I}$ is disjoint and $E_i \subseteq G_i \setminus G$ for every i. Now $\langle (G_i \setminus G) \setminus E_i\rangle_{i\in I}$ is an isolated family in $\mathcal{B}_\xi(X)$, so its union H belongs to $\mathcal{B}_\xi(X)$, and

$$\bigcup_{i\in I} E_i = ((\bigcup_{i\in I} G_i)\setminus G)\setminus H$$

belongs to $\mathcal{B}'_\xi(X)$.

1D Theorem (Spurný p05b) Let X be a hereditarily weakly θ-refinable Čech-complete space and $\langle E_i\rangle_{i\in I}$ a family of subsets of X such that $\bigcup_{i\in J} E_i$ is a G_δ set for every countable $J\subseteq I$. Then $\langle E_i\rangle_{i\in I}$ is σ-isolatedly refinable.

proof ? Suppose, if possible, otherwise.

(a) Express X as $\bigcap_{n\in\mathbb{N}} U_n$ where $\langle U_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of open sets in a compact Hausdorff space Z starting from $U_0=Z$. Note that every G_δ subset of X is also a G_δ subset of Z. Let $\mathcal{I}$ be the set of those $A\subseteq Z$ such that $\langle E_i\cap A\rangle_{i\in I}$ is σ-isolatedly refinable, as in Lemma 1B. Set $E=\bigcup_{i\in I} E_i$; we are supposing that X, and therefore E, do not belong to $\mathcal{I}$. For each $x\in E$ choose $j(x)\in I$ such that $x\in E_{j(x)}$, and a sequence $\langle U_n(x)\rangle_{n\in\mathbb{N}}$ of open sets in Z such that $x\in U_n(x)$ and $\overline{U_{n+1}(x)}\subseteq U_n(x)\subseteq U_n$ for every n, and $\bigcap_{n\in\mathbb{N}} U_n(x)\subseteq E_{j(x)}$; it will be convenient once again to suppose that $U_0(x)=Z$ for every $x\in E$.

(b) Note that if $A\subseteq Z$ and $A\notin\mathcal{I}$, there is an $x\in A\cap E$ such that $H\notin\mathcal{I}$ for any open set H containing x. **P** I am not assuming that Z itself is hereditarily weakly θ-refinable, but of course $Z\setminus E\in\mathcal{I}$, so $A\cap E\notin\mathcal{I}$. We can therefore apply Lemma 1Bb in X to see that there is a relatively open set $G\subseteq X$ such that $G\cap A\cap E\in\mathcal{I}$ but $H\cap A\cap E\setminus G\notin\mathcal{I}$ whenever H is a relatively open subset of X not included in G. Now we can take any $x\in A\cap E\setminus G$. **Q**

(c) Set $S_2^*=\bigcup_{n\in\mathbb{N}}\{0,1\}^n$ and choose $\langle x_\sigma\rangle_{\sigma\in S_2^*}$, $\langle E_\sigma\rangle_{\sigma\in S_2^*}$ inductively, as follows. Start by taking any $x_\emptyset\in E$ such that $H\notin\mathcal{I}$ for any open set H containing $x_\emptyset$, and set $E_\emptyset=Z$. Given that for $\sigma\in\bigcup_{m\le n}\{0,1\}^m$ we have chosen $x_\sigma\in E$ and a closed subset E_σ of Z and that $H\cap E_\sigma\notin\mathcal{I}$ for any open set H containing x_σ, set $C_n=\bigcup_{m\le n,\sigma\in\{0,1\}^m} E_{j(x_\sigma)}$. Then C_n is a G_δ set belonging to $\mathcal{I}$. Take any $\sigma\in\{0,1\}^n$. Then $E_\sigma\setminus C_n$ is an F_σ set not belonging to $\mathcal{I}$, so includes a closed set F'_σ not belonging to $\mathcal{I}$. Let $x_{\sigma^\frown 1}$ be a point of $F'_\sigma\cap E$ such that $H\cap F'_\sigma\notin\mathcal{I}$ for any neighbourhood H of $x_{\sigma^\frown 1}$. Let $E_{\sigma^\frown 1}$ be a closed relative neighbourhood of $x_{\sigma^\frown 1}$ in F'_σ such that $E_{\sigma^\frown 1}\subseteq U_{n+1}(x_{\sigma^\frown 1})$ and $x_\sigma\notin E_{\sigma^\frown 1}$. Next, set $x_{\sigma^\frown 0}=x_\sigma$ and let $E_{\sigma^\frown 0}$ be a closed relative neighbourhood of x_σ in E_σ disjoint from $E_{\sigma^\frown 1}$ and included in $U_{n+1}(x_\sigma)$. Continue.

(d) The effect of this construction is to ensure that

— for any $\alpha\in\{0,1\}^{\mathbb{N}}$, $\langle E_{\alpha\restriction n}\rangle_{n\in\mathbb{N}}$ is a decreasing sequence of closed sets in Z with $E_{\alpha\restriction n}\subseteq U_n$ for every n;

— $x_\sigma\in E_\sigma$ and $E_{\sigma^\frown 0}$ is disjoint from $E_{\sigma^\frown 1}$ for every $\sigma\in S_2^*$.

So if we set

$$K=\bigcap_{n\in\mathbb{N}}\bigcup_{\sigma\in\{0,1\}^n} E_\sigma,$$

$K\subseteq X$ is a compact set and we have a continuous surjection $h:K\to\{0,1\}^{\mathbb{N}}$ defined by saying that $h(x)=\alpha$ whenever $x\in K_\alpha=\bigcap_{n\in\mathbb{N}} E_{\alpha\restriction n}$. Let $L\subseteq K$ be a compact set such that $h[L]=h[K]$ and $h\restriction L$ is irreducible.

Let $D\subseteq\{0,1\}^{\mathbb{N}}$ be the set of eventually-zero sequences. Because D is dense and meager in $\{0,1\}^{\mathbb{N}}$, $L\cap h^{-1}[D]=\bigcup_{\alpha\in D} L\cap K_\alpha$ must be dense and meager in L. So $M=\bigcup_{\alpha\in D} K_\alpha$ cannot be a G_δ set in K.

