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Notes on Farah's 'Analytic Quotients'

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1. Representations of Boolean homomorphisms

1A. Definition Let $\mathcal{I}$, $\mathcal{J}$ be ideals in $\mathcal{P}\mathbb{N}$, and $\pi : \mathcal{P}\mathbb{N}/\mathcal{I} \to \mathcal{P}\mathbb{N}/\mathcal{J}$ a Boolean homomorphism. I say that a function $F : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ **represents** π on $A \subseteq \mathcal{P}\mathbb{N}$ if $\pi a^\bullet_{\mathcal{I}} = F(a)^\bullet_{\mathcal{J}}$ for every $a \in A$. (If $A = \mathcal{P}\mathbb{N}$, I will say simply that 'F represents π'. In the terminology of Farah 00, F is a 'lifting' of π.)

Note that if $\mathcal{I}'$ is another ideal, included in $\mathcal{I}$, and F represents $\pi : \mathcal{P}\mathbb{N}/\mathcal{I} \to \mathcal{P}\mathbb{N}/\mathcal{J}$, then F also represents $\pi\psi : \mathcal{P}\mathbb{N}/\mathcal{I}' \to \mathcal{P}\mathbb{N}/\mathcal{J}$, where $\psi : \mathcal{P}\mathbb{N}/\mathcal{I}' \to \mathcal{P}\mathbb{N}/\mathcal{I}$ is the canonical map defined by the formula

$$\psi a^\bullet_{\mathcal{I}'} = a^\bullet_{\mathcal{I}} \text{ for every } a \subseteq \mathbb{N}.$$

1B. Lemma Suppose that $K \subseteq \mathcal{P}\mathbb{N}$ is a set of measure greater than $\frac{1}{2}$ for the usual measure μ on $\mathcal{P}\mathbb{N}$. Then there is a continuous function $f : \mathcal{P}\mathbb{N} \to K$ such that $a \triangle f(a) \in K$ for every $a \subseteq \mathbb{N}$.

proof We may suppose that K is compact. For $a \subseteq \mathbb{N}$ set $a \triangle K = \{a \triangle b : b \in K\}$. Then $\mu(a \triangle K) = \mu K > \frac{1}{2}$, so $\mu(K \cap (a \triangle K)) > 0$ for every a. For $\sigma \in \bigcup_{n\in\mathbb{N}} \{0,1\}^n$ set $U_\sigma = \{a : a \subseteq \mathbb{N},\ a \cap n = \{i : \sigma(i) = 1\}\}$. Choose V_σ inductively, for $\sigma \in \bigcup_{n\in\mathbb{N}} \{0,1\}^n$, as follows. $V_\emptyset = \mathcal{P}\mathbb{N}$. Given that V_σ is an open-and-closed set and that $\mu(U_\sigma \cap K \cap (a \triangle K)) > 0$ for every $a \in V_\sigma$, then

$$G = \{a : \mu(U_{\sigma^\frown 0} \cap K \cap (a \triangle K)) > 0\}, \quad H = \{a : \mu(U_{\sigma^\frown 1} \cap K \cap (a \triangle K)) > 0\}$$

are open sets and $V_\sigma \subseteq G \cup H$, so there are disjoint open-and-closed sets $V_{\sigma^\frown 0} \subseteq G$, $V_{\sigma^\frown 1} \subseteq H$ with union V_σ. Continue.

Define $f : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ by saying that $f(a) \in U_\sigma$ if $a \in V_\sigma$. Because every V_σ is open, f is continuous. If $a \subseteq \mathbb{N}$, then for every $n \in \mathbb{N}$ we have $a \in V_{f(a)\restriction n}$, so $U_{f(a)\restriction n} \cap K \cap (a \triangle K) \neq \emptyset$, and $f(a) \in K \cap (a \triangle K)$; that is, $f(a)$ and $a \triangle f(a)$ both belong to K.

1C. Proposition Suppose that $\mathcal{I}$ and $\mathcal{J}$ are ideals in $\mathcal{P}\mathbb{N}$. Let $\pi : \mathcal{P}\mathbb{N}/\mathcal{I} \to \mathcal{P}\mathbb{N}/\mathcal{J}$ be a Boolean homomorphism which can be represented by a measurable function $H : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$. Then π can also be represented by a continuous function $G : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$.

proof Let $K \subseteq \mathbb{N}$ be a set of measure greater than $\frac{1}{2}$ such that $H\restriction K$ is continuous. Let $f : \mathcal{P}\mathbb{N} \to K$ be a continuous function such that $a \triangle f(a) \in K$ for every $a \subseteq \mathbb{N}$, as in Lemma 1B. Set $G(a) = H(f(a)) \triangle H(a \triangle f(a))$ for $a \subseteq \mathbb{N}$; this works.

1D. Definitions (a) If ν is a submeasure on $\mathcal{P}\mathbb{N}$, write $\nu_\infty(a) = \inf_{n\in\mathbb{N}} \nu(a \setminus n)$, for $a \subseteq \mathbb{N}$, and $\mathrm{Exh}(\nu)$ for $\{a : a \subseteq \mathbb{N},\ \nu_\infty(a) = 0\}$.

Note that a submeasure ν defined on $\mathcal{P}\mathbb{N}$ is lower semi-continuous, with respect to the usual topology on $\mathcal{P}\mathbb{N}$, iff $\nu a = \sup\{\nu I : I \subseteq a \text{ is finite}\}$ for every $a \subseteq \mathbb{N}$.

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(b) An ideal $\mathcal{I}$ of $\mathcal{P}\mathbb{N}$ is a **P-ideal** if it includes $[\mathbb{N}]^{<\omega}$ and for every sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{I}$ there is an $a\in\mathcal{I}$ such that $a\setminus a_n$ is finite for every $n\in\mathbb{N}$.

1E. Proposition (see Farah 00, 1.1.5) Let ν be a submeasure on $\mathcal{P}\mathbb{N}$.

(a)(i) ν_∞ is a submeasure, and $(\nu_\infty)_\infty=\nu_\infty$.

(ii) $\mathcal{I}=\mathrm{Exh}(\nu)$ is an ideal in $\mathcal{P}\mathbb{N}$ including $[\mathbb{N}]^{<\omega}$.

(iii) On $\mathfrak{A}=\mathcal{P}\mathbb{N}/\mathcal{I}$ we have a strictly positive submeasure $\bar\nu$ defined by setting $\bar\nu a^\bullet=\nu_\infty(a)$ for every $a\subseteq\mathbb{N}$.

(b) If ν is Borel measurable, so is ν_∞.

(c) Now suppose that ν is lower semi-continuous.

(i) ν is countably subadditive.

(ii) $\mathcal{I}$ is an analytic P-ideal.

(iii) $\bar\nu$ is a Maharam submeasure and $\mathfrak{A}$ is complete under the metric defined by $\bar\nu$.

1F. Proposition Let ν, ν' be lower semi-continuous submeasures on $\mathcal{P}\mathbb{N}$. Then the following are equivalent:

(i) $\mathrm{Exh}(\nu)\subseteq\mathrm{Exh}(nu')$;

(ii) for every $\epsilon>0$ there is a $\delta>0$ such that $\nu'_\infty(a)\le\delta$ whenever $\nu_\infty(a)\le\epsilon$.

proof Of course (ii)$\Rightarrow$(i). Suppose that (ii) is false; that there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{P}\mathbb{N}$ such that $\nu_\infty(a_n)\le 2^{-n}$ but $\nu'_\infty(a_n)>\epsilon$ for every $n\in\mathbb{N}$. Choose $\langle k_n\rangle_{n\in\mathbb{N}}$ inductively so that

$$\nu(a_n\setminus k_n)\le 2^{-n+1},\quad \nu'(a_n\cap k_{n+1}\setminus k_n)>\epsilon$$

for every n. Set $b=\bigcup_{n\in\mathbb{N}}a_n\cap k_{n+1}\setminus k_n$; then $b\in\mathrm{Exh}(\nu)\setminus\mathrm{Exh}(\nu')$.

1G. Theorem (Solecki 96, Fremlin n99) If $\mathcal{I}$ is an analytic P-ideal in $\mathcal{P}\mathbb{N}$, there is a lower semi-continuous submeasure ν on $\mathcal{P}\mathbb{N}$ such that $\mathcal{I}=\mathrm{Exh}(\nu)$.

1H. Proposition Suppose that $\mathcal{I}\lhd\mathcal{P}\mathbb{N}$ is an analytic P-ideal. Then any lower semi-continuous submeasure ν on $\mathcal{P}\mathbb{N}$ such that $\mathcal{I}=\mathrm{Exh}(\nu)$ gives rise to the same uniformity on $\mathfrak{A}=\mathcal{P}\mathbb{N}/\mathcal{I}$.

proof Use 1F.

1I. Definition A set $A\subseteq\mathcal{P}\mathbb{N}$ is **hereditary** if $a\in A$ whenever $a\subseteq b\in A$.

1J. Lemma (a) If $W\subseteq\mathcal{P}\mathbb{N}$, the following are equivalent:

(i) W is comeager;

(ii) there are a strictly increasing sequence $\langle n_i\rangle_{i\in\mathbb{N}}$ in $\mathbb{N}$ and a $\langle c_i\rangle_{i\in\mathbb{N}}$ such that $c_i\subseteq n_{i+1}\setminus n_i$ for every i and

$$W\supseteq\{a:a\subseteq\mathbb{N},\ \{i:a\cap n_{i+1}\setminus n_i=c_i\}\text{ is infinite}\}.$$

(b) If $A\subseteq\mathcal{P}\mathbb{N}$ is hereditary, then the following are equivalent:

(i) A is not meager;

(ii) whenever $\langle K_i\rangle_{i\in\mathbb{N}}$ is a disjoint sequence of finite subsets of $\mathbb{N}$, and $c_i\subseteq K_i$ for every i, then there is an infinite set $M\subseteq\mathbb{N}$ such that $\bigcup_{i\in M}c_i\in A$.

(c) If A, $B\subseteq\mathcal{P}\mathbb{N}$ are hereditary and nowhere meager, then $A\cap B$ is nowhere meager.

(d) If $\mathcal{K}\lhd\mathcal{P}\mathbb{N}$ is a non-meager ideal with the Baire property including $[\mathbb{N}]^{<\omega}$, it is the whole of $\mathcal{P}\mathbb{N}$.

1K. Proposition (Farah 00, 1.1.1) Let $\mathcal{I}$, $\mathcal{J}$ be ideals in $\mathcal{P}\mathbb{N}$, and $\pi : \mathcal{P}\mathbb{N}/\mathcal{I} \to \mathcal{P}\mathbb{N}/\mathcal{J}$ a Boolean homomorphism. If π has a representation with the Baire property, it has a continuous representation.

proof Let $H : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ be a function with the Baire property representing π. Let $Q \subseteq \mathcal{P}\mathbb{N}$ be a comeager set on which H is continuous. Let $\langle n_i \rangle_{i\in\mathbb{N}}$, $\langle c_i \rangle_{i\in\mathbb{N}}$ be such that $\langle n_i \rangle_{i\in\mathbb{N}}$ is a strictly increasing sequence in $\mathbb{N}$, $n_0 = 0$, $c_i \subseteq n_{i+1} \setminus n_i$ for every i, and $\{a : a \cap n_{i+1} \setminus c_i \text{ for infinitely many } i\} \subseteq Q$. Set

$$a_0^* = \bigcup_{i\in\mathbb{N}} n_{2i+1} \setminus n_{2i}, \quad c_0^* = \bigcup_{i\in\mathbb{N}} c_{2i+1},$$

$$a_1^* = \bigcup_{i\in\mathbb{N}} n_{2i+2} \setminus n_{2i+1}, \quad c_1^* = \bigcup_{i\in\mathbb{N}} c_{2i},$$

$$G(a) = \bigl(H((a \cap a_0^*) \cup c_0^*) \cap H(a_0^*)\bigr) \cup \bigl(H((a \cap a_1^*) \cup c_1^*) \cap H(a_1^*)\bigr)$$

for $a \subseteq \mathbb{N}$; then G is continuous and represents π.

1L. Definition A function $H : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ is **asymptotically additive** if there are disjoint sequences $\langle K_i \rangle_{i\in\mathbb{N}}$, $\langle L_i \rangle_{i\in\mathbb{N}}$ in $[\mathbb{N}]^{<\omega}$ and functions $H_i : \mathcal{P}K_i \to \mathcal{P}L_i$, for $i \in \mathbb{N}$, such that $H(a) = \bigcup_{i\in\mathbb{N}} H_i(a \cap K_i)$ for every $a \subseteq \mathbb{N}$.

1M. Proposition (Farah 00, 1.4.2 and 3.10.6) Let $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ be ideals of $\mathcal{P}\mathbb{N}$ including $[\mathbb{N}]^{<\omega}$, of which $\mathcal{J}$ is an analytic P-ideal and $\mathcal{K}$ is non-meager. Let $\pi : \mathcal{P}\mathbb{N}/\mathcal{I} \to \mathcal{P}\mathbb{N}/\mathcal{J}$ be a Boolean homomorphism which can be represented on $\mathcal{K}$ by a continuous function $H : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$. Then there is an asymptotically additive function $G : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ also representing π on $\mathcal{K}$.

proof Let ν be a lower semi-continuous submeasure on $\mathcal{P}\mathbb{N}$ such that $\mathcal{J} = \operatorname{Exh}(\nu)$ (Theorem 1F).

(a) Because $[\mathbb{N}]^{<\omega} \subseteq \mathcal{I} \cap \mathcal{K}$, $H(a) \in \operatorname{Exh}(\nu)$ for every finite set $a \subseteq \mathbb{N}$. Because $\operatorname{Exh}(\nu)$ is a P-ideal, there is a $d_0 \in \operatorname{Exh}(\nu)$ such that $H(a) \setminus d_0$ is finite for every $a \in [\mathbb{N}]^{<\omega}$; now $a \mapsto H(a) \setminus d_0$ is another a continuous function representing π on $\mathcal{K}$, and in addition takes finite sets to finite sets.

We may therefore suppose, without losing generality, that $H(a)$ is finite for every $a \in [\mathbb{N}]^{<\omega}$.

(b) For any $n \in \mathbb{N}$ and $\epsilon > 0$ there are a $k \geq n$ and a $c \subseteq k \setminus n$ such that

$\nu((H(a) \triangle H(b)) \setminus k) \leq \epsilon$ whenever a, $b \subseteq \mathbb{N}$, $a \cap k \setminus n = c$ and $a \triangle b \subseteq n$.

P For each $k \in \mathbb{N}$, set

$$V_k = \{a : a \subseteq \mathbb{N},\ \exists\, s,\ t \subseteq n,\ \nu((H(a \triangle s) \triangle H(a \triangle t)) \setminus k) > \epsilon\}.$$

Because H is continuous and ν is lower semi-continuous, V_k is open. If $a \in \mathcal{K}$ then $a \triangle s$, $a \triangle t \in \mathcal{K}$ and

$$H(a \triangle s)^\bullet = \pi(a \triangle s)^\bullet = \pi(a \triangle t)^\bullet = H(a \triangle t)^\bullet,$$

so $\nu_\infty((H(a \triangle s) \triangle H(a \triangle t)) = 0$ and there is a $k \in \mathbb{N}$ such that $\nu(((H(a \triangle s) \triangle H(a \triangle t)) \setminus k) \leq \epsilon$. Because $\mathcal{P}n$ is finite, there is a $k \in \mathbb{N}$ such that $\nu(((H(a \triangle s) \triangle H(a \triangle t)) \setminus k) \leq \epsilon$ for all s, $t \subseteq n$, that is, $a \notin V_k$. So $\bigcap_{k\in\mathbb{N}} V_k \cap \mathcal{K}$ is empty; because $\mathcal{K}$ is non-meager, $\bigcap_{k\in\mathbb{N}} V_k$ is not comeager, and there must be some $l \in \mathbb{N}$ such that V_l is not dense. Let $m \geq n$, $c' \subseteq m$ be such that $V_l \cap \{a : a \cap m = c'\} = \emptyset$.

Try $k = \max(l, m)$, $c = c' \setminus n \subseteq k \setminus n$. If a, $b \subseteq \mathbb{N}$ and $a \cap k \setminus n = c$, $a \triangle b \subseteq n$, then

$$a' = (a \setminus m) \cup c' \notin G_l \supseteq G_k$$

and $s = a \triangle a'$, $t = b \triangle a'$ are both subsets of n. So

$$\nu((H(a) \triangle H(b)) \setminus k) = \nu((H(a' \triangle s) \triangle H(a' \triangle t)) \setminus k) \leq \epsilon,$$

as required. **Q** (Cf. Farah 00, p. 97, Claim 3.5.)

(c) Choose sequences $\langle n_i \rangle_{i \in \mathbb{N}}$, $\langle k_i \rangle_{i \in \mathbb{N}}$ and $\langle c_i \rangle_{i \in \mathbb{N}}$ inductively, as follows. $n_0 = 0$. Given n_i, take $k_i \geq n_i$ and $c_i \subseteq k_i \setminus n_i$ such that

$\nu((H(a) \triangle H(b)) \setminus k_i) \leq 2^{-i}$ whenever a, $b \subseteq \mathbb{N}$, $a \cap k_i \setminus n_i = c_i$ and $a \triangle b \subseteq n_i$.

