Problem AC

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This note supplements Fremlin n82 and replaces the Note of 10.10.89, 'More on A.Bellow's problem'.

1. Notations If X is a set, then for $x \in X$ write $\hat{x}$ for the function from $\mathbb{R}^X$ to $\mathbb{R}$ defined by setting $\hat{x}(f) = f(x)$ for every $f \in \mathbb{R}^X$. Let $\mathfrak{T}_p$ be the topology on $\mathbb{R}^X$ generated by $\{\hat{x} : x \in X\}$.

If X is a topological space, write $C(X)$ for the space of continuous real-valued functions on X.

If (Z, Σ, μ) is a measure space, write $\mathcal{L}^0(\Sigma)$ for the space of real-valued measurable functions on Z.

2. Proposition Let X be a completely regular Hausdorff space. Then the following are equivalent:

(i) there is a probability μ on $C(X)$ such that $\hat{x}{\restriction}C(X)$ is μ-measurable for every $x \in X$ and $\mu^*G > 0$ for every non-empty $G \subseteq C(X)$ which is open for the topology on $C(X)$ induced by the topology $\mathfrak{T}_p$ on $\mathbb{R}^X$;

(ii) there is a probability space (Z, Σ, ν) with a set $Y \subseteq \mathcal{L}^0(\Sigma)$ such that Y, with the topology induced by $\mathfrak{T}_p$ on $\mathcal{L}^0(\Sigma)$, is homeomorphic to X, and also

$$\nu\{z : f(z) \neq g(z)\} \ > \ 0$$

whenever f, g are distinct members of Y.

proof (a) (i) $\Rightarrow$ (ii) Take $Z = C(X)$, $\nu = \mu$, $Y = \{\hat{x}{\restriction}C(X) : x \in X\}$.

(b) (ii) $\Rightarrow$ (i) Let $\phi : X \to Y$ be any homeomorphism, and define $\psi : Z \to C(X)$ by setting $\psi(z) = \hat{z} \circ \phi$. Let T be the σ-algebra of subsets of $C(X)$ generated by $\{\hat{x}{\restriction}C(X) : x \in X\}$, and set $\mu_0 = \nu\psi^{-1}{\restriction}\,\mathrm{T}$. Then μ_0 is a probability with domain T, and

$$\mu_0\{f : f \in C(X), f(x) \neq f(y)\} > 0$$

for all distinct $x, y \in X$. Let λ be any Borel probability measure on $\mathbb{R}$ with support $\mathbb{R}$, and define $\theta : \mathbb{R} \to C(X)$ by writing $\theta(\alpha)(x) = \alpha$ for $x \in X, \alpha \in \mathbb{R}$. Set $\mu_1 = \mu_0 + \lambda\theta^{-1}{\restriction}\,\mathrm{T}$; then μ_1 is a totally finite measure with domain T, $\mu_1 \geq \mu_0$, and

$$\mu_1\{f : f \in C(X), \alpha < f(x) < \beta\} > 0$$

whenever $\alpha < \beta$ in $\mathbb{R}$.

Now let $\langle q_n \rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathbb{Q} \cup \{-\sqrt{2}, \sqrt{2}\}$ with cofinal repetitions, and define $\langle \mu_n \rangle_{n\geq 2}$ as follows. Given μ_n, where $n \geq 1$, then

if $q_n \in \mathbb{Q}$, set $\mu'_n = \mu_n\chi_{q_n}^{-1}{\restriction}\,\mathrm{T}$, where $\chi_\alpha : C(X) \to C(X)$ is defined by writing $\chi_\alpha(f)(x) = \alpha f(x)$ for $x \in X, f \in C(X)$;

if $q_n = -\sqrt{2}$, set $\mu'_n = (\mu_n \times \mu_n)S^{-1}{\restriction}\,\mathrm{T}$, where $S : C(X) \times C(X) \to C(X)$ is defined by writing $S(f, g)(x) = f(x) + g(x)$ for $x \in X, f, g \in C(X)$;

if $q_n = \sqrt{2}$, set $\mu'_n = (\mu_n \times \mu_n)J^{-1}{\restriction}\,\mathrm{T}$, where $J : C(X) \times C(X) \to C(X)$ is defined by setting $J(f, g)(x) = \max(f(x), g(x))$ for $x \in X, f, g \in C(X)$.

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(Of course both S and J are $(\mathrm{T}\widehat{\otimes}_\sigma\mathrm{T},\ \mathrm{T})$-measurable.) Now set $\mu_{n+1} = \mu_n + 2^{-n}\mu_n'$, and continue.