(e) Suppose that $\alpha\in D$ and $\sigma\in S_2^*$ are such that $\sigma\subseteq\alpha$ and $\alpha(i)=0$ for $i\ge\#(\sigma)$. Then

$$K_\alpha\subseteq K\cap E_{j(x_\sigma)}\subseteq M.$$

P For $n\ge\#(\sigma)$, $x_{\alpha\restriction n}=x_\sigma$ and $E_{\alpha\restriction n}\subseteq U_n(x_\sigma)$; so

$$K_\alpha=\bigcap_{n\ge\#(\sigma)} E_{\alpha\restriction n}\subseteq\bigcap_{n\ge\#(\sigma)} U_n(x_\sigma)=\bigcap_{n\in\mathbb{N}} U_n(x_\sigma)\subseteq E_{j(x_\sigma)}.$$

On the other hand, if $y\in K\setminus M$, there is some β such that $y\in K_\beta$ and $\beta(n)=1$ for some $n\ge\#(\sigma)$. In this case, $E_{\beta\restriction n+1}\subseteq F'_{\beta\restriction n}$ was chosen to be disjoint from C_n, so $y\notin C_n\supseteq E_{j(x_\sigma)}$. As y is arbitrary, $K\cap E_{j(x_\sigma)}\subseteq M$. **Q**

(f) But this means that, setting $J=\{j(x_\sigma):\sigma\in S_2^*\}$,

$$M=K\cap\bigcup_{\sigma\in S_2^*} E_{j(x_\sigma)}=K\cap\bigcup_{i\in J} E_i$$

is relatively G_δ in K, by the additivity hypothesis on $\langle E_i\rangle_{i\in I}$; which contradicts the last sentence of (d). **X**

1E Lemma (Spurný p05a) Let X be a non-empty Čech-complete space and $\langle E_i\rangle_{i\in I}$ a point-countable cover of of X such that $\bigcup_{i\in J} E_i$ is an F_σ set for every countable $J\subseteq X$. Then there is an $i\in I$ such that E_i has non-empty interior.

proof ? Suppose, if possible, otherwise.

(a) Express X as $\bigcap_{n\in\mathbb{N}} U_n$ where $\langle U_n\rangle_{n\in\mathbb{N}}$ is a sequence of dense open sets in a compact Hausdorff space Z. Let M be a countable set containing enough objects and closed under enough operations, as specified in the footnotes[1] to come, to make the rest of this argument work.

(b) $M\cap C(Z)$ is not empty[2]; let $\langle f_n\rangle_{n\in\mathbb{N}}$ run over $M\cap C(Z)$, and define $f:Z\to\mathbb{R}^{\mathbb{N}}$ by setting $f(z)=\langle f_n(z)\rangle_{n\in\mathbb{N}}$ for $z\in Z$. Then f is continuous, so $f[Z]$ is compact and $f[Z\setminus X]$ is K_σ. Set $Y=f[Z]\setminus f[Z\setminus X]$, so that Y is a G_δ subset of $f[Z]$.

(c) If $H\subseteq Z$ is a closed set belonging to M, and $x\in M\cap Z\setminus H$, then there is an $m\in\mathbb{N}$ such that $f_m(x)\notin f_m[H]$[3], so $f(x)\notin f[H]$. Consequently $f(x)\notin f[Z\setminus U_n]$ for every $x\in M\cap X$ and $n\in\mathbb{N}$[4] and $f[M\cap X]\subseteq Y$.

(d) If $H\subseteq Z$ is a nowhere dense closed set belonging to M, then $f[M\cap X\setminus H]$ is dense in $f[Z]$. **P** Let V be an open subset of $\mathbb{R}^{\mathbb{N}}$ meeting $f[Z]$. Then there are an $n\in\mathbb{N}$ and $\alpha_0,\dots,\alpha_n,\beta_0,\dots,\beta_n\in\mathbb{Q}$ such that $V'=\{w:w\in\mathbb{R}^{\mathbb{N}},\ \alpha_k<w(k)<\beta_k \text{ for every } k\le n\}$ is included in V and meets $f[Z]$. Now $f^{-1}[V']\in M$[5], and is a non-empty open subset of Z. So $f^{-1}[V']\setminus H$ meets X and $M\cap X$[6]. Accordingly $f[M\cap X\setminus H]$ meets V' and V. **Q**

In particular, $f[M\cap X]$ is dense in $f[Z]$[7].

(e) Set $J=\{i:i\in I,\ M\cap E_i\ne\emptyset\}$. Then $f[E_i]$ is meager in $f[Z]$ for every $i\in J$. **P** There is an $x\in M$ such that $x\in E_i$, so $i\in M$[8] and $E_i\in M$[9]. We have a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ of closed subsets of Z such that $\langle H_n\rangle_{n\in\mathbb{N}}\in M$ and $E_i=X\cap\bigcup_{n\in\mathbb{N}}H_n$[10]; now every H_n is nowhere dense in Z (this is where we use the counter-hypothesis that E_i has empty interior in X) and belongs to M[11]. By (c), the closed set $f[H_n]$ is disjoint from $f[M\cap X\setminus H_n]$, while (d) tells us that $f[M\cap X\setminus H_n]$ is dense in $f[Z]$; so $f[H_n]$ is nowhere dense in $f[Z]$. This is true for every n, so $f[E_i]\subseteq\bigcup_{n\in\mathbb{N}}f[H_n]$ is meager. **Q**

(f) Set $K=I\setminus J$. Then $f[M\cap X]$ does not meet $f[\bigcup_{i\in K}E_i]$. **P** Take $x\in M\cap X$, $j\in K$ and $y\in E_j$. Then $F=\bigcup\{E_i:i\in I,\ x\notin E_i\}$ belongs to M[12], and F is of the form $X\cap\bigcup_{n\in\mathbb{N}}H_n$ where every H_n is a closed subset of Z belonging to M, as in (e) above. Since $x\in M\cap X\setminus H_n$, $f(x)\notin f[H_n]$ for every n, by (c); but $y\in E_j$ and $x\notin E_j$, so $y\in F\subseteq\bigcup_{n\in\mathbb{N}}H_n$ and $f(x)\ne f(y)$. **Q**

(g) Consequently $B=f[\bigcup_{i\in K}E_i]$ is meager in $f[Z]$. **P** Express $\bigcup_{i\in K}E_i$ as $X\cap\bigcup_{n\in\mathbb{N}}H_n$ where $H_n\subseteq Z$ is closed for every n. Since $f[M\cap X]$ does not meet either $f[Z\setminus X]$ or $f[\bigcup_{i\in K}E_i]$, $f[H_n]\cap f[M\cap X]=\emptyset$ for each n. But $f[M\cap X]$ is dense in $f[Z]$, by (d), so the closed sets $f[H_n]$ are nowhere dense and $B\subseteq\bigcup_{n\in\mathbb{N}}f[H_n]$ is meager. **Q**

(h) Because $\langle E_i\rangle_{i\in I}$ is point-countable, J is countable. Accordingly, using (e) and (g),

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[1] Watch out for integer superscripts.

[2] The constant function χZ belongs to M.

[3] If $H\subseteq Z$ is a closed set belonging to M and $x\in M\cap Z\setminus H$, then there is a $g\in M\cap C(Z)$ such that $g(x)=0$ and $g(z)=1$ for $z\in H$.

[4] $Z\setminus U_n\in M$ for every $n\in\mathbb{N}$.