Now take $n_{i+1} > k_i$ so that

$(H(a) \triangle H(b)) \cap k_i = \emptyset$ whenever a, $b \subseteq \mathbb{N}$ and $(a \triangle b) \cap n_{i+1} = \emptyset$;

this will be possible because H is continuous and $\mathcal{P}\mathbb{N}$ is compact.

(d) Because $\mathcal{K}$ is a non-meager ideal, there is an infinite set $M \subseteq \mathbb{N}$ such that $\bigcup_{i \in M} c_i \in \mathcal{K}$, by Lemma 1J(b). Enumerate M in increasing order as $\langle m_i \rangle_{i \in \mathbb{N}}$, and set

$$n'_i = n_{m_i}, \quad c'_i = c_{m_i}, \quad k'_i = k_{m_i}$$

for $i \in \mathbb{N}$; then $\langle n'_i \rangle_{i \in \mathbb{N}}$ is strictly increasing, $c'_i \subseteq k'_i \setminus n'_i$ for every i,

if a, $b \subseteq \mathbb{N}$, $a \triangle b \subseteq n'_i$, $a \cap n'_{i+1} \setminus n'_i = c'_i$ then $\nu((H(a) \triangle H(b)) \setminus k'_i) \leq 2^{-i}$

(because $a \cap n_{m_i+1} \setminus n_{m_i} = c_{m_i}$ and $2^{-m_i} \leq 2^{-i}$),

if a, $b \subseteq \mathbb{N}$ and $(a \triangle b) \cap n'_{i+1} = \emptyset$, then $(H(a) \cap H(b)) \cap k'_i = \emptyset$

(because $(a \triangle b) \cap n_{m_i+1} = \emptyset$); and, moreover, $\bigcup_{i \in \mathbb{N}} c'_i \in \mathcal{K}$.

(e) Fix any $F : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ representing π. Set

$$a^*_0 = \bigcup_{i \in \mathbb{N}} n'_{2i+1} \setminus n'_{2i}, \quad c^*_0 = \bigcup_{i \in \mathbb{N}} c'_{2i+1},$$

$$a^*_1 = \bigcup_{i \in \mathbb{N}} n'_{2i+2} \setminus n'_{2i+1}, \quad c^*_1 = \bigcup_{i \in \mathbb{N}} c'_{2i},$$

$$d = (F(a^*_0) \cup H(c^*_1)) \cap ((F(a^*_1) \cup H(c^*_0)) \in \mathrm{Exh}(\nu).$$

Because c^*_0 and c^*_1 belong to $\mathcal{K}$, $H(c^*_l)^\bullet = \pi(c^*_l)^\bullet$ for both l, and $d \in \mathrm{Exh}(\nu)$. Set

$$\tilde{G}(a) = \bigcup_{l=0}^{1} H((a \cap a^*_l) \cup c^*_l) \cap F(a^*_l) \setminus d.$$

If $a \in \mathcal{K}$ then

$$\begin{aligned}\tilde{G}(a) &= \bigcup_{l=0}^{1} H((a \cap a^*_l) \cup c^*_l) \cap F(a^*_l) \setminus d \\ &\equiv_{\nu_\infty} \bigcup_{l=0}^{1} F((a \cap a^*_l) \cup c^*_l) \cap F(a^*_l)\end{aligned}$$

(writing $u \equiv_{\nu_\infty} v$ when $\nu_\infty(u \triangle v) = 0$, and noting that $(a \cap a^*_l) \cup c^*_l \in \mathcal{K}$ for both l, while $d \in \mathrm{Exh}(\nu)$)

$$\equiv_{\nu_\infty} \bigcup_{l=0}^{1} F(a) \cap F(a_l^*)$$
$$\equiv_{\nu_\infty} F(a).$$

So $\tilde{G}$ represents π on $\mathcal{K}$.

(f) Set $l(i) = 0$ for even i, 1 for odd i. For $i \in \mathbb{N}$ set

$$L_i = (F(a^*_{l(i)}) \setminus d) \cap k'_{i+1} \setminus k'_{i-1}$$

(interpreting k'_{-1} as 0). Then $\langle L_i \rangle_{i\in\mathbb{N}}$ is disjoint, because $F(a_0^*) \cap F(a_1^*) \subseteq d$. Set

$$G(a) = \bigcup_{i\in\mathbb{N}} \tilde{G}(a \cap n'_{i+1} \setminus n_i) \cap L_i$$

for $a \subseteq \mathbb{N}$. Then G is asymptotically additive.

(g) Note that if $a \subseteq n'_{i+1} \setminus n'_i$, then

$$G(a) = \tilde{G}(a) \cap k'_{i+1} \setminus k'_{i-1}.$$

P $a \subseteq a^*_{l(i)}$ so $(a \cap a^*_{1-l(i)}) \cup c^*_{1-l(i)} = c^*_{1-l(i)}$ and

$$H((a \cap a^*_{1-l(i)}) \cup c^*_{1-l(i)}) \cap F(a^*_{1-l(i)}) \subseteq d, \quad \tilde{G}(a) = H(a \cup c^*_{l(i)}) \cap F(a^*_{l(i)}) \setminus d,$$

$$\tilde{G}(a) \cap k'_{i+1} \setminus k'_{i-1} \subseteq (F(a^*_{l(i)}) \setminus d) \cap k'_{i+1} \setminus k'_{i-1} = L_i. \text{ } \mathbf{Q}$$

(h) If $a \subseteq \mathbb{N}$ and $i \geq 1$, then

$$\nu((\tilde{G}(a) \triangle G(a)) \cap L_i) \leq 2^{-i+1}.$$

P We know that

$$\begin{aligned} G(a) \cap L_i &= \tilde{G}(a \cap n'_{i+1} \setminus n'_i) \cap L_i \\ &= (H((a \cap n'_{i+1} \setminus n'_i) \cup c^*_{l(i)}) \cap F(a^*_{l(i)}) \setminus d) \cap L_i, \end{aligned}$$

(because $L_i \cap F(a^*_{1-l(i)}) = \emptyset$), while

$$\tilde{G}(a) \cap L_i = H((a \cap a_0^*) \cup c_0^*) \cap L_i.$$

Setting

$$a' = (a \cap a^*_{l(i)}) \cup c^*_{l(i)}, \quad a'' = (a \cap n'_{i+1} \setminus n'_i) \cup c^*_{l(i)},$$

we have

$$(\tilde{G}(a) \triangle G(a)) \cap L_i = (H(a') \triangle H(a'')) \cap L_i.$$

Now if we write $b = (a \cap a^*_{l(i)} \cap n'_{i+1}) \cup c^*_{l(i)}$, $(a' \triangle b) \cap n'_{i+1} = \emptyset$, so $(a' \triangle b) \cap n'_{i+2} = \emptyset$ (since $a^*_{l(i)} \cap n'_{i+2} \setminus n'_{i+1}$ is empty), and

$$(H(a') \triangle H(b)) \cap L_i \subseteq ((H(a') \triangle H(b)) \cap k'_{i+1} = \emptyset$$

by the choice of n'_{i+1} (or, more exactly, of n'_{m_i+1}). On the other hand, $b \triangle a'' \subseteq n'_i$ so $b \triangle a'' \subseteq n'_{i-1}$ (this time because $a^*_{l(i)} \cap n'_i \setminus n'_{i-1}$ is empty), while

$$b \cap k'_{i-1} \setminus n'_{i-1} = c^*_{l(i)} \cap k'_{i-1} \setminus n'_{i-1} = c_{i-1},$$

so

$$\begin{aligned}\nu((\tilde{G}(a) \triangle G(a)) \cap L_i) &= \nu((H(a') \triangle H(a'')) \cap L_i) = \nu((H(b) \triangle H(a'')) \cap L_i) \\ &\le \nu((H(b) \triangle H(a'')) \setminus k'_{i-1}) \le 2^{-i+1},\end{aligned}$$

as required. **Q**

(i) It follows that $\nu_\infty(G(a) \triangle \tilde{G}(a)) = 0$ for every $a \subseteq \mathbb{N}$. **P** Set

$$e = \bigcup_{i \in \mathbb{N}} L_i = (F(a_0^*) \cup F(a_1^*)) \setminus d,$$

so that $\mathbb{N} \setminus e \in \mathrm{Exh}(\nu)$, and

$$\begin{aligned}\nu_\infty(G(a) \triangle \tilde{G}(a)) &= \nu_\infty(G(a) \triangle (e \cap \tilde{G}(a))) = \nu_\infty(\bigcup_{i \in \mathbb{N}} (G(a) \triangle \tilde{G}(a)) \cap L_i) \\ &\le \inf_{j \in \mathbb{N}} \nu(\bigcup_{i \ge j} (G(a) \triangle \tilde{G}(a)) \cap L_i)\end{aligned}$$

(because every L_i is finite)

$$\le \inf_{j \in \mathbb{N}} \sum_{i=j}^{\infty} 2^{-i+1} = 0. \text{ **Q**}$$

Since $\tilde{G}$ represents π on $\mathcal{K}$, so does G.

This completes the proof.

1N. Corollary Let $\mathcal{I}$ be an ideal of $\mathcal{P}\mathbb{N}$ including $[\mathbb{N}]^{<\omega}$, $\mathcal{J} \lhd \mathcal{P}\mathbb{N}$ an analytic P-ideal, and $a \subseteq \mathbb{N}$. Let $\pi : \mathcal{P}\mathbb{N}/\mathcal{I} \to \mathcal{P}\mathbb{N}/\mathcal{J}$ be a Boolean homomorphism which can be represented on $\mathcal{P}a$ by a continuous function $H : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$. Then there is an asymptotically additive function $G : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ also representing π on $\mathcal{P}a$, and we may suppose further that $G(b) \subseteq G(a)$ for every $b \subseteq a$.

proof Let ν be a lower semi-continuous submeasure on $\mathcal{P}\mathbb{N}$ such that $\mathcal{J} = \mathrm{Exh}(\nu)$ (Theorem 1F). If a is finite, this is trivial; we can set $G(d) = \emptyset$ for every d. Otherwise, set $\mathcal{I}_a = \mathcal{I} \cap \mathcal{P}a$, so that $\mathcal{I}_a$ is an ideal in $\mathcal{P}a$ including $[a]^{<\omega}$, and define $\pi_a : \mathcal{P}a/\mathcal{I}_a \to \mathcal{P}\mathbb{N}/\mathrm{Exh}(\nu)$ by setting $\pi_a(b^\bullet_{\mathcal{I}_a}) = \pi(b^\bullet_{\mathcal{I}})$ for every $b \subseteq a$. Then $H \restriction \mathcal{P}a : \mathcal{P}a \to \mathcal{P}\mathbb{N}$ is a continuous representation of π_a, so Proposition 1M tells us that there is an asymptotically additive $G_a : \mathcal{P}a \to \mathcal{P}\mathbb{N}$ representing π_a. Now $b \mapsto G_a(b \cap a) \cap G_a(a) : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ is an asymptotically additive function representing π on $\mathcal{P}a$.

1O. Lemma Let $\mathcal{I}$ and $\mathcal{J}$ be ideals of $\mathcal{P}\mathbb{N}$ and $\pi : \mathcal{P}\mathbb{N}/\mathcal{I} \to \mathcal{P}\mathbb{N}/\mathcal{J}$ be a Boolean homomorphism. Let $\mathcal{K}$ be the set of those $a \subseteq \mathbb{N}$ such that there is a continuous function from $\mathcal{P}\mathbb{N}$ to $\mathcal{P}\mathbb{N}$ representing π on $\mathcal{P}a$. Then $\mathcal{K} \lhd \mathcal{P}\mathbb{N}$.

1P. Lemma (Farah 00, p. 108) Suppose that $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K} \lhd \mathcal{P}\mathbb{N}$ are ideals including $[\mathbb{N}]^{<\omega}$, of which $\mathcal{K}$ is non-meager and $\mathcal{J}$ is an analytic P-ideal. Let $\pi : \mathcal{P}\mathbb{N}/\mathcal{I} \to \mathcal{P}\mathbb{N}/\mathcal{J}$

be a Boolean homomorphism. Suppose that $H : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ is an asymptotically additive function which represents π on $\mathcal{K}$. Then

$$H(a)^\bullet = \pi(a^\bullet) \cap H(\mathbb{N})^\bullet$$

for every $a \subseteq \mathbb{N}$.

proof Let ν be a lower semi-continuous submeasure on $\mathcal{P}\mathbb{N}$ such that $\mathcal{J} = \mathrm{Exh}(\nu)$ (Theorem 1G).

(a) Writing $u \equiv_{\mathcal{J}} v$ if $u \triangle v \in \mathcal{J}$, we have $H(a \cap b) \equiv_{\mathcal{J}} H(a) \cap H(b)$ for all a, $b \subseteq \mathbb{N}$. **P** Let $\langle K_i \rangle_{i\in\mathbb{N}}$, $\langle L_i \rangle_{i\in\mathbb{N}}$ be disjoint sequences of finite sets witnessing that H is asymptotically additive. **?** If $\nu_\infty(H(a \cap b) \triangle (H(a) \cap H(b))) > \epsilon > 0$, then we can find disjoint finite sets $M_i \subseteq \mathbb{N}$ such that

$$\nu((H(a \cap b) \triangle (H(a) \cap H(b))) \cap \bigcup_{j \in M_i} L_j) > \epsilon$$

for every $i \in \mathbb{N}$. Set $K'_i = \bigcup_{j\in M_i} K_i$; then

$$\nu(H(a \cap b \cap K'_i) \triangle (H(a \cap K'_i) \cap H(b \cap K'_i))) \ge \epsilon$$

for every i. But now there is an infinite $M \subseteq \mathbb{N}$ such that $c = \bigcup_{i\in M} K'_i \in \mathcal{K}$, and in this case

$$\nu_\infty(H((a \cap c) \cap (b \cap c)) \triangle (H(a \cap c) \cap H(b \cap c))) \ge \epsilon,$$

which is impossible if H represents π on $\mathcal{K}$. **XQ**

(b) Similarly, $H(\mathbb{N} \setminus a) \equiv_{\mathcal{J}} H(\mathbb{N}) \setminus H(a)$ for every $a \subseteq \mathbb{N}$. Consequently (or directly) $H(a \cup b) \equiv_{\mathcal{J}} H(a) \cup H(b)$ for all a, $b \subseteq \mathbb{N}$.

(c) Fix any $F : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ representing π. For $a \subseteq \mathbb{N}$, set

$$\mathcal{K}_a = \{b : b \subseteq \mathbb{N},\ H(b) \cap (H(a) \triangle F(a)) \in \mathrm{Exh}(\nu)\}.$$

Then $\mathcal{K}_a \lhd \mathcal{P}\mathbb{N}$ (using (a) and (b) above). Also $\mathcal{K} \subseteq \mathcal{K}_a$. **P** If $b \in \mathcal{K}$, then

$H(b) \equiv_{\mathcal{J}} F(b)$ because H represents π on $\mathcal{K}$,

$H(a \cap b) \equiv_{\mathcal{J}} H(a) \cap H(b)$ by (a) above,

$F(a \cap b) \equiv_{\mathcal{J}} F(a) \cap F(b)$.

So

$$\begin{aligned} H(b) \cap (H(a) \triangle F(a)) &= (H(b) \cap H(a)) \triangle (H(b) \cap F(a)) \\ &\equiv_{\mathcal{J}} (H(b) \cap H(a)) \triangle (F(b) \cap F(a)) \\ &\equiv_{\mathcal{J}} H(b \cap a) \triangle F(b \cap a) \in \mathrm{Exh}(\nu) \end{aligned}$$

because $b \cap a \in \mathcal{K}$. So $b \in \mathcal{K}_a$. **Q**

(c) But $\mathcal{K}_a$ is always a Borel set, because $\mathrm{Exh}(\nu)$ is. So $\mathcal{K}_a$ is a non-meager Borel ideal including $[\mathbb{N}]^{<\omega}$, and must be the whole of $\mathcal{P}\mathbb{N}$. In particular, $\mathbb{N} \in \mathcal{K}_a$, that is, $H(\mathbb{N}) \cap (H(a) \triangle F(a)) \in \mathrm{Exh}(\nu)$, and

$$H(a) \equiv_{\mathcal{J}} H(a \cap \mathbb{N}) \equiv_{\mathcal{J}} H(a) \cap H(\mathbb{N}) \equiv_{\mathcal{J}} F(a) \cap H(\mathbb{N}),$$

that is,

$$H(a)^\bullet = \pi(a^\bullet) \cap H(\mathbb{N})^\bullet,$$

as claimed.

1Q. Proposition Let $\mathcal{I}$, $\mathcal{J} \lhd \mathcal{P}\mathbb{N}$ be ideals, and $\pi : \mathcal{P}\mathbb{N}/\mathcal{I} \to \mathcal{P}\mathbb{N}/\mathcal{J}$ a Boolean isomorphism. If $\mathcal{J}$ is analytic and π can be represented by a continuous function, so can π^{-1}.

proof Let $H : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ be a continuous function representing π. Set

$$W = \{(a, b) : a,\ b \subseteq \mathcal{P}\mathbb{N},\ H(a)\triangle b \in \mathcal{J}\}.$$

Then W is an analytic set in $\mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{N}$ and every $b \subseteq \mathbb{N}$ is the second member of an element of W, so by the von Neumann-Jankow selection theorem there is a measurable function $G : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ such that $(G(b), b) \in W$ for every $b \subseteq \mathbb{N}$. Now $\pi(G(b)^\bullet) = b^\bullet$, that is, $G(b)^\bullet = \pi^{-1}(b^\bullet)$, for every $b \subseteq \mathbb{N}$, and G represents π^{-1}. By Proposition 1C, π^{-1} can be represented by a continuous function.