The sequence $\langle\mu_n\rangle_{n\in\mathbb{N}}$ is increasing and uniformly bounded, so has a totally finite limit measure μ_∞; set $\mu = \mu_\infty/\mu_\infty C(X)$. Then μ is a probability with domain T. I claim that its support is $C(X)$. For suppose that $G \subseteq C(X)$ is a non-empty set which is open for the topology induced by $\mathfrak{T}_p$. Then there are a finite non-empty $I \subseteq X$ and a non-empty open $H \subseteq \mathbb{R}^I$ such that $G \supseteq \pi_I^{-1}[H]$, where $\pi_I(f) = f\restriction I$ for $f \in C(X)$. Let λ_I be the Borel measure $\mu\pi_I^{-1}$, so that $\mu^*G \geq \lambda_I H$. Because λ_I is a Borel probability on a Polish space, it has a non-empty support F. But the construction of μ means that F is closed under multiplication by any rational constant function, under addition, and under $\vee$; also F contains the constants and separates the points of I; so (by a very special case of the Stone-Weierstrass theorem) it is the whole of $\mathbb{R}^I$, and $\lambda_I H > 0$, as required.

So μ witnesses (i).

3. Corollary If one of the following statements is true, so is the other:

(i) there is a non-metrizable compact Hausdorff space X such that $C(X)$ carries a probability measure for which all the functionals $\hat{x}\restriction C(X)$ are measurable and for which all non-empty $\mathfrak{T}_p$-open sets are non-negligible;

(ii) there is a probability space (Z, Σ, ν) with a $Y \subseteq \mathcal{L}^0(\Sigma)$ such that Y is compact and non-metrizable for $\mathfrak{T}_p$, and any pair of distinct members of Y differ on a non-negligible set.

Remark Whether (ii) is true is the "problem of A.Bellow" referred to in the title.

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4. Proposition 5 of FREMLIN N82 can be extended as follows.

Proposition Let (X, Σ, μ) be a probability space and K a $\mathfrak{T}_p$-compact subset of $\mathcal{L}^0 = \mathcal{L}^0(\Sigma)$ separated by the topology $\mathfrak{T}_m$ of convergence in measure. If an infinite compact Hausdorff space L is a continuous image of a closed subset of K, then L has a non-trivial convergent sequence.

proof We may suppose that L is a continuous image of K itself; let $f : K \to L$ be a continuous surjection. **?** If L has no non-trivial convergent sequence, let $\mathcal{K}$ be the set of closed sets $K' \subseteq K$ such that $f[K']$ is infinite. Then, as in §5 of FREMLIN N82, the intersection of any non-increasing sequence in $\mathcal{K}$ belongs to $\mathcal{K}$, so there is a $K^* \in \mathcal{K}$ such that $\Delta(K') = \Delta(K^*)$ for any $K' \subseteq K^*$ belonging to $\mathcal{K}$. Now $f[K^*]$ is an infinite compact space with no non-trivial convergent subsequences, so there is a continuous surjection $g : f[K^*] \to [0, 1]$. Shrinking K^* if necessary, we may suppose that $gf : K^* \to [0, 1]$ is irreducible. But in this case there is a non-empty open set $U \subseteq [0, 1]$ such that $\Delta(K^* \cap (gf)^{-1}[U]) < \Delta(K^*)$; and in this case there is a $K' \in \mathcal{K}$ such that $K' \subseteq K^* \cap (gf)^{-1}[U]$, so that $\Delta(K') < \Delta(K^*)$, which is impossible. **X**

6. Tidying up the ideas of Proposition 6 in FREMLIN N82, we get the following.

Proposition Suppose that (X, Σ, μ) is a probability space and $K \subseteq \mathcal{L}^0(\Sigma)$ a $\mathfrak{T}_p$-compact set which is separated by $\mathfrak{T}_m$ but not metrizable for $\mathfrak{T}_p$. Then there are a closed subset L_0 of K and a continuous surjection $f : L_0 \to [0, 1]^{\omega_1}$.

proof Take L as in §6 of FREMLIN N82, so that we have an uncountable set $A \subseteq X$ and two functions v, w such that $v(t) < w(t)$ for every $t \in A$ and, for every $B \subseteq A$, there is an $x \in K$ such that $x \restriction B = v \restriction B$, $x \restriction A \setminus B = w \restriction A \setminus B$. We may suppose that $\#(A) = \omega_1$; setting $L_0 = \{x : x \in K, x(t) \in \{v(t), w(t)\}$ for every $t \in A\}$ we have a continuous surjection from L_0 onto $\{0,1\}^{\omega_1}$. Now of course there is a continuous surjection from $\{0,1\}^{\omega_1}$ onto $[0,1]^{\omega_1}$, so we have the result.

7. Accordingly we may add to the list in Proposition 10 of FREMLIN N82, as follows:

Corollary Suppose that there are a probability space (X, Σ, μ) and a $\mathfrak{T}_p$-compact set $K \subseteq \mathcal{L}^0(\Sigma)$ which is separated by $\mathfrak{T}_m$ but is not $\mathfrak{T}_p$-metrizable. Then

(f) every infinite compact Hausdorff space of weight at most ω_1 has a non-trivial convergent sequence.

proof Put §5 and §6 together.

8. Remark A.Dow has observed that in any model obtained by adding random reals to a model of ZFC+CH, there will be an infinite compact Hausdorff space of weight ω_1 which has no non-trivial convergent sequence; so that in such models the problem has a positive answer.

References

Fremlin D.H. [n82] 'On a problem of A.Bellow', Note of 11.5.82