[5] If $g_0,\dots,g_n\in M\cap C(Z)$ and $\alpha_0,\dots,\alpha_n,\beta_0,\dots,\beta_n\in\mathbb{Q}$ then $\{z:z\in Z,\ \alpha_k<g_k(z)<\beta_k \text{ for every } k\le n\}$ belongs to M.

[6] If A, $B\in M$ then $A\cap X\setminus B\in M$; if $A\in M$ is not empty then $M\cap A\ne\emptyset$.

[7] $\emptyset\in M$.

[8] If $x\in M$ then $\{i:i\in I,\ x\in E_i\}\in M$; if $A\in M$ is countable then $A\subseteq M$.

[9] If $i\in M\cap I$ then $E_i\in M$.

[10] If $F\in M$ is expressible as $X\cap H$ where $H\subseteq Z$ is an F_σ set, then there is a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$, belonging to M, such that every H_n is a closed subset of Z and $F=X\cap\bigcup_{n\in\mathbb{N}}H_n$.

[11] If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence belonging to M, then $A_n\in M$ for every $n\in\mathbb{N}$.

[12] If $x\in M$ then $\{i:i\in I,\ x\notin E_i\}\in M$; if $L\subseteq I$ and $L\in M$ then $\bigcup_{i\in L}E_i\in M$.

$$Y \subseteq f[X] \subseteq \bigcup_{i\in J} f[E_i] \cup B$$

is meager in $f[Z]$; but it is a G_δ set and also dense, by (c) and (d), while X, Z and $f[Z]$ are not empty, so this is impossible. **X**

1F Theorem (J.Spurný; compare HANSELL 86) Let X be a hereditarily weakly θ-refinable Čech-complete space, and $\langle E_i\rangle_{i\in I}$ a point-countable family of subsets of X such that $\bigcup_{i\in J} E_i$ is an F_σ set for every $J \subseteq I$. Then $\langle E_i\rangle_{i\in I}$ is σ-isolatedly refinable.

proof Set $E = \bigcup_{i\in I} E_i$, and express E as $\bigcup_{n\in\mathbb{N}} F_n$ where each F_n is closed. For each $n \in \mathbb{N}$, $\langle F_n \cap E_i\rangle_{i\in I}$ is σ-isolatedly refinable. **P** There is an open set $G \subseteq X$ such that $\langle F_n \cap E_i \cap G\rangle_{i\in I}$ is σ-isolatedly refinable but $\langle F_n \cap E_i \cap H \setminus G\rangle_{i\in I}$ is not σ-isolatedly refinable for any open set H not included in G, by 1Bb. In particular, $E_i' = E_i \cap F_n \setminus G$ has empty interior in $F_n \setminus G$ for every $i \in I$, while $\langle E_i'\rangle_{i\in I}$ is point-countable and $\bigcup_{i\in J} E_i'$ is an F_σ set in $F_n \setminus G$ for every $J \subseteq I$. Moreover, $F_n \setminus G$, being closed in X, is Čech-complete, and is the union of $\langle E_i'\rangle_{i\in I}$. So Lemma 1E tells us that $F_n \setminus G$ is empty, so that $\langle F_n \cap E_i\rangle_{i\in I} = \langle F_n \cap G \cap E_i\rangle_{i\in I}$ is σ-isolatedly refinable. **Q**

By 1B(a-i), $\langle E_i\rangle_{i\in I}$ is σ-isolatedly refinable.

Remark Following HANSELL 71 it has long been known that many results of this type about complete metric spaces can be extended to more general classes of metric space, in particular, those which are Souslin-F in their completions. For such extensions see the references at the end of this note.

2 Selectors

2A Theorem Let X be a topological space, κ an infinite cardinal and $\mathcal{E}$ a family of subsets of X such that

(i) every open set and every closed set belongs to $\mathcal{E}$,

(ii) $E \cap F \in \mathcal{E}$ for all E, $F \in \mathcal{E}$,

(iii) $\bigcup_{i\in I} E_i \in \mathcal{E}$ for every isolated family $\langle E_i\rangle_{i\in I}$ in $\mathcal{E}$,

(iv) if $\langle A_i\rangle_{i\in I}$ is an $\mathcal{E}$-additive family of subsets of X and $\#(\{i : i \in I, x \in A_i\}) < \kappa$ for every $x \in X$, then $\langle A_i\rangle_{i\in I}$ is σ-isolatedly refinable.

Let Σ be the σ-algebra of subsets of X generated by $\mathcal{E}$. Suppose that (Y, ρ) is a complete metric space and $R \subseteq X \times Y$ is such that

(v) $R^{-1}[H] \in \mathcal{E}$ for every open subset H of Y,

(vi) $L^+(R[\{x\}]) \le \kappa$ for every $x \in X$,

(vii) $R[\{x\}]$ is closed in Y for every $x \in X$.

Then R has a Σ-measurable selector.

proof (a) The first step is to show that, under the hypotheses (i)-(vii), there is for any $\epsilon > 0$ a sequence $\langle R_n\rangle_{n\in\mathbb{N}}$ of subsets of $X \times Y$ such that

(α) $R_n \subseteq R$ and $\operatorname{diam} R_n[\{x\}] \le \epsilon$ for every $n \in \mathbb{N}$ and $x \in X$,

(β) $R_n^{-1}[H] \in \mathcal{E}$ for every $n \in \mathbb{N}$ and open $H \subseteq Y$,

(γ) $\bigcup_{n\in\mathbb{N}} R_n^{-1}[Y] = R^{-1}[Y]$,

(δ) $R_n[\{x\}]$ is closed for every $x \in X$.

P Let $\langle V_i\rangle_{i\in I}$ be a locally finite cover of Y by open sets of Y of diameter at most ϵ. Then (v) and (vi) ensure that $\langle R^{-1}[V_i]\rangle_{i\in I}$ satisfies the conditions of (iv), so has a σ-isolated refinement $\langle C_{in}\rangle_{i\in I,n\in\mathbb{N}}$ as in 1Ab. For $i \in I$ and $n \in \mathbb{N}$, set $C_{in}' = \overline{C}_{in} \setminus \bigcup_{m<n} \overline{C}_{im}$. Then $C_{in}' \in \mathcal{E}$, being the intersection of a closed set with an open set.