1R. Problem Under what circumstances can we represent a Boolean homomorphism $\pi : \mathcal{P}\mathbb{N}/\mathcal{I} \to \mathcal{P}\mathbb{N}/\mathcal{J}$ by an order-preserving function from $\mathcal{P}\mathbb{N}$ to itself?

2. Entirely non-pathological submeasures

2A. Definition A submeasure ν on a Boolean algebra $\mathfrak{A}$ is **entirely non-pathological** ('non-pathological' in the terminology of Farah 00) if

$$\nu a = \sup\{\mu a : \mu \le \nu \text{ is a non-negative finitely additive functional on } \mathfrak{A}\}$$

for every $a \in \mathfrak{A}$.

Note that if ν is entirely non-pathological, so is its restriction to any principal ideal of $\mathfrak{A}$.

2B. Lemma (Farah 00, 1.7.3) Let I be a finite set and ν the usual measure on $\mathcal{P}I$. Write ν^2 for the product measure on $(\mathcal{P}I)^2$. Let $\delta \in \left]0, \frac{1}{500}\right]$, $A \subseteq \mathcal{P}I$ be such that

$$\nu^2(\{(a, b) : a \in A,\ b \in A\}\triangle\{(a, b) : a \cap b \in A\}) \le \delta,$$

$$\nu(\{a : a \in A\}\triangle\{a : I \setminus a \notin A\}) \le \delta.$$

Then there is a $k \in I$ such that

$$\nu(A\triangle D_k) \le 53\delta, \text{ where } D_k = \{a : k \in a \subseteq I\},$$

$$\nu\{a : a \in A,\ a\triangle J \in I\} \ge \frac{1}{2} - 12\delta \text{ whenever } J \subseteq I \text{ and } k \notin J.$$

proof (a) For any $k \in I$, set $F_k = \{a : a \in A,\ a \cup \{k\} \notin A\}$. Then $\nu F_k \le \frac{4\delta}{1-2\delta}$. **P** Set $L = \{b : b \in A \text{ or } b \cup \{k\} \in A\} \setminus D_k$. Observe that

$$\{b : b \notin L,\ (I \setminus \{k\}) \setminus b \notin L\} \setminus D_k \subseteq \{b : b \notin A,\ I \setminus b \notin A\}$$

has measure at most δ, while $b \mapsto (I \setminus \{k\}) \setminus b$ is a bijection on the set $\mathcal{P}I \setminus D_k$, which has measure $\frac{1}{2}$; so that $2\nu L \ge \frac{1}{2} - \delta$ and $\nu L \ge \frac{1}{4}(1 - 2\delta)$. Now consider $F_k \times L$. If $a \in F_k$ and

$b \in L$ then either $a \cap b \in A$ or $a \cap b \notin A$. For $b \in L$ set $b' = b$ if $b \in A$, $b \cup \{k\}$ otherwise; then $b \mapsto b'$ is a bijection, and $a \cap b = a \cap b'$ for every $a \in F_k$, $b \in L$, so

$$\begin{aligned}\nu^2\{(a,b) : a \in F_k,\ b \in L,\ a \cap b \notin A\} &= \nu^2\{(a,b') : a \in F_k,\ b \in L,\ a \cap b \notin A\} \\ &\le \nu^2\{(a,b') : a \in A,\ b' \in A,\ a \cap b' \notin A\}.\end{aligned}$$

On the other hand, setting $a'' = a \cup \{k\}$ for every $a \in F_k$,

$$\begin{aligned}\nu^2\{(a,b) : a \in F_k,\ b \in L,\ a \cap b \in A\} &= \nu^2\{(a'',b) : a \in F_k,\ b \in L,\ a \cap b \in A\} \\ &\le \nu^2(\{(a,b) : a \cap b \in A\} \setminus \{(a,b) : a \in A,\ b \in A\}).\end{aligned}$$

So

$$\nu^2(F_k \times L) \le \nu^2(A^2 \setminus \{(a,b) : a \cap b \in A\}) + \nu^2(\{(a,b) : a \cap b \in A\} \setminus A^2) \le \delta.$$

So $\nu F_k \le \frac{\delta}{\nu L} \le \frac{4\delta}{1-2\delta}$. **Q**

(b) It will help to know that

$$\nu^2\{(a,b) : a \in A,\ b \in A,\ a \triangle b \in A\} \le 2\delta.$$

P Consider the sets

$$E_1 = \{(a,b) : a \triangle b \in A,\ I \setminus (a \triangle b) \in A\},$$

$$E_2 = \{(a,b) : a \in A,\ b \in A,\ a \cap b \notin A\},$$

$$E_3 = \{(a,b) : I \setminus (a \triangle b) \notin A,\ a \cap (I \setminus (a \triangle b)) \in A\}.$$

Because $(a,b) \mapsto a \triangle b : (\mathcal{P}I)^2 \to \mathcal{P}I$ is inverse-measure-preserving, $\nu^2 E_1 \le \delta$. Because $(a,b) \mapsto (a, I \setminus (a \triangle b)) : (\mathcal{P}I)^2 \to (\mathcal{P}I)^2$ is a bijection,

$$\nu^2 E_3 \le \nu^2(\{(a,b) : a \cap b \in A\} \setminus A^2).$$

But this means that

$$\nu^2 E_2 + \nu^2 E_3 \le \nu^2(A^2 \triangle \{(a,b) : a \cap b \in A\}) \le \delta;$$

and as $\{(a,b) : a \in A,\ b \in A,\ a \triangle b \in A\}$ is included in $E_1 \cup E_2 \cup E_3$, its measure is at most 2δ. **Q**

(c) For $J \subseteq I$ set $A_J = \{a : a \in A,\ a \triangle J \in A\}$. Then $A_\emptyset = A$ and $A_I = \{a : a \in A,\ I \setminus a \in A\}$. Note that $2\nu A \ge \nu\{a : a \in A \text{ or } I \setminus a \in A\} \ge 1 - \delta$, so $\nu A_\emptyset \ge \frac{1-\delta}{2}$, while $\nu A_I \le \delta$. Next, for any $J \subseteq I$,

$$\nu A_J + \nu A_{I \setminus J} \ge \frac{1-2\delta}{2}.$$

P Set

$$E_1 = \{a : a \in A \text{ or } I \setminus a \in A\}, \quad E_2 = \{a : a \triangle J \in A \text{ or } I \setminus (a \triangle J) \in A\}.$$

Then $\nu E_1 \geq 1-\delta$ and $\nu E_2 \geq 1-\delta$ so $\nu(E_1 \cap E_2) \geq 1-2\delta$. Now if $a \in E_1 \cap E_2$, either

$$a \in A,\ a\triangle J \in A \text{ so } a \in A_J$$

$$\text{or } a \in A,\ I \setminus (a\triangle J) \in A \text{ so } a \in A_{I\setminus J}$$

$$\text{or } I \setminus a \in A,\ a\triangle J \in A \text{ so } I \setminus a \in A_{I\setminus J}$$

$$\text{or } I \setminus a \in A,\ I \setminus (a\triangle J) \in A \text{ so } I \setminus a \in A_J.$$

But this means that $\nu(E_1 \cap E_2) \leq 2\nu(A_J \cup A_{I\setminus J})$, and $\nu A_J + \nu A_{I\setminus J} \geq \frac{1-2\delta}{2}$. **Q**

Moreover,

$$\nu A_J \cdot \nu A_{I\setminus J} \leq 5\delta.$$

P Consider $A_J \times A_{I\setminus J}$. If $a \in A_J$ and $b \in A_{I\setminus J}$, then a, $a\triangle J$, b and $b\triangle(I \setminus J)$ all belong to A. Consider the sets

$$E_1 = \{(a,b) : a\triangle J \in A,\ b \in A,\ a\triangle b\triangle J \in A\},$$

$$E_2 = \{(a,b) : a\triangle b\triangle J \notin A,\ a\triangle b\triangle J\triangle I \notin A\},$$

$$E_3 = \{(a,b) : a \in A,\ b\triangle J\triangle I \in A,\ a\triangle b\triangle J\triangle I \in A\}.$$

Because $(a,b) \mapsto (a\triangle J, b)$ is a bijection, $\nu^2 E_1 \leq 2\delta$, by (b); because $(a,b) \mapsto a\triangle b\triangle J$ is inverse-measure-preserving, $\nu^2 E_2 \leq \delta$; and similarly $\nu^2 E_3 \leq 2\delta$, because $(a,b) \mapsto (a, b\triangle J\triangle I)$ is a bijection. But $A_J \times A_{I\setminus J} \subseteq E_1 \cup E_2 \cup E_3$, so

$$\nu A_J \cdot \nu A_{I\setminus J} = \nu^2(A_I \times A_{I\setminus J}) \leq 5\delta,$$

as claimed. **Q**

(d) Set $\gamma = \frac{1}{2}\sqrt{1-84\delta+\delta^2}$. Because $\delta \leq \frac{1}{500}$, $\gamma \geq \frac{1}{2} - 22\delta$. Then, for any $J \subseteq I$, either $\nu A_J \leq \frac{1-2\delta-2\gamma}{4}$ or $\nu A_J \geq \frac{1-2\delta+2\gamma}{4}$. **P** Setting $x = \nu A_J$, $y = \nu A_{I\setminus J}$ we have $x+y \geq \frac{1-2\delta}{2}$ while $xy \leq 5\delta$. So $x^2 - \frac{1-2\delta}{2}x + 5\delta \geq 0$ and $|x - \frac{1-2\delta}{4}| \geq \frac{1}{4}\sqrt{1-84\delta+4\delta^2}$. **Q**

(e) Since $\nu A_\emptyset \geq \frac{1-\delta}{2} > \frac{1-2\delta-2\gamma}{4}$ and $\nu A_I \leq \delta < \frac{1-2\delta+2\gamma}{4}$, there must be some $J \subseteq I$, $k \in I \setminus J$ such that $\nu A_J \geq \frac{1-2\delta+2\gamma}{4}$, $\nu A_{J\cup\{k\}} \leq \frac{1-2\delta-2\gamma}{4}$, so that

$$\gamma \leq \nu(A_J \setminus A_{J\cup\{k\}}) \leq \nu\{a : a \in A,\ a\triangle J \in A,\ a\triangle(J \cup \{k\}) \notin A\}.$$

Since $\nu\{a : a \in A,\ a \cup \{k\} \notin A\} \leq \frac{4\delta}{1-2\delta}$ ((a) above),

$$\nu\{a : a \in A,\ (a\triangle J) \cup \{k\} \in A,\ a\triangle(J \cup \{k\}) \notin A\} \geq \gamma - \frac{4\delta}{1-2\delta}.$$

But (because $k \notin J$) $(a\triangle J) \cup \{k\} \neq a\triangle(J \cup \{k\})$ iff $k \in a$, so

$$\nu\{a : a \in A,\, k \in a\} \ge \gamma - \tfrac{4\delta}{1-2\delta} \ge \frac{1}{2} - 26\delta,$$

and

$$\nu(A\triangle D_k) = \frac{1}{2} + \nu A - 2\nu(A \cap D_k) \le \frac{1}{2} + \frac{1+\delta}{2} - 2(\frac{1}{2} - 26\delta) \le 53\delta,$$

as claimed. (To see that $\nu A \le \frac{1}{2}(1+\delta)$, note that $\nu\{a : a \in A,\, I \setminus a \in A\} \le \delta$.)

(e) As for the final remark, there can be only one k satisfying the condition $\nu(A\triangle\{a : k \in A\}) \le 53\delta$, because $53\delta < \frac{1}{4}$; so the J in the first part of (d) might be any subset of I not containing this k, and we must have $\nu A_J \ge \frac{1-2\delta+2\gamma}{4} \ge \frac{1}{2} - 12\delta$ for any such J.

2C. Theorem (Farah 00, 1.7.2) Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and K a finite set. Let $\psi : \mathfrak{A} \to \mathcal{P}K$ be a function. For any submeasure θ on $\mathcal{P}K$, write ϵ_θ for the greater of

$$\sup_{a\in\mathfrak{A}} \theta(\psi(1 \setminus a)\triangle(K \setminus \psi a)),$$

$$\sup_{a,b\in\mathfrak{A}} \theta(\psi(a \cap b)\triangle(\psi(a) \cap \psi(b))).$$

Then there is a Boolean homomorphism $\phi : \mathfrak{A} \to \mathcal{P}K$ such that $\theta(\psi(c)\triangle\phi(c)) \le 1025\epsilon_\theta$ for every $c \in \mathfrak{A}$ and every entirely non-pathological submeasure θ on $\mathcal{P}K$.

proof (a) To begin with (down to the end of (d) below) suppose that $\mathfrak{A}$ is finite. Then we can suppose that $\mathfrak{A} = \mathcal{P}I$ where I is a non-empty finite set. Let ν be the usual measure on $\mathcal{P}I$. For each $k \in K$, write $A_k = \{a : a \subseteq I,\, k \in \psi(a)\}$. Setting $D_i = \{a : i \in a \subseteq I\}$ for $i \in I$, choose for each $k \in K$ an $i_k \in I$ such that $\nu(D_{i_k}\triangle A_k) \le \nu(D_i\triangle A_k)$ for every $i \in I$. Set $\phi(a) = \{k : i_k \in a\}$ for $a \subseteq I$; then $\phi : \mathcal{P}I \to \mathcal{P}K$ is a Boolean homomorphism.

(b) It will be helpful to know that if μ is any probability measure on $\mathcal{P}K$ and a, $b \subseteq I$, then

$$\mu(\psi(a)\triangle\psi(b)\triangle\psi(a\triangle b)) \le 7\epsilon_\mu.$$

P

$$a\triangle b = (I \setminus (a \cap b)) \cap (I \setminus ((I \setminus a) \cap (I \setminus b))),$$

so

$$\begin{aligned}\psi(a)\triangle\psi(b)\triangle\psi(a\triangle b) \subseteq{}& (\psi(a) \cap \psi(b))\triangle\psi(a \cap b)\\ &\cup (K \setminus \psi(a \cap b))\triangle\psi(I \setminus (a \cap b))\\ &\cup (K \setminus \psi(a))\triangle\psi(I \setminus a)\\ &\cup (K \setminus \psi(b))\triangle\psi(I \setminus b)\\ &\cup (\psi(I \setminus a) \cap \psi(I \setminus b))\triangle\psi(I \setminus (a \cup b))\\ &\cup (K \setminus \psi(I \setminus (a \cup b)))\triangle\psi(I \setminus (a \cup b))\\ &\cup (\psi(I \setminus (a \cap b)) \cap \psi(a \cup b))\triangle\psi((I \setminus (a \cap b)) \cap (a \cup b))\end{aligned}$$

has measure at most $7\epsilon_\mu$. **Q**

(c) Now take an entirely non-pathological submeasure θ on $\mathcal{P}K$ and $c \in I$. If $\theta(\psi(c)\triangle\phi(c)) = 0$ then of course it is less than or equal to $1025\epsilon_\theta$, and we can stop. Otherwise, take a non-negative finitely additive functional μ_0 on $\mathcal{P}K$ such that $\mu_0(\psi(c)\triangle\phi(c)) = \theta(\psi(c)\triangle\phi(c))$ and $\mu_0 \le \theta$, and set $\mu = (\mu_0 K)^{-1}\mu_0$, so that μ is a probability measure μ on $\mathcal{P}K$. Write

$$Q_0 = \{k : k \in K,\ \nu\{a : k \in \psi(I \setminus a)\triangle(K \setminus \psi(a))\} \ge \frac{1}{500}\},$$

$$Q_1 = \{k : k \in K,\ \nu^2\{(a,b) : k \in \psi(a \cap b)\triangle(\psi(a) \cap \psi(b))\} \ge \frac{1}{500}\},$$

where ν^2 is the product measure on $(\mathcal{P}I)^2$. Applying Fubini's theorem to $\nu \times \mu$ and $\nu^2 \times \mu$, we see that $\mu Q_0 \le 500\epsilon_\mu$ and $\mu Q_1 \le 500\epsilon_\mu$, so that, taking $Q = Q_0 \cup Q_1$, $\mu Q \le 1000\epsilon_\mu$.

If $k \in K \setminus Q$, then we have

$$\nu(A_k \triangle \{a : I \setminus a \notin A_k\}) = \nu\{a : k \in \psi(I \setminus a)\triangle(K \setminus \psi(a))\} \le \frac{1}{500},$$

$$\begin{aligned}\nu^2(\{(a,b) : a \in A_k,\ b \in A_k\}\triangle\{(a,b) : a \cap b \in A_k\}) \\ = \nu^2\{(a,b) : k \in (\psi(a) \cap \psi(b))\triangle\psi(a \cap b)\} \le \frac{1}{500}.\end{aligned}$$

By Lemma 2B, there is some $i \in I$ such that $\nu(A_k \triangle D_i) \le \frac{53}{500}$, so $\nu(A_k \triangle D_{i_k}) \le \frac{53}{500}$.

(d) $\mu((\psi(c)\triangle\phi(c)) \setminus Q) \le 13\epsilon_\mu$. **P** Consider the set

$$W = \{(a,k) : a \in A_k,\ c \in A_k \triangle D_{i_k},\ a\triangle c \in A_k \iff c \in A_k\}.$$

For any $k \in (\psi(c)\triangle\phi(c)) \setminus Q$, $c \in A_k$ iff $i_k \notin c$, that is, $c \in A_k \triangle D_{i_k}$, so

$$\begin{aligned}\nu\{a : (a,k) \in W\} &= \nu\{a : a \in A_k,\ a\triangle c \in A_k \iff c \in A_k\} \\ &= \nu\{a : a \in A_k,\ a\triangle c \in A_k \iff i_k \notin c\} \\ &\ge \nu D_{i_k} - \nu(D_{i_k} \setminus A_k) - \nu\{a : a\triangle c \in D_{i_k} \setminus A_k\}\end{aligned}$$

(because if $a \in D_{i_k}$, then $a\triangle c \in D_{i_k}$ iff $i_k \notin c$)

$$= \frac{1}{2} - 2\nu(D_{i_k} \setminus A_k) \ge \frac{1}{2} - \frac{106}{500} = \frac{36}{125}.$$

On the other hand, for any $(a,k) \in W$, $k \in \psi(a) \setminus (\psi(c)\triangle\psi(a\triangle c))$, which has measure at most $7\epsilon_\mu$, by (b). So $(\nu \times \mu)(W) \le 7\epsilon_\mu$ and

$$\mu((\psi(c) \setminus \phi(c)) \setminus Q) \le \frac{125}{36} \cdot 7\epsilon_\mu \le 25\epsilon_\mu. \ \mathbf{Q}$$

Consequently $\mu(\phi(c)\triangle\psi(c)) \le 1025\epsilon_\mu$. But as μ_0 is a multiple of μ, it follows that

$$\theta(\phi(c)\triangle\psi(c)) = \mu_0(\phi(c)\triangle\psi(c)) \le 1025\epsilon_{\mu_0} \le 1025\epsilon_\theta,$$

as required.

(e) Thus the theorem is true when $\mathfrak{A}$ is finite. In general, we have for every finite subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ a homomorphism $\phi_{\mathfrak{B}} : \mathfrak{B} \to \mathcal{P}K$ such that $\mu(\phi(b)\triangle\psi(b)) \le 1025\epsilon_\theta$ for every $b \in \mathfrak{B}$ and every entirely non-pathological submeasure θ on $\mathcal{P}K$. (The point is that ϵ_θ, when calculated from $\psi\restriction\mathfrak{B}$, is not greater than when it is calculated from ψ.) But, because $\mathcal{P}K$ is finite, $\langle\phi_{\mathfrak{B}}\rangle_{\mathfrak{B}\text{ is a finite subalgebra of }\mathfrak{A}}$ will have a cluster point ϕ as $\mathfrak{B}$ increases, and this ϕ will be a suitable homomorphism from $\mathfrak{A}$ to $\mathcal{P}K$.

2D. Theorem (Farah 00, 1.8.1) Let ν be an entirely non-pathological lower semi-continuous submeasure on $\mathcal{P}\mathbb{N}$, and $\mathcal{I}$ an ideal of $\mathcal{P}\mathbb{N}$ including $[\mathbb{N}]^{<\omega}$. Let $\pi : \mathcal{P}\mathbb{N}/\mathcal{I} \to \mathcal{P}\mathbb{N}/\operatorname{Exh}(\nu)$ be a Boolean homomorphism which can be represented by a measurable function $H_0 : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$. Then there is an $h : \mathbb{N} \to \mathbb{N}$ such that $\pi a^\bullet = h^{-1}[a]^\bullet$ for every $a \subseteq \mathbb{N}$.

proof (a) By Proposition 1C, we can represent π by a continuous function; by Proposition 1M, we can represent π by an asymptotically additive function H defined from sequences $\langle K_i\rangle_{i\in\mathbb{N}}$, $\langle L_i\rangle_{i\in\mathbb{N}}$ and $\langle H_i\rangle_{i\in\mathbb{N}}$, where $\langle K_i\rangle_{i\in\mathbb{N}}$ and $\langle L_i\rangle_{i\in\mathbb{N}}$ are disjoint sequences of finite sets, $H_i : \mathcal{P}K_i \to \mathcal{P}L_i$ is a function for each i, and $H(a) = \bigcup_{i\in\mathbb{N}} H_i(a\cap K_i)$ for every $a \subseteq \mathbb{N}$.

(b)(i) For $j \le k$ in $\mathbb{N}$, set $M_{jk} = \bigcup_{j\le i<k} K_i$, $N_{jk} = \bigcup_{j\le i<k} L_i$,

$$\epsilon_{jk} = \max\nolimits_{a\subseteq M_{jk}} \nu(H(M_{jk}\setminus a)\triangle(N_{jk}\setminus H(a))).$$

Then $\lim_{j\to\infty}\sup_{k\ge j}\epsilon_{jk} = 0$. **P?** Otherwise, we can find $\delta > 0$ and $j_0 \le k_0 \le j_1 \le k_1 \le \ldots$ such that $\epsilon_{j_ik_i} \ge \delta$ for every i. For each i, choose $a_i \subseteq M_{j_i,k_i}$ such that

$$\nu(H(M_{j_i,k_i}\setminus a_i)\triangle(N_{j_i,k_i}\setminus H(a_i))) \ge \delta.$$

Set $b = \bigcup_{i\in\mathbb{N}} M_{j_i,k_i}$, $a = \bigcup_{i\in\mathbb{N}} a_i$. Note that $H(b) = \bigcup_{i\in\mathbb{N}} H(M_{j_i,k_i})$, $H(a) = \bigcup_{i\in\mathbb{N}} a_i$ and $H(b\setminus a) = \bigcup_{i\in\mathbb{N}} H(M_{j_i,k_i}\setminus a_i)$, because H decomposes along the sequence $\langle K_i\rangle_{i\in\mathbb{N}}$. Note also that, setting $d = \bigcup_{i\in\mathbb{N}} N_{j_i,k_i}$, $d\setminus H(b) \subseteq \mathbb{N}\setminus H(\mathbb{N})$ must belong to $\operatorname{Exh}(\nu)$. But this means that

$$\begin{aligned}\nu_\infty(H(b\setminus a)\triangle(H(b)\setminus H(a))) &= \nu_\infty(H(b\setminus a)\triangle(d\setminus H(a)))\\ &\ge \limsup_{i\to\infty}\nu((H(b\setminus a)\triangle(d\setminus H(a)))\cap N_{j_i,k_i})\\ &= \limsup_{i\to\infty}\nu(H(M_{j_i,k_i}\setminus a_i)\triangle(N_{j_i,k_i}\setminus H(a_i))) \ge \delta,\end{aligned}$$

which is impossible if H represents a Boolean homomorphism into $\mathcal{P}\mathbb{N}/\operatorname{Exh}(\nu)$. **XQ**

(ii) Next, for $j \le k$ in $\mathbb{N}$, set

$$\epsilon'_{jk} = \max\nolimits_{a,b\subseteq M_{jk}} \nu((H(a)\cap H(b))\triangle(H(a)\cap H(b)).$$

Then $\lim_{j\to\infty}\sup_{k\ge j}\epsilon'_{jk} = 0$. **P** The idea of the argument is the same as in (i). **?** If not, we can find $\delta > 0$ and $j_0 \le k_0 \le j_1 \le k_1 \le \ldots$ such that $\epsilon'_{j_ik_i} \ge \delta$ for every i. For each i, choose a_i, $b_i \subseteq M_{j_i,k_i}$ such that

$$\nu(H(a_i\cap b_i)\triangle(H(a_i)\cap H(b_i))) \ge \delta.$$

Set $a = \bigcup_{i\in\mathbb{N}} a_i$, $b \in \bigcup_{i\in\mathbb{N}} b_i$ and observe that

$$\nu((H(a\cap b)\triangle(H(a)\cap H(b)))\cap N_{j_i,k_i})\ge\delta$$

for every i, so that $H(a\cap b)\triangle(H(a)\cap H(b))\notin\mathrm{Exh}(\nu)$ and once again H cannot represent a Boolean homomorphism into $\mathcal{P}\mathbb{N}/\mathrm{Exh}(\nu)$. **XQ**

(c) We can therefore find a strictly increasing sequence $\langle r_i\rangle_{i\in\mathbb{N}}$ such that $\epsilon_{r_ir_{i+1}}\le 2^{-i}$, $\epsilon'_{r_ir_{i+1}}\le 2^{-i}$ for every $i\in\mathbb{N}$. Set $K'_i=M_{r_ir_{i+1}}$, $L'_i=N_{r_ir_{i+1}}$, $G_i=H\restriction\mathcal{P}K'_i$ for each $i\in\mathbb{N}$; then G_i is a function from $\mathcal{P}K'_i$ to $\mathcal{P}L'_i$. Moreover, if $i\in\mathbb{N}$ and a, $b\subseteq K'_i$,

$$\nu(G_i(K'_i\setminus a)\triangle(L'_i\setminus G_i(a)))\le\epsilon_{r_ir_{i+1}}\le 2^{-i},$$

$$\nu(G_i(a\cap b)\triangle(G_i(a)\cap G_i(b)))\le\epsilon'_{r_ir_{i+1}}\le 2^{-i}.$$

But also we know that $\nu\restriction\mathcal{P}L'_i$ is entirely non-pathological, because ν is. By Theorem 2C, there is a Boolean homomorphism $F_i:\mathcal{P}K'_i\to\mathcal{P}L'_i$ such that $\nu(G_i(a)\triangle F_i(a))\le 1025\cdot 2^{-i}$ for every $a\subseteq K'_i$. Set $F(a)=\bigcup_{i\in\mathbb{N}}F_i(a\cap K'_i)$ for $a\subseteq\mathbb{N}$. Then $F(a)\triangle G(a)\in\mathrm{Exh}(\nu)$ for every a, so F represents π.

For each $i\in\mathbb{N}$, we have a function $h_i:L'_i\to K'_i$ such that $F_i(a)=f_i^{-1}[a]$ for $a\subseteq K'_i$. If $L=F[\mathbb{N}]=\bigcup_{i\in\mathbb{N}}L'_i$, $\mathbb{N}\setminus L\in\mathrm{Exh}(\nu)$. Define $h:\mathbb{N}\to\mathbb{N}$ by setting $h(j)=h_i(j)$ if $j\in L'_i$, 0 if $j\in\mathbb{N}\setminus L$; then $h^{-1}[a]\triangle F(a)\subseteq\mathbb{N}\setminus L$ for every $a\subseteq\mathbb{N}$, so h has the property required.

2E. Proposition (Farah 00, 1.9.3 and 1.3.4) Let $\mathcal{I}$ be an ideal of $\mathcal{P}\mathbb{N}$ including $[\mathbb{N}]^{<\omega}$, and ν an entirely non-pathological lower semi-continuous submeasure on $\mathcal{P}\mathbb{N}$.

(a) Suppose that $\pi:\mathcal{P}\mathbb{N}/\mathcal{I}\to\mathcal{P}\mathbb{N}/\mathrm{Exh}(\nu)$ is an injective Boolean homomorphism with a continuous representation $H:\mathcal{P}\mathbb{N}\to\mathcal{P}\mathbb{N}$. Then $\mathcal{I}=\mathrm{Exh}(\nu')$ for some entirely non-pathological submeasure ν' on $\mathcal{P}\mathbb{N}$.

(b) If moreover π is a Boolean isomorphism, then there is a bijection $h:J\to I$, where I and J are cofinite subsets of $\mathbb{N}$, such that $\pi(a^\bullet)=h^{-1}[a]^\bullet$ for every $a\subseteq\mathbb{N}$.

proof (a) By Theorem 2D, there is a function $h_1:\mathbb{N}\to\mathbb{N}$ such that $\pi(a^\bullet)=h_1^{-1}[a]^\bullet$ for every $a\subseteq\mathbb{N}$. Set $\nu'(a)=\nu h_1^{-1}[a]$ for every $a\subseteq\mathbb{N}$. It is easy to see that ν' is a submeasure. It is lower semi-continuous because the map $a\mapsto h_1^{-1}[a]:\mathcal{P}\mathbb{N}\to\mathcal{P}\mathbb{N}$ is continuous. It is entirely non-pathological because if $a\subseteq\mathbb{N}$, there is a non-negative additive functional $\mu:\mathcal{P}\mathbb{N}\to\mathcal{P}\mathbb{N}$ such that $\mu h_1^{-1}[a]=\nu h_1^{-1}[a]$ and $\mu\le\nu$; setting $\mu'(c)=\mu h_1^{-1}[c]$ for $c\subseteq\mathbb{N}$, $\mu'\le\nu'$ is additive and $\mu'(a)=\nu'(a)$.

If $a\in\mathcal{I}$, then $h_1^{-1}[a]\in\mathcal{J}$, so

$$\begin{aligned}\nu'_\infty(a)&=\inf_{k\in\mathbb{N}}\nu'(a\setminus k)=\inf_{k\in\mathbb{N}}\nu(h_1^{-1}[a\setminus k])\\&\le\inf_{l\in\mathbb{N}}\nu(h_1^{-1}[a]\setminus l)=\nu_\infty h_1^{-1}[a]=0.\end{aligned}$$

If $a\notin\mathcal{I}$, then

$$\begin{aligned}\nu'_\infty(a)&=\inf_{k\in\mathbb{N}}\nu(h_1^{-1}[a\setminus k])\ge\inf_{k\in\mathbb{N}}\nu_\infty(h_1^{-1}[a\setminus k])\\&=\inf_{k\in\mathbb{N}}\bar\nu(\pi(a\setminus k)^\bullet)=\bar\nu(\pi(a^\bullet))>0.\end{aligned}$$

So $\mathcal{I}=\mathrm{Exh}(\nu')$ is of the declared form.

(b) By Proposition 1Q, π^{-1} can be represented by a continuous function from $\mathcal{P}\mathbb{N}$ to itself; by Theorem 2D again, there is a function $h_2 : \mathbb{N} \to \mathbb{N}$ such that $\pi^{-1}(b^\bullet) = h_2^{-1}[b]^\bullet$ for every $b \subseteq \mathbb{N}$. Consider $g = h_1 h_2 : \mathbb{N} \to \mathbb{N}$. We have $a^\bullet = g^{-1}[a]^\bullet$, that is, $a \triangle g^{-1}[a] \in \mathcal{I}$ for every $a \subseteq \mathbb{N}$.

Consider $c = \{n : g(n) \neq n\}$. Then we can express c as $c_0 \cup c_1 \cup c_2$ where $g[c_j] \cap c_j = \emptyset$ for each j (Lemma B1). In this case, $c_j \cap g^{-1}[c_j] = \emptyset$, so $c_j \in \mathcal{I}$, for each j, and $c \in \mathcal{I}$. Similarly, $d = \{n : h_2 h_1(n) \neq n\} \in \mathcal{J}$. Set $I_0 = \mathbb{N} \setminus c$, $J_0 = \mathbb{N} \setminus d$; then $h_2(n) \in J_0$ for $n \in I_0$, $h_1(n) \in I_0$ for $n \in J_0$, so $h_0 = h_1 \restriction J_0$ is a bijection between J_0 and I_0. Let h be any extension of h_0 to a bijection between cofinite subsets of $\mathbb{N}$. If $a \subseteq \mathbb{N}$, then

$$\begin{aligned} h^{-1}[a]^\bullet &= (h^{-1}[a] \cap J_0)^\bullet = h_1^{-1}[a \cap I_0]^\bullet \\ &= \pi((a \cap I_0)^\bullet) = \pi(a^\bullet), \end{aligned}$$

as required.

3. Approximating homomorphisms

3A. Definition Let ν be a submeasure on $\mathcal{P}\mathbb{N}$, $\mathcal{I}$ an ideal in $\mathcal{P}\mathbb{N}$, $\epsilon \geq 0$ and $A \subseteq \mathcal{P}\mathbb{N}$. Then I say that a function $H : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ **ϵ-approximates** a homomorphism $\pi : \mathcal{P}\mathbb{N}/\mathcal{I} \to \mathcal{P}\mathbb{N}/\operatorname{Exh}(\nu)$ on A if $\bar{\nu}(H(b)^\bullet_{\operatorname{Exh}(\nu)} \triangle \pi(b^\bullet_{\mathcal{I}})) \leq \epsilon$ for every $b \in A$. (When $A = \mathcal{P}\mathbb{N}$, I will say simply that 'H ϵ-approximates π'.)

Note that if $F : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ is any function representing π, this will be equivalent to saying that $\nu_\infty(H(b) \triangle F(b)) \leq \epsilon$ for every $b \in A$.

3B. Lemma Let ν be a submeasure on $\mathcal{P}\mathbb{N}$, $\mathcal{I}$ an ideal in $\mathcal{P}\mathbb{N}$ including $[\mathbb{N}]^{<\omega}$, $\pi : \mathcal{P}\mathbb{N}/\mathcal{I} \to \mathcal{P}\mathbb{N}/\operatorname{Exh}(\nu)$ a Boolean homomorphism, and $\epsilon \geq 0$. Suppose that $E \subseteq \mathcal{P}\mathbb{N}$ is a set of positive measure and $H : E \to \mathcal{P}N$ a measurable function which ϵ-approximates π on E. Then there is a continuous function $H_1 : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ which 2ϵ-approximates π.

proof (a) There is an $n \in \mathbb{N}$ such that $F = \{a : \exists\, b \in E,\, a \setminus n = b \setminus n\}$ has measure greater than $\frac{1}{2}$. Enumerate $\mathcal{P}n$ as $\langle c_i \rangle_{i<2^n}$ and for $b \in F$ set $k(b) = \min\{i : b \triangle c_i \in F\}$, $H_0(b) = H(b \triangle c_{k(b)})$; note that $\{b : k(b) = i\}$ is measurable for every i, so H_0 is a measurable function. If $b \in F$, then, writing $a = b \triangle c_{k(b)}$,

$$\bar{\nu}(H_0(b)^\bullet \triangle \pi(b^\bullet) = \bar{\nu}(H(a)^\bullet \triangle \pi(a^\bullet)) \leq \epsilon.$$

By Lemma 1B there is a continuous function $f : \mathcal{P}N \to F$ such that $a \triangle f(a) \in F$ for every $a \subseteq \mathbb{N}$. Set

$$H_1(a) = H_0(f(a)) \triangle H_0(a \triangle f(a)).$$

This works.