For each $n \in \mathbb{N}$, set

$$Q_n = R \cap \bigcup_{i\in I} C_{in}' \times V_i, \quad R_n = \{(x, y) : x \in X,\, y \in \overline{Q_n[\{x\}]}\}$$

for each n. Of course $Q_n \subseteq R$; because $R[\{x\}]$ is always closed, $R_n \subseteq R$. $\operatorname{diam} R_n[\{x\}] = \operatorname{diam} Q_n[\{x\}] \le \epsilon$ for every $x \in X$ because $\langle C_{in}'\rangle_{i\in I}$ is disjoint. If $H \subseteq Y$ is open, then

$$R_n^{-1}[H] = Q_n^{-1}[H] = \bigcup_{i\in I} C_{in}' \cap R^{-1}[V_i \cap H]$$

belongs to $\mathcal{E}$ because $\langle C_{in}' \cap R^{-1}[V_i \cap H]\rangle_{i\in I}$ is an isolated family in $\mathcal{E}$. Finally,

$$\bigcup_{n\in\mathbb{N}} R_n^{-1}[Y] \supseteq \bigcup_{n\in\mathbb{N}} Q_n^{-1}[Y] = \bigcup_{i\in I,n\in\mathbb{N}} C'_{in} \cap R^{-1}[V_i] = \bigcup_{i\in I,n\in\mathbb{N}} \overline{C}_{in} \cap R^{-1}[V_i]$$
$$\supseteq \bigcup_{i\in I,n\in\mathbb{N}} C_{in} = \bigcup_{i\in I} R^{-1}[V_i] = R^{-1}[Y]. \textbf{Q}$$

(b) Set $S^* = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$. Starting from R as in the hypotheses of the proposition, construct inductively a family $\langle R_\sigma\rangle_{\sigma\in S^*}$ such that $R_\emptyset = R$ and, for every $n \in \mathbb{N}$ and $\sigma \in \mathbb{N}^n$,

$R_\sigma \subseteq R$,
$R_\sigma^{-1}[H] \in \mathcal{E}$ for every open set $H \subseteq Y$,
$R_\sigma[\{x\}]$ is a closed set of diameter at most 2^{-n} if $n \geq 1$ and $x \in X$,
$R_\sigma^{-1}[Y] = \bigcup_{i\in\mathbb{N}} R_{\sigma^\frown i}^{-1}[Y]$.

(Note that the Lindelöf-number condition (vi) is automatically inherited by the R_σ because $R_\sigma[\{x\}]$ is always a closed subset of $R[\{x\}]$.) Next, we can define $\langle E_\sigma\rangle_{\sigma\in S^*}$ by saying that

$E_\emptyset = R^{-1}[Y]$,
$E_{\sigma^\frown i} = E_\sigma \cap R_{\sigma^\frown i}^{-1}[Y] \setminus \bigcup_{j<i} R_{\sigma^\frown j}^{-1}[Y]$

for every $\sigma \in S^*$. Then every E_σ belongs to Σ. If $x \in R^{-1}[Y]$, define $\alpha_x \in \mathbb{N}^\mathbb{N}$ by saying that $x \in E_{\alpha_x\restriction n}$ for every n. Consider $\langle R_{\alpha_x\restriction n}[\{x\}]\rangle_{n\in\mathbb{N}}$. This is a non-increasing sequence of non-empty closed subsets of Y with diameters converging to 0, so its intersection has just one point $f(x)$ say, and $f(x)$ belongs to the closed set $R[\{x\}]$. This defines a function $f \subseteq R$ such that $\operatorname{dom} f = R^{-1}[Y]$.

(c) Finally, f is Σ-measurable. $\mathbf{P}$ Suppose that $H \subseteq Y$ is open. For $n \geq 1$, set $H_n = \{y : y \in Y, \rho(y, Y \setminus H) > 2^{-n+1}\}$, and consider

$$E = \bigcup_{n\geq 1} \bigcup_{\sigma\in\mathbb{N}^n} E_\sigma \cap R_\sigma^{-1}[H_n] \in \Sigma.$$

If $x \in E$, then, for some $n \geq 1$, $x \in E_{\alpha_x\restriction n} \cap R_{\alpha_x\restriction n}^{-1}[H_n]$ and

$$f(x) \in R_{\alpha_x\restriction n}[\{x\}] \subseteq \{y : \rho(y, H_n) \leq 2^{-n}\}$$

(because $R_{\alpha_x\restriction n}[\{x\}]$ has diameter at most 2^{-n} and meets H_n)

$$\subseteq H.$$

Thus $E \subseteq f^{-1}[H]$. On the other hand, if $x \in f^{-1}[H]$, let $n \geq 1$ be such that $f(x) \in H_n$. Then $R_{\alpha_x\restriction n}[\{x\}]$ must meet H_n and

$$x \in E_{\alpha_x\restriction n} \cap R_{\alpha_x\restriction n}^{-1}[H_n] \subseteq E.$$

So $f^{-1}[H] = E \in \Sigma$. $\mathbf{Q}$

2B Remarks Compare Fremlin 87, 10B. Observe that in fact we have

$$f^{-1}[H] \in \mathcal{E}_\sigma \text{ for every open } H \subseteq Y,$$

where $\mathcal{F} = \{E \setminus F : E, F \in \mathcal{E}\}$ and $\mathcal{E}_\sigma = \{\bigcup_{n\in\mathbb{N}} E_n : \langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{F}\}$. In particular, if $\mathcal{E} = \mathcal{B}_\xi(X)$ where $1 \leq \xi < \omega_1$, then we shall have $f^{-1}[H] \in \mathcal{B}_{\xi+1}(X)$ for every open $H \subseteq Y$.

The standard allocations of κ are

— $\kappa = w(Y)^+$, so that (iv) declares that very many $\mathcal{E}$-additive families are σ-isolatedly refinable, and (vi) is automatically satisfied;

— $\kappa = \omega_1$, so that (iv) declares that point-countable $\mathcal{E}$-additive families are σ-isolatedly refinable, and (vi) that $R[\{x\}]$ is separable for every x;

— $\kappa = \omega$, so that (iv) declares that point-finite $\mathcal{E}$-additive families are σ-isolatedly refinable, and (vi) that $R[\{x\}]$ is compact for every x.

2C Corollary (Spurný) Let X be a perfectly normal hereditarily weakly θ-refinable Čech-complete space, (Y, σ) a complete metric space, and $R \subseteq X \times Y$ a relation with closed vertical sections such that $R^{-1}[H]$ is a G_δ set for every open $H \subseteq Y$. Then R has a Borel measurable selector.

proof Let $\mathcal{E}$ be the family of G_δ subsets of X. Then $\mathcal{E}$ satisfies conditions (i)-(iv) of 2A. **P** Open sets are always G_δ, closed sets are G_δ because X is perfectly normal, and the intersection of two G_δ sets is certainly G_δ. Again because all closed sets are G_δ, $\mathcal{E}$ is precisely $\mathcal{B}'_1(X)$ and 1Cb tells us that (iii) is satisfied. And (iv) is satisfied, for any κ, by 1D. **Q**

Now we can read off the result from Theorem 2A.