3C. Lemma Suppose that $\langle a_n \rangle_{n \in \mathbb{N}}$ is an almost disjoint family of subsets of $\mathbb{N}$. Let ν be a Borel measurable submeasure on $\mathcal{P}\mathbb{N}$, $\mathcal{I}$ an ideal of $\mathcal{P}\mathbb{N}$ including $[\mathbb{N}]^{<\omega}$, and $\pi : \mathcal{P}\mathbb{N}/\mathcal{I} \to \mathcal{P}N/\operatorname{Exh}(\nu)$ a Boolean homomorphism. Suppose that $\epsilon \geq 0$ and that $H : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ is a function which ϵ-approximates π and is covered by a sequence of Borel measurable functions. Then for some $n \in \mathbb{N}$ there is a continuous function $G : \mathcal{P}a_n \to \mathcal{P}N$ which 4ϵ-approximates π on $\mathcal{P}a_n$.

proof (Compare Farah 00, 3.11.3.)

(a) To begin with (down to the end of (d) below) I will suppose that $\langle a_n\rangle_{n\in\mathbb{N}}$ is actually a partition of $\mathcal{P}\mathbb{N}$.

? Suppose, if possible, that the result is false.

Let $\langle H_n\rangle_{n\in\mathbb{N}}$ be a sequence of Borel measurable functions from $\mathcal{P}\mathbb{N}$ to itself covering H. The notation will be a little simpler (and will be less plagued by $^\bullet$s) if we suppose that $\nu=\nu_\infty$ (since $(\nu_\infty)_\infty=\nu_\infty$, $\mathrm{Exh}(\nu_\infty)=\mathrm{Exh}(\nu)$ and there is no loss of generality here – of course ν_∞ is still Borel measurable), and take an arbitrary function $F:\mathcal{P}\mathbb{N}\to\mathcal{P}\mathbb{N}$ representing π; then the hypothesis that H ϵ-approximates π just says that $\nu(H(a)\triangle F(a))\le\epsilon$ for every $a\subseteq\mathbb{N}$, and we are trying to find n, G such that $\nu(G(a)\triangle F(a))\le 4\epsilon$ for every $a\subseteq a_n$.

(c) Think of the Radon measure space $\mathcal{P}\mathbb{N}$ as a product $\prod_{n\in\mathbb{N}}\mathcal{P}a_n$, identifying $\langle x_n\rangle_{n\in\mathbb{N}}\in\prod_{n\in\mathbb{N}}\mathcal{P}a_n$ with $\bigcup_{n\in\mathbb{N}}x_n\subseteq\mathbb{N}$. Setting $c_n=\bigcup_{i\ge n}a_i$, we can also think of $\mathcal{P}\mathbb{N}$ as $\mathcal{P}a_0\times\ldots\times\mathcal{P}a_n\times\mathcal{P}c_{n+1}$ and $\mathcal{P}c_n$ as $\mathcal{P}a_n\times\mathcal{P}c_{n+1}$ for each n. The strategy of the proof is to choose inductively $x_n\subseteq a_n$, for $n\in\mathbb{N}$, and compact sets $K_n\subseteq\mathcal{P}c_n$, such that

(α) $\mu_{c_n}(K_n)>0$, where μ_{c_n} is the usual measure on $\mathcal{P}c_n$,

(β) $K_{n+1}\subseteq\{y:y\subseteq c_{n+1},\,x_n\cup y\in K_n\}$,

(γ) if $y\in K_{n+1}$ then $\nu((H_n(z_n\cup y)\cap F(a_n))\triangle F(x_n))>\epsilon$, where $z_n=\bigcup_{i\le n}x_i$

for every $n\in\mathbb{N}$.

(c) The induction starts with $c_0=\mathbb{N}$, $K_0=\mathcal{P}\mathbb{N}$. For the inductive step, given that $\mu_{c_n}(K_n)>0$, start by taking non-negligible compact sets $L\subseteq\mathcal{P}a_n$ and $L'\subseteq\mathcal{P}c_{n+1}$ such that

$$\mu_{c_{n+1}}\{y:y\subseteq c_{n+1},\,x\cup y\in K_n\}>\frac{1}{2}\mu_{c_{n+1}}L'$$

for every $x\in L$. Consider the set

$$W=\{(x,y,w):x\in L,\,y\in L',\,w\subseteq\mathbb{N},\,\nu((H_n(z_{n-1}\cup x\cup y)\cap F(a_n))\triangle w)\le\epsilon\}.$$

Because ν and H_n are Borel measurable, this is a Borel set in $L\times L'\times\mathcal{P}\mathbb{N}$. So

$$V=\{(x,w):x\in L,\,w\subseteq\mathbb{N},\,\mu_{c_{n+1}}\{y:(x,y,w)\in W\}>\frac{1}{2}\mu_{c_{n+1}}(L')\}$$

is also a Borel set. By the von Neumann-Jankow selection theorem (Fremlin 00*, 423G), there is a measurable selector $G_0:U\to\mathcal{P}\mathbb{N}$ for V, where $U=\{x:\exists\,z,\,(x,z)\in V\}$ is an analytic set, therefore measurable.

Now recall that we are assuming that there is no continuous 4ϵ-approximation of π on $\mathcal{P}a_n$; that is, for any continuous $G:\mathcal{P}a_n\to\mathcal{P}\mathbb{N}$, there must be an $x\subseteq\mathcal{P}a_n$ such that $\nu(G(x)\triangle F(x))>4\epsilon$. Applying Lemma 3B, we see that there is an $x\in L$ such that either $x\notin U$ or $x\in U$ and $\nu(G_0(x)\triangle F(x))>2\epsilon$. Set

$$E=\{y:y\in L',\,x\cup y\in K_n,\,\nu(H_n(z_{n-1}\cup x\cup y)\cap F(a_n))\triangle F(x))>\epsilon\};$$

of course E is a Borel set because H_n and ν are Borel measurable functions. The point is that $\mu_{c_{n+1}}(E)>0$.

P case 1 If $x\notin U$, then

$$\mu_{c_{n+1}}\{y:(x,y,w)\in W\}\le\frac{1}{2}\mu_{c_{n+1}}(L')$$

for every $w\subseteq\mathbb{N}$, and, in particular, when $w=F(x)$; so that

$$\begin{aligned}\mu_{c_{n+1}}\{y:y\in L',\,w\subseteq\mathbb{N},\,\nu((H_n(z_{n-1}\cup x\cup y)\cap F(a_n))\triangle F(x))\le\epsilon\}\\ <\frac{1}{2}\mu_{c_{n+1}}(L')<\mu_{c_{n+1}}\{y:y\in L',\,x\cup y\in K_n\},\end{aligned}$$

and the difference E of these sets is non-negligible.

case 2 If $x\in U$ and $\nu(G_0(x)\triangle F(x))>2\epsilon$, set $A=\{y:(x,y,G_0(x))\in W\}$, so that $\mu_{c_{n-1}}(A)>\frac{1}{2}\mu_{c_{n-1}}(L')$ and $A'=\{y:y\in A,\,x\cup y\in K_n\}$ is not negligible. But for any $y\in A'$, $\nu((H_n(z_{n-1}\cup x\cup y)\cap F(a_n))\triangle G_0(x))\le\epsilon$ so $\nu((H_n(z_{n-1}\cup x\cup y)\cap F(a_n))\triangle F(x))>\epsilon$ and $y\in E$. Thus in this case also E is not negligible. **Q**

We can therefore take $x_n=x$ and a non-negligible compact $K_{n+1}\subseteq E$, and the induction will proceed.

(d) At the end of the induction, consider $x=\bigcup_{n\in\mathbb{N}}x_n$. Then there is some $n\in\mathbb{N}$ such that $H(x)=H_n(x)$. We have $x=z_n\cup y$, where $y=\bigcup_{i>n}x_i$. Now $y\in K_{n+1}$. **P** Condition (β) tells us that, for each $m>n$, $K_m\subseteq\{z:z\subseteq c_m,\,\bigcup_{n<i<m}x_i\cup z\in K_{n+1}\}$. So these sets are never empty, and we have for each $m>n$ a z_m such that $y'_m=\bigcup_{n<i<m}x_i\cup z_m\in K_{n+1}$. But not $y'_m\to y$ as $m\to\infty$; since K_{n+1} is closed, $y\in K_{n+1}$. **Q**

By (γ),

$$\begin{aligned}\epsilon&<\nu((H_n(z_n\cup y)\cap F(a_n))\triangle F(x_n))=\nu((H(x)\cap F(a_n))\triangle F(x_n))\\&\le\nu(H(x)\triangle F(x))+\nu((F(x)\cap F(a_n))\triangle F(x_n))=\nu(H(x)\triangle F(x))\end{aligned}$$

because F represents a Boolean homomorphism into $\mathcal{P}\mathbb{N}/\mathrm{Exh}(\nu)$ and $x\cap a_n=x_n$. But this contradicts the hypothesis that H ϵ-approximates π. **X**

This contradiction proves the result in the case when $\langle a_n\rangle_{n\in\mathbb{N}}$ is disjoint and has union $\mathbb{N}$.

(e) In the general case, set $b=\bigcup_{n\in\mathbb{N}}a_n$ and $a'_n=a_n\setminus\bigcup_{i<n}a_i$, so that $\langle a'_n\rangle_{n\in\mathbb{N}}$ is a partition of b. The function $H\restriction\mathcal{P}b$ is covered by a sequence of Borel measurable functions and ϵ-approximates π on $\mathcal{P}b$. We can think of π as a homomorphism from $\mathcal{P}b/\mathcal{P}b\cap\mathcal{I}$ to $\mathcal{P}\mathbb{N}/\mathcal{J}$, and $\mathcal{P}_b\cap\mathcal{I}\supseteq[b]^{<\omega}$. So (a)-(d) tell us that there are $n\in\mathbb{N}$ and a continuous function $G:\mathcal{P}a'_n\to\mathcal{P}\mathbb{N}$ which 4ϵ-approximates π on $\mathcal{P}a'_n$. (Strictly speaking we ought to consider the trivial case of finite b separately.) But, because $\langle a_n\rangle_{n\in\mathbb{N}}$ is almost disjoint, $a_n\setminus a'_n$ is finite; so, setting $G'(a)=G(a\cap a'_n)$ for every $a\subseteq a_n$, we obtain a continuous function G' which 4ϵ-approximates π on $\mathcal{P}a_n$, as required.

3D. Lemma (Farah 00, 3.10.4) Suppose that $\mathcal{I}$, $\mathcal{K}\triangleleft\mathcal{P}\mathbb{N}$ are ideals including $[\mathbb{N}]^{<\omega}$, and that $\mathcal{K}$ is non-meager. Let ν be a lower semi-continuous submeasure on $\mathcal{P}\mathbb{N}$ and $\pi:\mathcal{P}\mathbb{N}/\mathcal{I}\to\mathcal{P}\mathbb{N}/\mathrm{Exh}(\nu)$ a Boolean homomorphism. Suppose that $\epsilon\ge0$ and that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of hereditary subsets of $\mathcal{P}\mathbb{N}$, covering $\mathcal{K}$, such that for each $n\in\mathbb{N}$ there is a function with the Baire property from $\mathcal{P}\mathbb{N}$ to itself which ϵ-approximates π on A_n. Then there is a continuous 12ϵ-approximation to π on $\mathcal{K}$.

proof Start by fixing on a function $F:\mathcal{P}\mathbb{N}\to\mathcal{P}\mathbb{N}$ which represents π.

(a) (FARAH 00, p. 95, Claim 1) We can suppose that for each $n \in \mathbb{N}$ there is a continuous function from $\mathcal{P}\mathbb{N}$ to itself which 2ϵ-approximates π on A_n.

P For each $n \in \mathbb{N}$, let $H_n : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ be a function with the Baire property which ϵ-approximates π on A_n. Let $Q \subseteq \mathcal{P}\mathbb{N}$ be a comeager set such that $H_n \restriction Q$ is continuous for every n. Let $\langle n_i \rangle_{i\in\mathbb{N}}$ and $\langle c_i \rangle_{i\in\mathbb{N}}$ be such that $\langle n_i \rangle_{i\in\mathbb{N}}$ is a strictly increasing sequence in $\mathbb{N}$, $c_i \subseteq n_{i+1} \setminus n_i$ for every i, and

$$\{a : a \cap n_{i+1} \setminus n_i = c_i \text{ for infinitely many } i\} \subseteq Q.$$

Because $\mathcal{K}$ is non-meager, there is an infinite set $M \subseteq \mathbb{N}$ such that $\bigcup_{i\in M} c_i \in \mathcal{K}$. Enumerate M in increasing order as $\langle m_i \rangle_{i\in\mathbb{N}}$, and set $n'_i = n_{m_i}$, $c'_i = c_{m_i}$ for $i \in \mathbb{N}$, so that

$$\{a : a \cap n'_{i+1} \setminus n'_i = c'_i \text{ for infinitely many } i\} \subseteq Q, \quad \bigcup_{i\in\mathbb{N}} c'_i \in \mathcal{K}.$$

Set

$$a_0^* = \bigcup_{i\in\mathbb{N}} n'_{2i+1} \setminus n'_{2i}, \quad c_0^* = \bigcup_{i\in\mathbb{N}} c'_{2i},$$

$$a_1^* = \bigcup_{i\in\mathbb{N}} n'_{2i+2} \setminus n'_{2i+1}, \quad c_1^* = \bigcup_{i\in\mathbb{N}} c'_{2i+1},$$

and for m, $n \in \mathbb{N}$ set

$$G_{mn}(a) = (H_m((a \cap a_0^*) \cup c_0^*) \cap F(a_0^*)) \cup (H_n((a \cap a_1^*) \cup c_1^*) \cap F(a_1^*)),$$

$$A_{mn} = \{a : a \subseteq \mathbb{N},\ (a \cap a_0^*) \cup c_0^* \in A_m,\ (a \cap a_1)^* \cup c_1^* \in A_n\}.$$

Then every G_{mn} is continuous and $\mathcal{K} \subseteq \bigcup_{m,n\in\mathbb{N}} A_{mn}$. Moreover, if $a \in A_{mn}$,

$$\nu_\infty(G_{mn}(a) \triangle F(a)) = \nu_\infty(G_{mn}(a) \triangle (F(a \cap a_0^*) \cup F(a \cap a_1^*)))$$

(because $a_0^* \cup a_1^* = \mathbb{N} \setminus n'_0$ is cofinite)

$$\begin{aligned} &\le \nu_\infty(((H_m((a \cap a_0^*) \cup c_0^*) \cap F(a_0^*)) \triangle F(a \cap a_0^*)) \\ &\qquad + \nu_\infty(((H_n((a \cap a_1^*) \cup c_1^*) \cap F(a_1^*)) \triangle F(a \cap a_1^*)) \\ &\le 2\epsilon + \nu_\infty(((F((a \cap a_0^*) \cup c_0^*) \cap F(a_0^*)) \triangle F(a \cap a_0^*)) \\ &\qquad + \nu_\infty(((F((a \cap a_1^*) \cup c_1^*) \cap F(a_1^*)) \triangle F(a \cap a_1^*)) \end{aligned}$$

(because H_m, H_n ϵ-approximate π on A_m, A_n respectively)

$$= 2\epsilon.$$

So every G_{mn} 2ϵ-approximates π on A_{mn}, and we can replace $\langle (H_n, A_n) \rangle_{n\in\mathbb{N}}$ by $\langle (G_{mn}, A_{mn}) \rangle_{m,n\in\mathbb{N}}$ to get a sequence of the appropriate form. **Q**

(c) (FARAH 00, p. 95, Claim 2) We can suppose that each A_n is either meager or nowhere meager. **P** Let $H_n : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ be a continuous function which 2ϵ-approximates π on A_n. Define A'_n, H'_n as follows. If A_n is meager, set $A'_n = A_n$ and $H'_n = H_n$. Otherwise, there is a non-empty open set where A is nowhere meager, and this includes $\{a : a \cap k_n = c_n\}$ for some $k_n \in \mathbb{N}$, $c_n \subseteq k_n$. Set $A'_n = \{a : a \subseteq \mathbb{N},\ a \setminus k_n \in A_n\}$. Then A'_n is hereditary. Moreover, it is nowhere meager, because it includes the nowhere meager set $\{a : (a \setminus k_n) \cup c_n \in A_n\}$. Set $H'_n(a) = H_n((a \setminus k_n) \cup c_n)$ for $a \subseteq \mathbb{N}$; then H'_n is continuous, and if $a \in A'_n$ then

$$\nu_\infty(H_n'(a)\triangle F(a)) = \nu_\infty(H_n((a\setminus k_n)\cup c_n)\triangle F((a\setminus k_n)\cup c_n)) \le 2\epsilon,$$

so H_n' is a 2ϵ-approximation to π on A_n'. Finally, $A_n \subseteq A_n'$ for every n, so $\mathcal{K} \subseteq \bigcup_{n\in\mathbb{N}} A_n'$. Thus $\langle (A_n', H_n')\rangle_{n\in\mathbb{N}}$ is a sequence of the type required. **Q**

(d) Henceforth, therefore, let us suppose that every A_n is either meager or nowhere meager, and that $H_n : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ is a continuous 2ϵ-approximation to π on A_n.

Set $L = \{n : n \in \mathbb{N},\ A_n \text{ is not meager}\}$. (Of course L cannot be empty, because $\bigcup_{n\in\mathbb{N}} A_n \supseteq \mathcal{K}$ is not meager.) For m, $n \in L$, $Q_{mn} = \{a : \nu_\infty(H_m(a)\triangle H_n(a)) \le 4\epsilon\}$ is a Borel set including $A_m \cap A_n$, which is nowhere meager, by Lemma 1Jc; so Q_{mn} is comeager. Let Q be the comeager set

$$\bigcap_{m,n\in L} Q_{mn} \setminus \bigcup_{n\in\mathbb{N}\setminus L} A_n.$$

Let $\langle n_i\rangle_{i\in\mathbb{N}}$, $\langle c_i\rangle_{i\in\mathbb{N}}$ be such that $n_i < n_{i+1}$ and $c_i \subseteq n_{i+1}\setminus n_i$ for every i, and $\{a : a\cap n_{i+1}\setminus n_i = c_i \text{ for infinitely many } i\} \subseteq Q$. Then there is an infinite $M \subseteq \mathbb{N}$ such that $\bigcup_{i\in\mathbb{N}} c_i \in \mathcal{K}$; enumerate M in increasing order as $\langle m_i\rangle_{i\in\mathbb{N}}$, and set

$$n_i' = n_{m_i}, \quad c_i' = c_{m_i} \text{ for } i \in \mathbb{N},$$

$$a_0^* = \bigcup_{i\in\mathbb{N}} n_{2i+1}'\setminus n_{2i}', \quad c_0^* = \bigcup_{i\in\mathbb{N}} c_{2i+1}',$$

$$a_1^* = \bigcup_{i\in\mathbb{N}} n_{2i+2}'\setminus n_{2i+1}', \quad c_1^* = \bigcup_{i\in\mathbb{N}} c_{2i}'.$$

Fix on any $p \in L$; try

$$H(a) = \bigcup_{l=0}^1 H_p((a\cap a_l^*)\cup c_l^*)\cap F(a_l^*)$$

for $a \subseteq \mathbb{N}$. Of course H is continuous.

(d) If $a \in \mathcal{K}$, there are m_0, $m_1 \in \mathbb{N}$ such that $(a\cap a_l^*)\cup c_l^* \in A_{m_l}$ for both l. But $(a\cap a_l^*)\cup c_l^*$ belongs to Q, so m_0 and m_1 must both belong to L, and

$$\begin{aligned}
\nu_\infty(H(a)\triangle F(a)) &= \nu_\infty(H(a)\triangle \bigcup_{l=0}^1 F((a\cap a_l^*)\cup c_l^*)\cap F(a_l^*))\\
&\le \sum_{l=0}^1 \nu_\infty(H_p((a\cap a_l^*)\cup c_l^*)\cap F(a_l^*))\triangle(F((a\cap a_l^*)\cup c_l^*)\cap F(a_l^*))\\
&\le \sum_{l=0}^1 \nu_\infty(H_p((a\cap a_l^*)\cup c_l^*)\triangle F((a\cap a_l^*)\cup c_l^*))\\
&\le \sum_{l=0}^1 \nu_\infty(H_p((a\cap a_l^*)\cup c_l^*)\triangle H_{m_l}((a\cap a_l^*)\cup c_l^*))\\
&\qquad + \nu_\infty(H_{m_l}((a\cap a_l^*)\cup c_l^*)\triangle F((a\cap a_l^*)\cup c_l^*))\\
&\le \sum_{l=0}^1 4\epsilon + 2\epsilon
\end{aligned}$$

(because m_l and p both belong to L, while $(a\cap a_l^*)\cup c_l^* \in Q_{m_l p}\cap A_{m_l}$)

$$= 12\epsilon.$$

Thus H is a continuous 12ϵ-approximation to π on $\mathcal{K}$.

3E. Lemma Suppose that $\mathcal{I}$, $\mathcal{K} \lhd \mathcal{P}\mathbb{N}$ are ideals including $[\mathbb{N}]^{<\omega}$, of which $\mathcal{K}$ is non-meager. (For instance, $\mathcal{K}$ could be $\mathcal{P}\mathbb{N}$.) Let ν be a lower semi-continuous submeasure on $\mathcal{P}\mathbb{N}$, and $\pi : \mathcal{P}\mathbb{N}/\mathcal{I} \to \mathcal{P}\mathbb{N}/\operatorname{Exh}(\nu)$ a Boolean homomorphism. Suppose that for every $\epsilon > 0$ there is a continuous ϵ-approximation to π on $\mathcal{K}$. Then there is a continuous $H : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ representing π on $\mathcal{K}$.

proof For each $n \in \mathbb{N}$ let $H_n : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ be a continuous 2^{-n}-approximation to π on $\mathcal{K}$. Set

$$W = \{(a, d) : a,\ d \subseteq \mathbb{N},\ \nu_\infty(d, H_n(a)) \leq 2^{-n} \text{ for every } n \in \mathbb{N}\}.$$

Then W is a Borel set in $\mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{N}$, so by the von Neumann-Jankow selection theorem has a selector G with the Baire property. Observe that if $F : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ is any function representing π, then $(a, F(a)) \in W$ for every $a \in \mathcal{K}$; accordingly the analytic set $\operatorname{dom} G$ must include $\mathcal{K}$ and be comeager. Extending G arbitrarily to a function $G' : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$, we have a function G' with the Baire property representing π on $\mathcal{K}$. Applying Lemma 3D with $A_n = \mathcal{K}$, $\epsilon = 0$ we see that there is a continuous function representing π on $\mathcal{K}$.

3F. Lemma Let ν be a lower semi-continuous submeasure on $\mathcal{P}\mathbb{N}$, and $\epsilon > 0$. Let X be a family of pairs (H, d) where $d \subseteq \mathbb{N}$ and $H : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ is continuous, and suppose that $\nu(H(a) \triangle H'(a)) \leq \epsilon$ whenever (H, d) and $(H', d') \in X$ and $a \subseteq d \cap d'$. Then there is a function $G : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$, with the restricted Baire property, such that $\nu(G(a) \triangle H(a)) \leq \epsilon$ whenever $(H, d) \in X$ and $a \subseteq d$.

proof (a) Let $X_0 \subseteq X$ be a countable set such that whenever $(H, d) \in X$ and $k \in \mathbb{N}$, there is an $(H', d') \in X_0$ such that $d' \cap k = d \cap k$ and $H'(s) \cap k = H(s) \cap k$ for every $s \subseteq k$. Let W be the set of pairs $(a, b) \in \mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{N}$ such that

for every $k \in \mathbb{N}$ there are $(H, d) \in X_0$, $k' \geq k$ such that $a \cap k' \subseteq d$ and $H(a \cap k') \cap k = b \cap k$;

then W is a Borel set because all the functions H involved are continuous.

There is therefore a function $G : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$, with the restricted Baire property, such that $(a, G(a)) \in W$ whenever there is any $b \subseteq \mathbb{N}$ such that $(a, b) \in W$.

(b) If $(H, d) \in X$ and $a \subseteq d$ then $(a, H(a)) \in W$. **P** Let $k \in \mathbb{N}$. Then there is a $k' \geq k$ such that $H(a \cap k') \cap k = H(a) \cap k$, because H is continuous. By the choice of X_0, there is an $(H', d') \in X_0$ such that $d' \cap k' = d \cap k'$ and $H'(s) \cap k' = H(s) \cap k'$ for every $s \subseteq k'$. Now

$$d' \supseteq a \cap k',$$

$$H(a) \cap k = H(a \cap k') \cap k = H'(a \cap k') \cap k.$$

As k is arbitrary, $(a, H(a)) \in W$. **Q**

It follows that, under these circumstances, $(a, G(a)) \in W$.

(c) If $(H, d) \in X$ and $a \subseteq d$ then $\nu(H(a) \triangle G(a)) \leq \epsilon$. **P?** Otherwise, there is a $k_0 \in \mathbb{N}$ such that $\nu((H(a) \triangle G(a)) \cap k_0) > \epsilon$. Because H is continuous, there is a $k \geq k_0$ such that

$\nu((H(a\cap k')\triangle G(a))\cap k_0)>\epsilon$ for every $k'\ge k$, so that $\nu((H(a\cap k')\triangle G(a))\cap k)>\epsilon$ for every $k'\ge k$. Now there are $(H',d')\in X_0$ and $k'\ge k$ such that

$$d'\supseteq a\cap k',\quad G(a)\cap k=H'(a\cap k')\cap k.$$

So $\nu((H(a\cap k')\triangle H'(a\cap k'))\cap k)>\epsilon$, which is impossible, because (H,d) and (H',d') both belong to X and $a\cap k'\subseteq d\cap d'$. **XQ**

3G. Lemma (Farah 00, p. 109, Claim 2) Let H, $H':\mathcal{P}N\to\mathcal{P}N$ be asymptotically additive functions and ν a lower semi-continuous measure on $\mathcal{P}\mathbb{N}$. Suppose that $a\subseteq\mathbb{N}$ is such that $\nu_\infty(H(b)\triangle H'(b))=0$ for every $b\subseteq a$. Then for every $\epsilon>0$ there is an $m\in\mathbb{N}$ such that $\nu(H(b)\triangle H'(b))\le\epsilon$ for every $b\subseteq a\setminus m$.

proof For each $k\in\mathbb{N}$, set

$$U_k=\{b:b\subseteq a,\,\nu((H(b)\triangle H'(b))\setminus k)>\epsilon\}.$$

Then each U_k is an open subset of $\mathcal{P}a$, and $\bigcap_{k\in\mathbb{N}}U_k=\emptyset$, so there is some $k\in\mathbb{N}$ such that U_k is not dense; say $m_0\in\mathbb{N}$, $s\subseteq m_0\cap a$ are such that $b\cap m_0\ne s$ for every $b\in U_k$. Now, because H and H' are asymptotically additive, there is an $m\ge m_0$ such that

$$H(b\cup s)=H(b)\cup H(s),\quad H'(b\cup s)=H'(b)\cup H'(s),$$

$$(H(b)\cup H'(b))\cap(H(s)\cup H'(s)\cup k)=\emptyset$$

whenever $b\subseteq\mathbb{N}\setminus m$.

Take any $b\subseteq a\setminus m$. Then

$$H(b\cup s)\triangle H'(b\cup s)=(H(b)\triangle H'(b))\cup(H(s)\triangle H'(s)),$$

so

$$\nu(H(b)\triangle H'(b))=\nu((H(b)\triangle H'(b))\setminus k)\le\nu((H(b\cup s)\triangle H'(b\cup s))\setminus k)\le\epsilon$$

because $b\cup s\notin U_k$. So we have found a suitable m.

4. A class of large ideals

4A. Definition An ideal $\mathcal{K}$ of $\mathcal{P}\mathbb{N}$ is **ccc over** $[\mathbb{N}]^{<\omega}$ if there is no uncountable almost disjoint family $A\subseteq\mathcal{P}\mathbb{N}\setminus\mathcal{K}$.

Note that an ideal including $\mathcal{K}$ will again be ccc over $[\mathbb{N}]^{<\omega}$.

4B. Lemma If $\mathcal{K}\lhd\mathcal{P}\mathbb{N}$ is ccc over $[\mathbb{N}]^{<\omega}$, it is not meager.

proof ? If $\mathcal{K}$ is meager, there is a disjoint sequence $\langle I_n\rangle_{n\in\mathbb{N}}$ of finite sets such that $\bigcup_{n\in J}I_n\notin\mathcal{K}$ for any infinite $J\subseteq\mathbb{N}$. But now if we take any uncountably almost disjoint family $A\subseteq\mathcal{P}\mathbb{N}$, $\{\bigcup_{n\in a}I_n:a\in A\}$ is an uncountable almost disjoint family in $\mathcal{P}\mathbb{N}\setminus\mathcal{K}$. **X**

Remark $\mathcal{K}$ can be negligible; for instance, take any additive functional ν on $\mathcal{P}\mathbb{N}$ extending density (that is, such that $\nu a=\lim_{n\to\infty}\frac{1}{n}\#(a\cap n)$ whenever the limit is defined) and set $\mathcal{K}=\{a:\nu a=0\}$. The point is that $\nu a=\frac{1}{2}$ for almost every a, so that $\mathcal{K}$ is negligible, while any almost disjoint family in $\mathcal{P}\mathbb{N}$ can meet any $\{a:\nu a\ge 2^{-n}\}$ in at most 2^n points.

5. The Open Colouring Axiom

5A. (a) Definition Let X be a set. I will say that a set $S \subseteq [X]^2$ is a **countable union of rectangles** if there are sequences $\langle A_n\rangle_{n\in\mathbb{N}}$, $\langle B_n\rangle_{n\in\mathbb{N}}$ of subsets of X such that $S = \bigcup_{n\in\mathbb{N}}\{\{x, y\} : x \in A_n, y \in B_n, x \neq y\}$.

(b) The **Open Colouring Axiom** or **OCA** is the assertion

whenever X is a set and $S \subseteq [X]^2$ is a countable union of rectangles, then

either there is an uncountable set $A \subseteq X$ such that $[A]^2 \subseteq S$

or there is a sequence $\langle X_n\rangle_{n\in\mathbb{N}}$ of subsets of X, covering X, such that $[X_n]^2 \cap S = \emptyset$ for every $n \in \mathbb{N}$.

5B. Theorem [OCA + $\mathfrak{m} > \omega_1$] (Farah 3.11.1) Let $\mathcal{I}$ and $\mathcal{J}$ be ideals of $\mathcal{P}\mathbb{N}$ such that $[\mathbb{N}]^{<\omega} \subseteq \mathcal{I}$ and $\mathcal{I}$ is an analytic P-ideal. Let $\pi : \mathcal{P}\mathbb{N}/\mathcal{I} \to \mathcal{P}\mathbb{N}/\mathcal{J}$ be a Boolean homomorphism. Write $\mathcal{K}$ for the ideal of those $a \subseteq \mathbb{N}$ such that there is a continuous function from $\mathcal{P}\mathbb{N}$ to itself representing π on $\mathcal{P}a$. Then $\mathcal{K}$ is ccc over $[\mathbb{N}]^{<\omega}$.

proof ? Suppose, if possible, otherwise. Let ν be a lower semi-continuous submeasure on $\mathcal{P}\mathbb{N}$ such that $\mathcal{J} = \mathrm{Exh}(\nu)$ (Proposition 1G). Fix a function $F : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ which represents π.

(a) For $\epsilon > 0$, write $\mathcal{Y}_\epsilon$ for the family of those $a \subseteq \mathbb{N}$ such that π_a cannot be 4ϵ-approximated by any function which is covered by a sequence of Borel measurable functions. Then there are an $\epsilon > 0$ and an uncountable treelike family $\langle a_i\rangle_{i\in I}$ of sets in $\mathcal{Y}_\epsilon$. **P** We are supposing that there is an uncountable almost disjoint family $\langle b_j\rangle_{j\in J}$ in $\mathcal{P}\mathbb{N} \setminus A$. For each $j \in J$, there must be an $n \in \mathbb{N}$ such that π_{b_j} cannot be 2^{-n}-approximated by a continuous function (Lemma 3E), so there are an $\epsilon > 0$ and an uncountable set $J' \subseteq J$ such that π_{b_j} cannot be 32ϵ-approximated by a continuous function for any $j \in J'$. We are assuming that $\mathfrak{m} > \omega_1$, so there is an uncountable almost disjoint family $\langle c_k\rangle_{k\in K}$ such that $\{j : j \in J', b_j \setminus c_k \text{ is finite}\}$ is infinite for every $k \in K$ (Lemma A3). We may suppose that $\#(K) = \omega_1$.

If $j \in J'$, $k \in K$ and $b_j \setminus c_k$ is finite, then $\pi_{b_j\setminus c_k} = \pi_{b_j}$ cannot be 32ϵ-approximated by any continuous function. By Lemma 3C, π_{c_k} cannot be 8ϵ-approximated by any function which is covered by a sequence of Borel measurable functions.