2D Proposition Let (X,ρ) be a complete metric space, $\mathcal{B}(X)$ its Borel σ-algebra, and κ an infinite cardinal. Then the following are equiveridical:

(i) whenever $\langle E_i\rangle_{i\in I}$ is a $\mathcal{B}(X)$-additive family of subsets of X such that $\#(\{i : i\in I,\, x\in E_i\})<\kappa$ for every $x\in X$, then $\langle E_i\rangle_{i\in I}$ is σ-isolatedly refinable;

(ii) whenever (Y,σ) is a complete metric space, $R\subseteq X\times Y$ is a relation with closed vertical sections such that $R^{-1}[H]\in\mathcal{B}(X)$ for every open $H\subseteq Y$ and $L^+(R[\{x\}])\le\kappa$ for every $x\in X$, then R has a Borel measurable selector.

proof (i)$\Rightarrow$(ii) is covered by Proposition 2A with $\mathcal{E}=\Sigma=\mathcal{B}(X)$. In the reverse direction, assuming (ii), suppose that $\langle E_i\rangle_{i\in I}$ is as in (i). Give I its $\{0,1\}$-valued metric and consider $R=\{(x,i) : i\in I,\, x\in E_i\}$. Then R satisfies the conditions of (ii) so has a Borel measurable selector f say. Now $\langle f^{-1}[\{i\}]\rangle_{i\in I}$ is a disjoint $\mathcal{B}(X)$-additive family so by Hansell's theorem (Hansell 71, Kaniewski & Pol 75 or Fremlin 87, 3H) it is σ-isolatedly refinable. But any σ-isolated refinement of $\langle f^{-1}[\{i\}]\rangle_{i\in I}$ is also a σ-isolated refinement of $\langle E_i\rangle_{i\in I}$. So (i) is true.

3 Point-finite families and relations with compact sections

3A Lemma Let X be a set, Σ a σ-algebra of subsets of X and $\langle E_i\rangle_{i\in I}$ a point-finite Σ-additive family of subsets of X of cardinal at most $\mathfrak{c}$. Then there is a family $\langle E_{in}\rangle_{i\in I,n\in\mathbb{N}}$ such that

(i) $E_i=\bigcup_{n\in\mathbb{N}}E_{in}$ for every $i\in I$,

(ii) $\langle E_{in}\rangle_{i\in I}$ is disjoint for every $n\in\mathbb{N}$,

(iii) if $\langle E'_i\rangle_{i\in I}$ is a family of subsets of X such that $\langle E_i\cap E'_i\rangle_{i\in I}$ is Σ-additive, then $\langle E_{in}\cap E'_i\rangle_{i\in I}$ is Σ-additive.

proof (a) We can suppose that $I=\mathcal{P}\mathbb{N}$. For $K\subseteq m\in\mathbb{N}$ set $G_{mK}=\bigcup_{A\subseteq\mathbb{N},A\cap m=K}E_A$. For $x\in X$, $n\in\mathbb{N}$ set $h_n(x)=\#(\{K : K\subseteq n,\, x\in G_{nK}\}$. Then every G_{nK} belongs to Σ and every $h_n : X\to\mathbb{N}$ is Σ-measurable. Also, for any $x\in X$, $\langle h_n(x)\rangle_{n\in\mathbb{N}}$ is non-decreasing with finite supremum $h(x)=\#(\{A : A\subseteq\mathbb{N},\, x\in E_A\})$. So if we set

$$m_x=\min\{n : h_n(x)=h_i(x)\text{ for every }i\ge n\}$$

$x\mapsto m_x : X\to\mathbb{N}$ is Σ-measurable. If $m_x=m$ then for each $K\subseteq m$ there can be at most one $A\subseteq\mathbb{N}$ such that $A\cap m=K$ and $x\in E_A$.

(b) For $K\subseteq m\in\mathbb{N}$ and $A\subseteq\mathbb{N}$ set

$$\begin{aligned}E_{mKA}&=\{x : x\in E_A,\, m_x=m\}\text{ if }A\cap m=K\\&=\emptyset\text{ otherwise.}\end{aligned}$$

Then $\bigcup_{K\subseteq m\in\mathbb{N}}E_{mKA}=E_A$ for $A\subseteq\mathbb{N}$. If $K\subseteq m\in\mathbb{N}$, A, $B\subseteq\mathbb{N}$ and $x\in E_{mKA}\cap E_{mKB}$, then $A\cap m_x=B\cap m_x$ and $x\in E_A\cap E_B$, so $A=B$; thus $\langle E_{mKA}\rangle_{A\subseteq\mathbb{N}}$ is disjoint.

(c) If now $\langle E'_A\rangle_{A\subseteq\mathbb{N}}$ is a family of sets such that $\langle E_A\cap E'_A\rangle_{A\subseteq\mathbb{N}}$ is Σ-additive, $K\subseteq m\in\mathbb{N}$ and $\mathcal{A}\subseteq\mathcal{P}\mathbb{N}$, then

$$\bigcup_{A\in\mathcal{A}}E_{mKA}\cap E'_A=\{x : x\in\bigcup_{A\in\mathcal{A},A\cap m=K}E_A\cap E'_A,\, m_x=m\}\in\Sigma,$$

so $\langle E_{mKA}\cap E'_A\rangle_{A\subseteq\mathbb{N}}$ is Σ-additive. Accordingly $\langle E_{mKA}\rangle_{K\subseteq m\in\mathbb{N},A\subseteq\mathbb{N}}$, re-indexed, will satisfy all the requirements.

3B Proposition (Spahn 83) Let X be a set, Σ a σ-algebra of subsets of X and Y a metrizable space of weight at most $\mathfrak{c}$. Let $R\subseteq X\times Y$ be a relation such that $R^{-1}[H]\in\Sigma$ for every open set $H\subseteq Y$ and $R[\{x\}]$ is compact for every $x\in X$. Then R has a Σ-measurable selector.

proof Let ρ be a metric on Y inducing its topology.

(a) If $\epsilon > 0$ there is a $Q \subseteq R$ such that $Q^{-1}[H] \in \Sigma$ for every open $H \subseteq Y$, $Q[\{x\}]$ is a compact set of diameter at most ϵ for every $x \in X$, and $Q^{-1}[Y] = R^{-1}[Y]$. **P** Because $w(Y) \le \mathfrak{c}$, Y has a locally finite cover $\langle V_i \rangle_{i\in I}$ by open sets of diameter at most ϵ, with $\#(I) \le \mathfrak{c}$. Set $E_i = R^{-1}[V_i]$ for $i \in I$; then $\langle E_i \rangle_{i\in I}$ satisfies the conditions of Lemma 3A. Let $\langle E_{in} \rangle_{i\in I, n\in\mathbb{N}}$ be as in 3A. Set $G_n = \bigcup_{i\in I} E_{in}$, $H_n = G_n \setminus \bigcup_{m<n} G_n$ for $n \in \mathbb{N}$, so that $\langle H_n \rangle_{n\in\mathbb{N}}$ is a partition of $R^{-1}[Y]$ into members of Σ, and $\langle H_n \cap E_{in} \rangle_{i\in I}$ is a partition of H_n for each $n \in \mathbb{N}$.