By Lemma A2a, there are treelike families $\langle c'_k\rangle_{k\in K}$ and $\langle c''_k\rangle_{k\in K}$ such that $c'_k \cup c''_k$ is a cofinite subset of c_k for each k. Now if $G' : \mathcal{P}c'_k \to \mathcal{P}\mathbb{N}$ and $G'' : \mathcal{P}c''_k \to \mathcal{P}\mathbb{N}$ are functions, each covered by countably many Borel measurable functions, set $G(c) = G'(c \cap c'_k) \cup G''(c \cap c''_k)$ for every $c \subseteq c_k$; then G can be covered by countably many Borel measurable functions, so cannot 8ϵ-approximate π_{c_k}, and either G' does not 4ϵ-approximate $\pi_{c'_k}$ or G'' does not 4ϵ-approximate $\pi_{c''_k}$. This means that for each $k \in K$ at least one of c'_k, c''_k belongs to $\mathcal{Y}_\epsilon$. So we may take $\langle a_i\rangle_{i\in I}$ to be either $\langle c'_k\rangle_{k\in K'}$ or $\langle c''_k\rangle_{k\in K'}$ for some uncountable $K' \subseteq K$. **Q**

(b) Let $f : I \to \{0, 1\}^{\mathbb{N}}$ be an injective function witnessing that $\langle a_i\rangle_{i\in I}$ is treelike ('neat', in the terminology of Veličkovi'c 93), that is, such that $K_j = \bigcup\{a_i : f(i)(j) = 0\} \cap \bigcup\{a_i : f(i)(j) = 1\}$ is finite for every $j \in \mathbb{N}$. Set $B = \bigcup_{i\in I}[a_i]^\omega$; for $b \in B$ there is only one $i \in I$ such that $b \subseteq a_i$, so we can define $f' : B \to \{0, 1\}^{\mathbb{N}}$ by setting $f'(b) = f(i)$ whenever $b \in B$ and $B \subseteq a_i$. Note that if $f'(b)(j) = 0$ and $f'(b')(j) = 1$ then $b \cap b' \subseteq K_j$.

(c) Let X be the set of pairs (b,c) where $c \subseteq b \in B$. Give X the second-countable topology generated by the function $(b,c) \mapsto (b,c,F(b),F(c),f'(b)) : X \to (\mathcal{P}\mathbb{N})^4 \times \{0,1\}^{\mathbb{N}}$. Let $S \subseteq [X]^2$ be the set of pairs $\{(b,c),(b',c')\}$ such that

(S1) $f'(b) \neq f'(b')$

(S2) $b \cap c' = b' \cap c$

(S3) $\nu((F(b) \cap F(c'))\triangle(F(b') \cap F(c))) > \epsilon$.

Then S is open. **P** S is the set of pairs $\{(b,c),(b',c')\}$ for which there are j, $m \in \mathbb{N}$ and sets b_0, c_0, b'_0, $c'_0 \subseteq K_j$, p_0, q_0, p'_0, $q'_0 \subseteq m$ such that $b_0 \cap c'_0 = b'_0 \cap c_0$, $\nu((p_0 \cap q'_0)\triangle(p'_0 \cap q_0)) > \epsilon$ and

$$f'(b)(j) = 0, \qquad f'(b')(j) = 1,$$

$$b \cap K_j = b_0, \qquad c \cap K_j = c_0, \qquad b' \cap K_j = b'_0, \qquad c' \cap K_j = c'_0,$$

$$F(b) \cap m = p_0, \qquad F(c) \cap m = q_0, \qquad F(b') \cap m = p'_0, \qquad F(c') \cap m = q'_0. \text{ **Q**}$$

(d) There is no uncountable S-homogeneous set. **P?** Suppose that $Y \subseteq X$ is an uncountable S-homogeneous set. Set $d = \bigcup\{c : (b,c) \in Y\}$. If $(b,c) \in Y$, then

$$d \cap b = \bigcup\{b \cap c' : (b',c') \in Y\} = \bigcup\{b' \cap c : (b',c') \in Y\} = c,$$

so $\nu_\infty((F(d) \cap F(b))\triangle F(c)) = 0$. We shall therefore be able to find $m \in \mathbb{N}$ and sets p, $q \subseteq m$ such that, for some uncountable set $Y' \subseteq Y$,

$$\nu(((F(d) \cap F(b))\triangle F(c)) \setminus m) \leq \frac{1}{2}\epsilon,$$

$$F(b) \cap m = p, \qquad F(c) \cap m = q$$

for all $(b,c) \in Y'$. Now take any distinct (b,c), $(b',c') \in Y'$. Then

$$\nu(((F(b) \cap F(c'))\triangle(F(b) \cap F(b') \cap F(d)) \setminus m) \leq \frac{1}{2}\epsilon,$$

$$\nu(((F(b') \cap F(c))\triangle(F(b) \cap F(b') \cap F(d)) \setminus m) \leq \frac{1}{2}\epsilon,$$

so

$$\nu((F(c') \cap F(b))\triangle(F(c) \cap F(b'))) \geq \nu(((F(c') \cap F(b))\triangle(F(c) \cap F(b')) \setminus m) \leq \epsilon.$$

But condition (S3) tells us that $\nu((F(c') \cap F(b))\triangle(F(c) \cap F(b'))) > \epsilon$. **XQ**

(e) OCA now tells us that X can be expressed as $\bigcup_{n\in\mathbb{N}} X_n$ where $[X_n]^2 \cap S = \emptyset$ for every n. We may suppose that the X_n are disjoint. For each $n \in \mathbb{N}$ let X'_n be a countable dense subset of X_n (for the topology on X described above). Fix $i \in I$ such that $f'(b) \neq f(a_i)$ for any $(b,c) \in \bigcup_{n\in\mathbb{N}} X'_n$, that is, $b \cap a_i$ is finite for every $(b,c) \in \bigcup_{n\in\mathbb{N}} X'_n$. Observe that

(*) if $(a,d) \in X_n$ and $a \subseteq a_i$ and $(b,c) \in X'_n$ and $b \cap d = a \cap c$, then $\nu((F(a_i) \cap F(c))\triangle(F(b) \cap F(d))) \leq \epsilon$,

because $\{(a,d),(b,c)\} \notin S$.

For k, $m \in \mathbb{N}$ let $D(k,m)$ be the set of pentuples (n,s_0,s_1,t_0,t_1) such that $n \leq k$ and there is a $(b,c) \in X'_n$ with

$$b \cap m = s_0, \quad c \cap m = s_1, \quad F(b) \cap k = t_0, \quad F(c) \cap k = t_1.$$

Since $b \cap a_i$ is finite whenever $(b, c) \in \bigcup_{n \in \mathbb{N}} X'_n$, we can find a sequence $\langle m_k \rangle_{k \in \mathbb{N}}$ such that $m_0 = 0$ and, for every $k \in \mathbb{N}$,

$a_i \cap m_{k+1} \setminus m_k \neq \emptyset$,

whenever $(n, s_0, s_1, t_0, t_1) \in D(k, m_k)$ there is a $(b, c) \in X'_n$ such that

$$b \cap m = s_0, \quad c \cap m = s_1, \quad F(b) \cap k = t_0, \quad F(c) \cap k = t_1$$

and $b \cap a_i \subseteq m_{k+1}$.

Set $a' = a_i \cap \bigcup_{k \in \mathbb{N}} m_{2k+1} \setminus m_{2k}$ and $a'' = a_i \cap \bigcup_{k \in \mathbb{N}} m_{2k+2} \setminus m_{2k+1}$. Note that, because $a_i \cap m_{k+1} \setminus m_k$ is never empty, both a' and a'' are infinite, and $f'(a') = f'(a'') = f'(a_i)$.

(f) There is a function G' which 2ϵ-approximates $\pi_{a'}$ and can be covered by countably many Borel measurable functions. **P** For each $n \in \mathbb{N}$, consider the set $A_n = \{d : d \subseteq a', (a', d) \in X_n\}$ and the set W_n of pairs (d, e) where $d \subseteq a'$, $e \subseteq \mathbb{N}$ and, for every $k \in \mathbb{N}$, there are $(b, c) \in X'_n$ such that

(†) $a' \cap c = b \cap d$,
$F(b) \cap 2k = F(a') \cap 2k$,
$F(c) \cap 2k = e \cap 2k$,
$\nu(((F(c) \cap F(a')) \triangle (F(b) \cap e)) \cap 2k) \leq \epsilon$.

Then $W_n = \bigcap_{k \in \mathbb{N}} \bigcup_{(b,c) \in X'_n} W_n(k, b, c)$, where $W_n(k, b, c)$ is the set of pairs (d, e) such that the four conditions in (†) are satisfied (of course this is empty if $F(b) \cap 2k \neq F(a') \cap 2k$). Now, for any k, $W_{n,k} = \bigcup_{(b,c) \in X'_n} W_n(k, b, c)$ has open-and-closed vertical sections. So, for any open-and-closed set $U \subseteq \mathcal{P}\mathbb{N}$,

$$W_n^{-1}[U] = \bigcap_{k \in \mathbb{N}} W_{n,k}^{-1}[U] = \bigcap_{k \in \mathbb{N}} \bigcup_{(b,c) \in X'_n} W_n(k, b, c)^{-1}[U].$$

At the same time, for any $k \in \mathbb{N}$ and any $(b, c) \in X'_n$, $W_n(k, b, c)^{-1}[U]$ is either $\{d : a' \cap c = b \cap d\}$ or empty, depending on whether there is or is not an $e \in U$ such that $F(c) \cap 2k = e \cap 2k$ and $\nu(((F(c) \cap F(a')) \triangle (F(b) \cap e)) \cap 2k) \leq \epsilon$ (provided, of course, that $F(b) \cap 2k = F(a') \cap 2k$). So these are all Borel sets and $W_n^{-1}[U]$ is a Borel set for every open-and-closed set $U \subseteq \mathcal{P}\mathbb{N}$. Accordingly W_n has a Borel measurable selector G_n, by Kechris 95[1].

If $d \in A_n$ and $k \geq n$, there is a $(b, c) \in X'_n$ such that

$$b \cap m_{2k+1} = a' \cap m_{2k+1}, \quad c \cap m_{2k+1} = d \cap m_{2k+1},$$

$$F(b) \cap (2k+1) = F(a') \cap (2k+1), \quad F(c) \cap (2k+1) = F(d) \cap (2k+1),$$

because X'_n is dense in X_n. But in this case, by the choice of m_{2k+2}, there is a $(b, c) \in X'_n$ such that

$$b \cap m_{2k+1} = a' \cap m_{2k+1}, \quad c \cap m_{2k+1} = d \cap m_{2k+1},$$

$$F(b) \cap (2k+1) = F(a') \cap (2k+1), \quad F(c) \cap (2k+1) = F(d) \cap (2k+1),$$

$$b \cap a_i \subseteq m_{2k+2}.$$

[1] query

Since, in this case, $b \cap a' \subseteq m_{2k+1}$, we shall have

$$b \cap d = b \cap a' \cap d = b \cap d \cap m_{2k+1} = a' \cap c \cap m_{2k+1} = a' \cap b \cap c = a' \cap c.$$

By (*) above, $\nu((F(a') \cap F(c)) \triangle (F(b) \cap F(d))) \leq \epsilon$, so that (b, c) witnesses that $(d, F(d)) \in W_n$.

What this means is that the domain $\{d : (d, e) \in W_n\}$ of G_n includes A_n. So we can define $G' : \mathcal{P}a' \to \mathcal{P}\mathbb{N}$ by saying that $G'(d) = G_n(d) \cap F(a')$ if $d \in A_n$, and G' will be a function which can be covered by a sequence of Borel measurable functions.

Now, for any $d \in A_n$, consider

$$\nu_\infty(G'(d) \triangle F(d)) = \nu_\infty((G_n(d) \cap F(a')) \triangle ((F(d) \cap F(a')))$$

(because $d \subseteq a'$, so $F(d) \setminus F(a') \in \mathrm{Exh}(\nu)$)

$$\leq \nu(F(a') \cap (G_n(d) \triangle F(d)).$$

For any $k \in \mathbb{N}$, there is a pair $(b, c) \in X'_n$ such that

$a' \cap c = b \cap d$,

$F(b) \cap 2k = F(a') \cap 2k$,

$\nu(((F(a') \cap F(c)) \triangle (F(b) \cap G_n(d)) \cap 2k) \leq \epsilon$.

By (*) again, $\nu((F(a') \cap F(c)) \triangle ((F(b) \cap F(d))) \leq \epsilon$, so that

$$\begin{aligned} 2\epsilon &\geq \nu((F(b) \cap F(d)) \triangle (F(b) \cap G_n(d))) \geq \nu(F(b) \cap (F(d) \triangle G_n(d)) \cap 2k) \\ &= \nu(F(a') \cap (F(d) \triangle G_n(d)) \cap 2k). \end{aligned}$$

As k is arbitrary,

$$\begin{aligned} \nu_\infty(G'(d) \triangle F(d)) &\leq \nu(F(a') \cap (F(d) \triangle G_n(d))) \\ &= \sup_{k \in \mathbb{N}} \nu(F(a') \cap (F(d) \triangle G_n(d)) \cap 2k) \leq 2\epsilon. \end{aligned}$$

As d is arbitrary, we have a suitable function G'. **Q**

(f) The same arguments show that we have a function $G'' : \mathcal{P}a'' \to \mathcal{P}\mathbb{N}$ which 2ϵ-approximates $\pi_{a''}$ and can be covered by a sequence of Borel measurable functions. Assembling these, we have a function $G : \mathcal{P}a_i \to \mathcal{P}\mathbb{N}$ which 4ϵ-approximates π_a and which can be covered by a sequence of Borel measurable functions. (Set $G(d) = G'(d \cap a') \cup G''(d \cap a'')$ for $d \subseteq a_i$.) But this means that $a_i \notin \mathcal{Y}_\epsilon$, contradicting the choice of $\langle a_j \rangle_{j \in I}$. **X**

This contradiction shows that the theorem is true.

Problem Can $\mathcal{K}$ be negligible?

5C. Lemma [OCA + $\mathfrak{m} > \omega_1$] (Farah 00, 3.12.4) Let $\mathcal{I}$ and $\mathcal{J}$ be ideals of $\mathcal{P}\mathbb{N}$ such that $[\mathbb{N}]^{<\omega} \subseteq \mathcal{I}$ and $\mathcal{I}$ is an analytic P-ideal. Let $\pi : \mathcal{P}\mathbb{N}/\mathcal{I} \to \mathcal{P}\mathbb{N}/\mathcal{J}$ be a Boolean homomorphism. Write $\mathcal{K}$ for the ideal of those $a \subseteq \mathbb{N}$ such that there is a continuous function from $\mathcal{P}\mathbb{N}$ to itself representing π on $\mathcal{P}a$, and $\mathcal{K}^*$ for the set of those $a \subseteq \mathbb{N}$ such that there is a continuous function from $\mathcal{P}\mathbb{N}$ to itself representing π on $\mathcal{K} \cap \mathcal{P}a$. Then $\mathcal{K}^*$ is a P-ideal.

proof (a) Of course $b \in \mathcal{K}^*$ whenever $b \subseteq a \in \mathcal{K}^*$. Also, if a, $b \in \mathcal{K}^*$ then $a \cup b \in \mathcal{K}^*$. **P** There are continuous H, $H' : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ representing π on $\mathcal{K} \cap \mathcal{P}a$, $\mathcal{K} \cap \mathcal{P}b$ respectively. If we set $G(c) = H(a \cap c) \cup H'(c \setminus a)$ for $c \subseteq \mathbb{N}$, then (because $\mathcal{K}$ is an ideal) G represents π on $\mathcal{P}(a \cup b)$. **Q**

Thus $\mathcal{K}^*$ is an ideal. Since it includes $\mathcal{K}$, it includes $[\mathbb{N}]^{<\omega}$.

(b) Let $\langle a_n \rangle_{n \in \mathbb{N}}$ be a sequence in $\mathcal{K}^*$. Write

$$B^* = \{b : b \subseteq \mathbb{N},\ b \cap a_n \text{ is finite for every } n \in \mathbb{N}\}.$$

Let $C \subseteq B^* \setminus \mathcal{K}$ be a maximal almost disjoint set. Then C is countable, by Theorem 5B, so there is an $a \subseteq \mathbb{N}$ such that $a_n \setminus a$ is finite for every $n \in \mathbb{N}$ and $c \cap a$ is finite for every $c \in C$. Set $a'_n = a \cap \bigcup_{i \le n} a_i$ for each n, so that $\langle a'_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in $\mathcal{K}^*$, and $B = B^* \cap \mathcal{P}a$, so that $B \subseteq \mathcal{K}$ (since all finite sets belong to $\mathcal{K}$).

(c) Let X be the set of pairs (H, b) where $b \in B$ and $H : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ is an asymptotically additive function representing π on $\mathcal{P}b$. Then every member of b is the second member of an element of X. **P** Because $B \subseteq \mathcal{K}$, there is certainly a continuous $G : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ representing π on $\mathcal{P}b$. But now, by Corollary 1N, there is an asymptotically additive $H : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ representing π on $\mathcal{P}b$, so that $(H, b) \in X$. **Q**

(d) For the moment (down to the end of (j) below) fix $\epsilon > 0$. Let $S \subseteq [X]^2$ be the set of pairs $\{(H, b), (H', b')\}$ such that $\sup_{d \subseteq b \cap b'} \nu(H(d) \triangle H'(d)) > \epsilon$. Then

$$S = \bigcup_{k \in \mathbb{N}, s, t, t' \in [\mathbb{N}]^{<\omega}, \nu(t \triangle t') > \epsilon} \{\{(H, b), (H', b')\} : s \subseteq b,\ H(s) \cap k = t,\ s \subseteq b',\ H'(s) \cap k = t'\}$$

is a countable union of rectangles. Now if $\langle (H_\xi, b_\xi) \rangle_{\xi < \omega_1}$ is any family in X, there are $\xi < \eta < \omega_1$ such that $\{(H_\xi, b_\xi), (H_\eta, b_\eta)\} \notin S$. **P** Because $\mathfrak{m} > \omega_1$, there is a $b \in B$ such that $b_\xi \setminus b$ is finite for every $\xi < \omega_1$ (by Fremlin 84, 21A, or otherwise). Let $D \subseteq \omega_1$ be an uncountable set such that $b_\xi \setminus b = b_\eta \setminus b$ for all ξ, $\eta \in D$, so that $b' = \bigcup_{\xi \in D} b_\xi$ belongs to B (since $b' \setminus b$ is finite).