Set

$$Q = \bigcup_{i\in I, n\in\mathbb{N}} \{(x, y) : x \in H_n \cap E_{in},\, y \in \overline{R[\{x\}] \cap V_i}\}.$$

Of course $Q \subseteq R$, $Q^{-1}[Y] = R^{-1}[Y]$ and $Q[\{x\}]$ is a compact set of diameter at most ϵ for every x. If $H \subseteq Y$ is open, then for $i \in I$ set $E'_i = R^{-1}[V_i \cap H]$. Then

$$\bigcup_{i\in J} E_i \cap E'_i = R^{-1}(H \cap \bigcup_{i\in J} V_i) \in \Sigma$$

for every $J \subseteq \mathbb{N}$, so

$$\begin{aligned} Q^{-1}[H] &= \bigcup_{i\in I, n\in\mathbb{N}} \{x : x \in H_n \cap E_{in},\, R[\{x\}] \cap V_i \cap H \ne \emptyset\} \\ &= \bigcup_{n\in\mathbb{N}} (H_n \cap \bigcup_{i\in I} E_{in} \cap E'_i) \in \Sigma \end{aligned}$$

by (iii) of 3A. So Q has the required properties. **Q**

(b) We can therefore find a non-increasing sequence $\langle R_n \rangle_{n\in\mathbb{N}}$ of subsets of $X \times Y$ such that $R_0 = R$ and for every $n \ge 1$

R_n has compact vertical sections of diameter at most 2^{-n},
$R_n^{-1}[Y] = R^{-1}[Y]$,
$R_n^{-1}[H] \in \Sigma$ for every open $H \subseteq Y$.

Taking $f(x)$ to be the unique point of $\bigcap_{n\in\mathbb{N}} R_n[\{x\}]$ for each x, f is Σ-measurable just as in (b)-(c) of the proof of Proposition 2A, and is an appropriate selector for R.

3C Theorem (Hansell 81) Let X and Y be metrizable spaces, $\xi < \omega_1$, and $R \subseteq X \times Y$ a relation with compact vertical sections such that $R^{-1}[H] \in \mathcal{B}_\xi(X)$ for every open set $H \subseteq Y$. Then R has a Borel measurable selector.

proof We may suppose throughout that $\xi \ge 1$.

(a) Suppose that $\langle E_i \rangle_{i\in I}$ is a point-finite $\mathcal{B}_\xi(X)$-additive family of sets.

(i) Let $\langle G_i \rangle_{i\in I}$ be any family of open subsets of X. Then $\bigcup_{i\in I} E_i \cap G_i \in \mathcal{B}_\xi(X)$. **P** Let $\langle \mathcal{V}_n \rangle_{n\in\mathbb{N}}$ be a sequence of discrete families such that $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a base for the topology of X (Fremlin 03, 4A2Lg). For $V \in \mathcal{V}$ set $I_V = \{i : i \in I,\, V \subseteq G_i\}$ and $H_V = V \cap \bigcup_{i\in I_V} E_i \in \mathcal{B}_\xi(X)$. For each n, $\langle H_V \rangle_{V\in\mathcal{V}_n}$ is an isolated family in $\mathcal{B}_\xi(X)$, so its union C_n belongs to $\mathcal{B}_\xi(X)$ (Lemma 1Ca). Now

$$\bigcup_{i\in I} E_i \cap G_i = \bigcup_{V\in\mathcal{V}, i\in I_V} E_i \cap V = \bigcup_{n\in\mathbb{N}} C_n \in \mathcal{B}_\xi(X). \textbf{ Q}$$

(ii) Let $\langle E_i \rangle_{i\in I}$ be a family of G_δ sets. Then $\bigcup_{i\in I} E_i \cap E_i \in \mathcal{B}'_{\xi+1}(X)$. **P** Express each E_i as as $\bigcap_{n\in\mathbb{N}} G_{in}$ where $\langle G_{in} \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of open sets. Then

$$\bigcup_{i\in I} E_i \cap E_i = \bigcap_{n\in\mathbb{N}} \bigcup_{i\in I} E_i \cap G_{in}$$

because $\langle E_i \rangle_{i\in I}$ is point-finite. By (i), $\bigcup_{i\in I} E_i \cap E_i$ is the intersection of a sequence in $\mathcal{B}_\xi(X)$ and belongs to $\mathcal{B}'_{\xi+1}(X)$. **Q**

(iii) Let $\langle E_i \rangle_{i\in I}$ be a family of F_σ sets. Then $\bigcup_{i\in I} E_i \cap E_i \in \mathcal{B}_{\xi+2}(X)$. **P** Express each E_i as as $\bigcup_{n\in\mathbb{N}} E_{in}$ where $\langle E_{in} \rangle_{n\in\mathbb{N}}$ is a sequence of closed sets. Then

$$\bigcup_{i\in I} E_i \cap E_i = \bigcup_{n\in\mathbb{N}} \bigcup_{i\in I} E_i \cap E_{in}$$

is the union of a sequence in $\mathcal{B}'_{\xi+1}(X)$ (by (ii)) and belongs to $\mathcal{B}_{\xi+2}(X)$. **Q**

(iv) Suppose that $1 \le \eta < \omega_1$. Then

— if $\langle E_i\rangle_{i\in I}$ is a family in $\mathcal{B}_\eta(X)$, $\bigcup_{i\in I} E_i \cap E_i \in \mathcal{B}_{\xi+1+\eta}(X)$,

— if $\langle E_i\rangle_{i\in I}$ is a family in $\mathcal{B}'_\eta(X)$, $\bigcup_{i\in I} E_i \cap E_i \in \mathcal{B}'_{\xi+1+\eta}(X)$.

P Induce on η. If $\eta = 1$ then $\mathcal{B}_1(X)$ is the family of F_σ sets, $\mathcal{B}'_1(X)$ is the family of G_δ sets and we can apply (iii) and (ii). For the inductive step to countable $\eta > 1$, suppose that $\langle E_i\rangle_{i\in I}$ is a family in $\mathcal{B}_\eta(X)$. Then for each i we can express E_i as the union of a sequence $\langle E_{in}\rangle_{n\in\mathbb{N}}$ in $\bigcup_{1\le\zeta<\eta} \mathcal{B}'_\zeta(X)$. For each $n \in \mathbb{N}$ and $\zeta < \eta$, the inductive hypothesis tells us that $\bigcup\{E_i \cap E_{in} : i \in I,\ E_{in} \in \mathcal{B}'_\zeta(X)\}$ belongs to $\mathcal{B}'_{\xi+1+\zeta}(X)$, while $\xi + 1 + \zeta < \xi + 1 + \eta$, so the union $\bigcup_{i\in I} E_i \cap E_i$ belongs to $\mathcal{B}_{\xi+1+\eta}(X)$.

On the other hand, if $\langle E_i\rangle_{i\in I}$ is a family in $\mathcal{B}'_\eta(X)$, let $\langle\zeta_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\eta \setminus \{0\}$ which is cofinal with η. For each $i \in I$, we can express E_i as the intersection of a non-increasing sequence $\langle E_{in}\rangle_{n\in\mathbb{N}}$ such that $E_{in} \in \mathcal{B}_{\zeta_n}(X)$ for each n. Now, just as in (ii),

$$\bigcup_{i\in I} E_i \cap E_i = \bigcap_{n\in\mathbb{N}} \bigcup_{i\in I} E_i \cap E_{in} \in \mathcal{B}'_{\xi+1+\eta}(X)$$

because $\bigcup_{i\in I} E_i \cap E_{in} \in \mathcal{B}_{\xi+1+\zeta_n}$ for each n. So the induction proceeds. **Q**

(b) Now suppose that R is as in the statement of the theorem. Give Y a metric inducing its topology. Then for any $\epsilon > 0$ we can find an $R_1 \subseteq R$ such that

R_1 has compact vertical sections of diameter at most ϵ, and $R_1^{-1}[Y] = R^{-1}[Y]$,

$R_1^{-1}[H] \in \mathcal{B}'_{\xi+1+\xi}(X)$ for every open $H \subseteq Y$.

P Let $\langle V_\eta\rangle_{\eta<\kappa}$ be a locally finite cover of Y by open sets of diameter at most ϵ, and set $E_\eta = R^{-1}[V_\eta]$ for each η; then $\langle E_\eta\rangle_{\eta<\kappa}$ is a point-finite $\mathcal{B}_\xi(X)$-additive family. For $\eta < \kappa$, set $E_\eta = X \setminus \bigcup_{\eta<\zeta<\kappa} E_\eta \in \mathcal{B}'_\xi(X)$ and $E'_\eta = E_\eta \cap E_\eta$. Because $\langle E_\eta\rangle_{\eta<\kappa}$ is point-finite, $\bigcup_{\eta<\kappa} E'_\eta = \bigcup_{\eta<\kappa} E_\eta = R^{-1}[Y]$, while $\langle E'_\eta\rangle_{\eta<\kappa}$ is disjoint. Set

$$R_1 = \bigcup_{\eta<\kappa}\{(x, y) : x \in E'_\eta,\ y \in \overline{V_\eta \cap R[\{x\}]}\}.$$

Then $R_1 \subseteq R$ has compact vertical sections of diameter at most ϵ, and $R_1^{-1}[Y] = R^{-1}[Y]$. If $H \subseteq Y$ is open, then

$$\begin{aligned} R^{-1}[H] &= \bigcup_{\eta<\kappa}\{x : x \in E'_\eta,\ H \cap V_\eta \cap R[\{x\}] \ne \emptyset\} \\ &= \bigcup_{\eta<\kappa} E_\eta \cap E_\eta \cap R^{-1}[H \cap V_\eta] \in \mathcal{B}'_{\xi+1+\xi}(X) \end{aligned}$$

by (a-iv). **Q**

(c) We can therefore find sequences $\langle\xi_n\rangle_{n\in\mathbb{N}}$ and $\langle R_n\rangle_{n\in\mathbb{N}}$ such that

$\xi_0 = \xi$, $R_0 = R$,

$\xi_{n+1} = \xi_n + 1 + \xi_n + 1 < \omega_1$,

$R_{n+1} \subseteq R_n$ has compact vertical sections of diameter at most 2^{-n},

$R_n^{-1}[H] \in \mathcal{B}_{\xi_n}(X)$ for every open set $H \subseteq Y$

for each $n \in \mathbb{N}$. The intersection $\bigcap_{n\in\mathbb{N}} R_n$ is now the graph of a Borel measurable selector for R.

4 Problems

4A Problem CC The problem as stated is

If X and Y are topological spaces, a relation $R \subseteq X \times Y$ is **lower Borel measurable** if $R^{-1}[H]$ is Borel measurable in X for every open set $H \subseteq Y$.

(a) Can there be metrizable spaces X and Y and a lower Borel measurable $R \subseteq X \times Y$ with compact vertical sections which does not have a Borel measurable selector?

(b) Can there be complete metric spaces (X, ρ) and (Y, σ) and a lower Borel measurable $R \subseteq X \times Y$ with closed separable vertical sections which does not have a Borel measurable selector?

In the language of Proposition 2A, we are looking at the case $\mathcal{E} = \Sigma = \mathcal{B}(X)$, the Borel σ-algebra of X, with $\kappa = \omega$ in (a) and $\kappa = \omega_1$ in (b).

In view of Proposition 2A, we find that we have questions

(a)$'$ Can there be a metrizable space X with a point-finite $\mathcal{B}(X)$-additive family of subsets of X which is not σ-isolatedly refinable?

(b)$'$ Can there be a complete metric space (X, ρ) with a point-countable $\mathcal{B}(X)$-additive family of subsets of X which is not σ-isolatedly refinable?

and Proposition 2D, with $\kappa = \omega_1$, tells us that (b) and (b)$'$ are equivalent. However the answer to (a)$'$ is known to be 'yes'; if $\mathfrak{m} > \omega_1$ (that is, MA(ω_1) is true) then we can take any $X \in [\mathbb{R}]^{\omega_1}$; the set of singletons in X is point-finite and $\mathcal{B}(X)$-additive (in fact, disjoint and $\mathcal{E}$-additive where $\mathcal{E}$ is the family of subsets of X which are both F_σ and G_δ in X; see Fremlin 84, 23B), but is surely not σ-isolatedly refinable.

A large number of special cases have been resolved. For instance, Hansell's theorem (quoted in the proof of Proposition 2D) tells us that

if X is a complete metric space, any point-finite $\mathcal{B}(X)$-additive family of subsets of X is σ-isolatedly refinable,

and therefore, by 2A, that

if (X, ρ) is a complete metric space, Y is a metrizable space and $R \subseteq X \times Y$ is a lower Borel measurable relation with compact vertical sections, then R has a Borel measurable selector.

(In 2A I supposed that Y was complete; but of course we can work with its completion if necessary.) In Fremlin 87, 3J, we have

if (X, ρ) is a complete metric space of weight at most ω_1, every point-countable $\mathcal{B}(X)$-additive family of subsets of X is σ-isolatedly refinable,

thereby dealing with (b) and (b)$'$ for X of weight at most ω_1. Theorems 1D and 1F can be regarded as other very special cases in which we can answer 'no' to (b)$'$. Note that if (X, ρ) and (Y, σ) are complete metric spaces and $R \subseteq X \times Y$ is a lower Borel measurable relation with closed separable vertical sections, then R does have a selector f such that $f^{-1}[H]$ has the Baire property in X for every open set $H \subseteq Y$ (Fremlin 87, 10H).

Under an appropriate combination of special axioms, we can have

if X is metrizable, a point-countable $\mathcal{B}(X)$-additive family of subsets of X is σ-isolatedly refinable

(Fremlin 87, 3N), disposing of (a), (b), (a)$'$ and (b)$'$ together.

I should perhaps remark that the hypothesis that X should be complete, in (b) and (b)$'$, is there to distance them from (a) and (a)$'$; as far as I know, the version

(b)$_1$ Can there be a metrizable space X, a complete metric space (Y, σ) and a lower Borel measurable $R \subseteq X \times Y$ with closed separable vertical sections which does not have a Borel measurable selector?

is equally open.

4B In view of Theorem 3C it is natural to ask whether

(*) whenever X and Y are metric spaces and $R \subseteq X \times Y$ is a lower Borel measurable relation with compact vertical sections, there is a $\xi < \omega_1$ such that $R^{-1}[H] \in \mathcal{B}_\xi(X)$ for every open $H \subseteq Y$.

This is examined in detail in Fremlin Hansell & Junnila 83. We showed there (Theorem 16) that if X and Y are metric spaces and $R \subseteq X \times Y$ is a lower Borel measurable relation with compact vertical sections, then either there is a $\xi < \omega_1$ such that $R^{-1}[H] \in \mathcal{B}_\xi(X)$ for every open $H \subseteq Y$ or

(i) there is a quasi-measurable cardinal (definition: Fremlin 08?, 542A) which is at most $\min(\mathfrak{c}, w(Y), w(X))$; consequently, the continuum hypothesis is wildly false;

(ii) $\mathcal{P}\omega_1$ is not countably generated; consequently, $\mathfrak{m} = \omega_1$;

(iii) $R[Z]$ is separable for every separable $Z \subseteq X$;

(iv) there is a $y \in Y$ such that $R^{-1}[H]$ is non-separable for every neighbourhood H of y;

(v) X is not a Souslin-F subset of its completion;

(vi) Fleissner's Proposition P (Fleissner 79) is false; the Product Measure Extension Axiom (Fremlin 08?, 545B) is false.

So (*) must be true in a wide variety of models of set theory. Furthermore, it can be deduced from a special case. Consider the statements

(†) whenever X is a metrizable space and $\langle E_i\rangle_{i\in I}$ is a point-finite $\mathcal{B}(X)$-additive family of subsets of X, there is a $\xi<\omega_1$ such that $\langle E_i\rangle_{i\in I}$ is $\mathcal{B}_\xi(X)$-additive,

$(\dagger)_{\mathfrak c}$ whenever X is a metrizable space, $\#(I)\le\mathfrak c$ and $\langle E_i\rangle_{i\in I}$ is a point-finite $\mathcal{B}(X)$-additive family of subsets of X, there is a $\xi<\omega_1$ such that $\langle E_i\rangle_{i\in I}$ is $\mathcal{B}_\xi(X)$-additive,

$(\ddagger)_{\mathfrak c}$ whenever X is a metrizable space, $\#(I)\le\mathfrak c$ and $\langle E_i\rangle_{i\in I}$ is a disjoint $\mathcal{B}(X)$-additive family of subsets of X, then there is a $\xi<\omega_1$ such that $\langle E_i\rangle_{i\in I}$ is $\mathcal{B}_\xi(X)$-additive.

Then (*), (†), $(\dagger)_{\mathfrak c}$ and $(\ddagger)_{\mathfrak c}$ are equiveridical. **P** (†) is the special case of (*) in which Y is discrete, and of course $(\dagger)\Rightarrow(\dagger)_{\mathfrak c}\Rightarrow(\ddagger)_{\mathfrak c}$. In the reverse direction, if $(\ddagger)_{\mathfrak c}$ is true, X is a metrizable space, $\#(I)\le\mathfrak c$ and $\langle E_i\rangle_{i\in I}$ is a point-finite $\mathcal{B}(X)$-additive family of subsets of X, let $\langle E_{in}\rangle_{i\in I,n\in\mathbb{N}}$ be as in Lemma 3A. Then for each $n\in\mathbb{N}$ the family $\langle E_{in}\rangle_{i\in I}$ is disjoint and $\mathcal{B}(X)$-additive, so there is a $\xi_n<\omega_1$ such that it is $\mathcal{B}_{\xi_n}(X)$-additive. Setting $\xi=\sup_{n\in\mathbb{N}}\xi_n$, $\langle E_i\rangle_{i\in I}$ is $\mathcal{B}_\xi$-additive.

Thus $(\ddagger)_{\mathfrak c}\Rightarrow(\dagger)_{\mathfrak c}$ Next, $(\dagger)_{\mathfrak c}\Rightarrow(\dagger)$ by Proposition 18 of Fremlin Hansell & Junnila 83. Finally, if (†) is true, X and Y are metrizable spaces and $R\subseteq X\times Y$ is a lower Borel measurable relation with compact vertical sections, then let $\mathcal{V}$ be a σ-discrete base for the topology of Y, indexed as $\langle V_{ni}\rangle_{i\in I_n,n\in\mathbb{N}}$ where $\langle V_{ni}\rangle_{i\in I_n}$ is disjoint for every n. Set $E_{ni}=R^{-1}[V_{ni}]$ for $n\in\mathbb{N}$ and $i\in I_n$; then $\langle E_{ni}\rangle_{i\in I_n}$ is point-finite and $\mathcal{B}(X)$-additive for each n. By (†), there is a $\xi<\omega_1$ such that $\langle E_{ni}\rangle_{i\in I_n}$ is $\mathcal{B}_\xi(X)$-additive for every $n\in\mathbb{N}$. But this means that if $H\subseteq Y$ is disjoint, and $J_n=\{i:i\in I_n,\,V_{ni}\subseteq H\}$ for each n,

$$R^{-1}[H]=\bigcup_{n\in\mathbb{N}}\bigcup_{i\in J_n}E_{ni}\in\mathcal{B}_\xi(X).$$

As X, Y and R are arbitrary, (*) is true, and the chain of implications is fully reversible. **Q**

4C Returning to Problem CC, we see that if

$\binom{*}{*}$ whenever X and Y are metrizable spaces and $f:X\to Y$ is Borel measurable, then there is a $\xi<\omega_1$ such that $f^{-1}[H]\in\mathcal{B}_\xi(X)$ for every open $H\subseteq Y$

is true, then $(\ddagger)_{\mathfrak c}$ is true (this being just the special case of $\binom{*}{*}$ in which Y is discrete and has cardinal at most $\mathfrak c$), so that (*) is true. But Theorem 3C tells us that if (*) is true, then (a) of 2E has a negative answer. Of course $\binom{*}{*}$ is an old problem of A.H.Stone, Problem CA in my list Fremlin n06, also examined in Fremlin Hansell & Junnila 83 without a successful resolution.

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