Let $H : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ be an asymptotically additive function representing π on $\mathcal{P}b'$. For each $\xi \in D$ there is a $k_\xi \in \mathbb{N}$ such that $\nu(H_\xi(d) \triangle H(d)) \le \frac{1}{2}\epsilon$ whenever $d \subseteq b_\xi \setminus k_\xi$, by Lemma 3G. Next, there is a finite set $K_\xi \subseteq \mathbb{N}$ such that $k_\xi \subseteq K_\xi$ and $H_\xi(s \cup d) = H_\xi(s) \cup H_\xi(d)$ whenever $s \subseteq K_\xi$ and $d \subseteq \mathbb{N} \setminus K_\xi$. Note that $H_\xi(s) \in [\mathbb{N}]^{<\omega}$ for every $s \subseteq K_\xi$.

Because D is uncountable, there are distinct ξ, $\eta \in D$ such that $k_\xi = k_\eta$, $K_\xi = K_\eta$ and $H_\xi(s) = H_\eta(s)$ for every $s \subseteq K_\xi$. Take any $d \subseteq b_\xi \cap b_\eta$, and set $s = d \cap K_\xi$, $d' = d \setminus K_\xi$. Then $H_\xi(d) \triangle H_\eta(d) \subseteq H_\xi(d') \triangle H_\eta(d')$; and because $d' \cap k_\xi = d' \cap k_\eta = \emptyset$, we have

$$\nu(H_\xi(d) \triangle H_\eta(d)) \le \nu(H_\xi(d') \triangle H(d')) + \nu(H_\eta(d') \triangle H(d')) \le \epsilon.$$

As d is arbitrary, $\{(H_\xi, b_\xi), (H_\eta, b_\eta)\} \notin S$, as claimed. **Q**

(e) OCA now tells us that X is expressible as $\bigcup_{n \in \mathbb{N}} X_n$ where $[X_n]^2 \cap S = \emptyset$ for every $n \in \mathbb{N}$. Set $B_n = \{b : (H, b) \in X_n\}$ for each n. Then there is an $m \in \mathbb{N}$ such that every member of B is included in $a'_m \cup b'$ for some $b' \in B_m$. **P?** Otherwise, for each $i \in \mathbb{N}$ choose $b_i \in B$ such that b_i is not included in $a'_i \cup b'$ for any $b' \in B_i$. Set $b = \bigcup_{i \in \mathbb{N}} b_i \setminus a'_i$;

then $b \in B$. But there must now be some k such that $b \in B_k$, in which case $b_k \subseteq a'_k \cup b$, contrary to the choice of b_k. **XQ**

(f) Whenever $c \in \mathcal{K} \cap \mathcal{P}a$ and $G : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ is an asymptotically additive function representing π on $\mathcal{P}c$, there is some $n \geq m$ such that $\nu(G(d)\triangle H(d)) \leq 2\epsilon$ whenever $(H, b) \in X_m$ and $d \subseteq c \cap b \setminus a'_n$. **P?** Otherwise, choose a sequence $\langle (H_k, b_k)\rangle_{k\in\mathbb{N}}$ in B_m and a sequence $\langle s_k\rangle_{k\in\mathbb{N}}$ in $[\mathbb{N}]^{<\omega}$ such that $s_k \subseteq c \cap b_k \setminus a'_k$ and $\nu(G(s_k)\triangle H_k(s_k)) > 2\epsilon$. Set $b = \bigcup_{k\in\mathbb{N}} s_k \in B$, and let $b' \in B_m$ be such that $b \subseteq a'_m \cup b'$. Let H' be such that $(H', b') \in X_m$. For all k large enough, $\nu(G(s_k)\triangle H'(s_k)) \leq \epsilon$, by Lemma 3G; but also $\nu(H_k(s_k)\triangle H'(s_k)) \leq \epsilon$ for every k, because $[X_m]^2 \cap S = \emptyset$, so $\nu(G(s_k)\triangle H_k(s_k)) \leq 2\epsilon$ for all large k, which is impossible. **XQ**

(g) Write C_n for the set of those $c \in \mathcal{K} \cap \mathcal{P}a$ such that there is a continuous function $G : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$, representing π on $\mathcal{P}c$, such that $\nu(G(d)\triangle H(d)) \leq 2\epsilon$ whenever $(H, b) \in X_m$ and $d \subseteq b \cap c \setminus a'_n$. Then $\mathcal{K} \cap \mathcal{P}a = \bigcup_{n\geq m} C_n$. **P** In view of (f), we have only to note that if $c \in C_n$ then there is an asymptotically additive function G representing π on $\mathcal{P}c$; and this follows from Corollary 1N, just as in (c) above. **Q**

At this point, recall that each a'_n is supposed to belong to $\mathcal{K}^*$, so there is a continuous function $F_n : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ such that F_n represents π on $\mathcal{K} \cap \mathcal{P}a'_n$. Let Y_n be the set of pairs (G, c) such that $c \in \mathcal{K} \cap \mathcal{P}a$, $G : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ is a continuous function representing π on $\mathcal{P}c$, $\nu(G(d) \cap H(d)) \leq 2\epsilon$ whenever $(H, b) \in X_m$ and $d \subseteq b \cap c \setminus a'_n$, and $G(d) = G(d \setminus a'_n) \cup F_n(d \cap a'_n)$ for every $d \subseteq \mathbb{N}$. Then for every $c \in C_n$ there is a G such that $(G, d) \in Y_n$. **P** As just remarked, there is a G' satisfying all the conditions except perhaps the last. But if we now set $G(d) = F_n(d \cap a'_n) \cup G'(d \setminus a'_n)$ for every $d \subseteq \mathbb{N}$, we shall have $(G, c) \in Y_n$, because F_n represents π on $\mathcal{P}(c \cap a'_n)$. **Q**

(h) If $n \geq m$, (G, c), (G', c') are two members of Y_n and $d \subseteq c\cap c'$, then $\nu(G(d)\triangle G'(d)) \leq 4\epsilon$. **P** Because ν is lower semi-continuous and G, G' are continuous, it is enough to consider finite d. By the definition of Y_n, $G(d)\triangle G'(d) \subseteq G(d \setminus a'_n)\triangle G'(d \setminus a'_n)$. But $d \setminus a'_n$ is finite, so belongs to B, and there is some $b \in B_m$ such that $d \setminus a'_n \subseteq b \cup a'_m$, that is, $d \setminus a'_n \subseteq b$. If H is such that $(H, b) \in X_m$, then

$$\begin{aligned}\nu(G(d \setminus a'_n)\triangle G'(d \setminus a'_n)) &\leq \nu(G(d \setminus a'_n)\triangle H(d \setminus a'_n)) + \nu(H(d \setminus a'_n)\triangle G'(d \setminus a'_n))\\ &\leq 4\epsilon. \ \mathbf{Q}\end{aligned}$$

(i) By Lemma 3F, there is for each $n \geq m$ a function $G^*_n : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$, with the restricted Baire property, such that $\nu(G^*_n(d)\triangle G(d)) \leq 4\epsilon$ whenever $(G, c) \in Y_n$ and $d \subseteq c$. Set $A_n = \bigcup_{c\in C_n} \mathcal{P}c$; then A_n is hereditary and G^*_n 4ϵ-approximates π on A_n, while $\bigcup_{n\geq m} A_n \supseteq \mathcal{K} \cap \mathcal{P}a$.

(j) It follows that there is a continuous function $F : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ which is a 48ϵ-approximation to π on $\mathcal{P}a$. **P** If a is finite, this is trivial. Otherwise, apply Lemma 3D to $\mathcal{I}_a = \mathcal{I} \cap \mathcal{P}a$ and the induced homomorphism $\pi_a : \mathcal{P}a/\mathcal{I}_a \to \mathcal{P}\mathbb{N}/\operatorname{Exh}(\nu)$, as in the proof of Lemma 1N. The functions $G^*_n\restriction\mathcal{P}a : \mathcal{P}a \to \mathcal{P}\mathbb{N}$ have the Baire property with respect to the subspace topology of $\mathcal{P}a$, which is the natural topology on $\mathcal{P}a$, and 4ϵ-approximate π_a on the hereditary sets A_n, at least for $n \geq m$. At the same time, $\mathcal{K} \cap \mathcal{P}a$

is ccc over $[a]^{<\omega}$ and therefore non-meager when regarded as a subset of $\mathcal{P}a$. So Lemma 3D gives the result. **Q**

(k) At this point, recall that ϵ was arbitrary. So we can apply Lemma 3E to $\mathcal{P}a$, $\mathcal{I}_a = \mathcal{P}a \cap \mathcal{I}$ and $\pi_a = \pi \restriction \{b^\bullet : b \subseteq a\}$ to see that there is a continuous function $F^* : \mathcal{P}a \to \mathcal{P}N$ which represents π_a on $\mathcal{K} \cap \mathcal{P}a$. Setting $F(d) = F^*(a \cap d)$, we see that $F : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ is a continuous function representing π on $\mathcal{K} \cap \mathcal{P}a$, so that $a \in \mathcal{K}^*$. And $a_n \setminus a$ is finite for every $n \in \mathbb{N}$. As $\langle a_n \rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{K}^*$ is a P-ideal, as claimed.

5D. Lemma (Compare FARAH 00, p. 112, Claim 5) Suppose that ν is a lower semi-continuous submeasure on $\mathcal{P}\mathbb{N}$. Let $\epsilon > 0$, and let $\langle (H_\xi, a_\xi \rangle_{\xi<\omega_1}$ be a family such that

(i) for each $\xi < \omega_1$, a_ξ is a subset of $\mathbb{N}$ and $H_\xi : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ is an asymptotically additive function,

(ii) whenever $\xi < \eta < \omega_1$, $a_\xi \setminus a_\eta$ is finite and there is a $d \subseteq a_\xi \cap a_\eta$ such that $\nu(H_\xi(d) \triangle H_\eta(d)) > 2\epsilon$.

Then for any $k \in \mathbb{N}$ and uncountable sets C, $D \subseteq \omega_1$, there are a finite set $s \subseteq \mathbb{N} \setminus k$ and uncountable sets $C' \subseteq C$, $D' \subseteq D$ such that $s \subseteq a_\xi \cap a_\eta$ and $\nu(H_\xi(s) \triangle H_\eta(s)) > \epsilon$ for every $\xi \in C'$, $\eta \in D'$.

proof (a) Because C and D are uncountable, we can find strictly increasing families $\langle \alpha_\xi \rangle_{\xi<\omega_1}$ and $\langle \beta_\xi \rangle_{\xi<\omega_1}$ such that $\alpha_\xi < \beta_\xi$, $\alpha_\xi \in C$ and $\beta_\xi \in D$ for every $\xi < \omega_1$.

(b) Now we can find $k' \geq k$ and a finite set $K \supseteq k'$ and a function $F : \mathcal{P}K \to [\mathbb{N}]^{<\omega}$ such that

$$\begin{aligned} E = \{\xi : \xi < \omega_1, a_{\alpha_\xi} \setminus a_{\beta_\xi} \subseteq k', \\ H_{\alpha_\xi}(d) = F(d \cap K) \cup H_{\alpha_\xi}(d \setminus K) \text{ for every } d \subseteq \mathbb{N}\} \end{aligned}$$

is uncountable. Let $\gamma < \omega_1$ be such that

whenever s, $t \in [\mathbb{N}]^{<\omega}$ and $E_{st} = \{\xi : \xi \in E, s \subseteq a_{\alpha_\xi}, H_{\alpha_\xi}(s) = t\}$ is countable, then $E_{st} \subseteq \gamma$,

whenever s, $t \in [\mathbb{N}]^{<\omega}$ and $E'_{st} = \{\xi : \xi \in E, s \subseteq a_{\beta_\xi}, H_{\beta_\xi}(s) = t\}$ is countable, then $E'_{st} \subseteq \gamma$,

and set $E^* = E \setminus \gamma$.

(c) Take any $\xi < \eta$ in E^*. Then there is a finite set $s_0 \subseteq a_{\alpha_\xi} \cap a_{\alpha_\eta}$ such that

$$\begin{aligned} 2\epsilon &< \nu(H_{\alpha_\xi}(s_0) \triangle H_{\alpha_\eta}(s_0)) \\ &= \nu((F(s_0 \cap K) \cup H_{\alpha_\xi}(s)) \triangle (F(s_0 \cap K) \cup H_{\alpha_\eta}(s))) \\ &\leq \nu(H_{\alpha_\xi}(s)) \triangle H_{\alpha_\eta}(s)), \end{aligned}$$

where $s = s_0 \setminus K$. Note that as $s \cap k' = \emptyset$, $s \subseteq a_{\beta_\eta}$. At least one of

$$\nu(H_{\alpha_\xi}(s) \triangle H_{\beta_\eta}(s)), \quad \nu(H_{\alpha_\eta}(s) \triangle H_{\beta_\eta}(s))$$

is greater than ϵ, so we must have a $\zeta \in E^*$ such that $s \subseteq a_{\alpha_\zeta} \cap a_{\beta_\eta}$ and

$$\nu(H_{\alpha_\zeta}(s) \triangle H_{\beta_\eta}(s)) > \epsilon.$$

By the choice of γ, we now see that

$$C' = \{\theta : \theta \in C,\ s \subseteq a_\theta,\ H_\theta(s) = H_{\alpha_\zeta}(s)\},$$

$$D' = \{\theta : \theta \in D,\ s \subseteq a_\theta,\ H_\theta(s) = H_{\beta_\eta}(s)\}$$

are both uncountable, so we have appropriate sets s, C' and D'.

5E. Lemma [OCA + $\mathfrak{m} > \omega_1$] Let $\mathcal{I}$ be an ideal of $\mathcal{P}\mathbb{N}$ including $[\mathbb{N}]^{<\omega}$, and ν a lower semi-continuous submeasure on $\mathcal{P}\mathbb{N}$. Let $\pi : \mathcal{P}\mathbb{N}/\mathcal{I} \to \mathcal{P}\mathbb{N}/\operatorname{Exh}(\nu)$ be a Boolean homomorphism. Write $\mathcal{K}$ for the ideal of those $a \subseteq \mathbb{N}$ such that there is a continuous function from $\mathcal{P}\mathbb{N}$ to itself representing π on $\mathcal{P}a$. Suppose that $\epsilon > 0$ and that $\langle (H_\xi, a_\xi)\rangle_{\xi<\omega_1}$ is a family such that (α) for each $\xi < \omega_1$, $a_\xi \subseteq \mathbb{N}$ and $H_\xi : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ is an asymptotically additive function representing π on $\mathcal{K} \cap \mathcal{P}a_\xi$ (β) $a_\xi \setminus a_\eta$ is finite whenever $\xi < \eta < \omega_1$. Then there are distinct ξ, $\eta < \omega_1$ such that $\nu(H_\xi(d) \triangle H_\eta(d)) \le 2\epsilon$ whenever $d \subseteq a_\xi \cap a_\eta$.

proof ? Suppose, if possible, otherwise.

(a) Let $\gamma < \omega_1$ be such that

(*) whenever $K' \subseteq K \in [\mathbb{N}]^{<\omega}$, $F : \mathcal{P}K \to \mathcal{P}\mathbb{N}$ are such that

$$D_{KK'F} = \{\xi : K' = K \cap a_\xi, H_\xi(s) = F(s) \text{ for every } s \subseteq K,$$
$$H_\xi(d) = H_\xi(d \cap K) \cup H_\xi(d \setminus K) \text{ for every } d \subseteq \mathbb{N}\}$$

is countable, then it is included in γ.

(b) Let P be the set of $p = (I_p, k_p, \sigma_p, J_p, f_p)$ such that

$I_p \subseteq \omega_1$, $J_p \subseteq \omega_1 \setminus \gamma$ are finite,

$k_p \in \mathbb{N}$,

$\sigma_p : I_p \to \mathcal{P}k_p$ and $f_p : J_p \to I_p$ are functions,

$f_p(\xi) \le \xi$ for every $\xi \in J_p$,

whenever ξ, η are distinct members of J_p such that $f_p(\xi) = f_p(\eta)$, there is an $s \subseteq \sigma_p(f_p(\xi)) \cap a_\xi \cap a_\eta$ such that $\nu(H_\xi(s) \triangle H_\eta(s)) \ge \epsilon$.

Order P by saying that $p \le q$ if

$I_p \subseteq I_q$, $J_p \subseteq J_q$, $k_p \le k_q$,

$\sigma_p(\xi) = \sigma_q(\xi) \cap k_p$ for every $\xi \in I_p$,

$\sigma_q(\xi) \cap \sigma_q(\eta) \subseteq k_p$ for all distinct ξ, $\eta \in I_p$,

f_q extends f_p.

It is straightforward to check that P is a partially ordered set.

(c) P is upwards-ccc. **P** Let $\langle p_\theta\rangle_{\theta<\omega_1}$ be a family in P. Using the Δ-system lemma twice, then thinning out, we can find an uncountable set $D \subseteq \omega_1$, finite sets I, $J \subseteq \omega_1$, $k \in \mathbb{N}$, $\sigma : I \to \mathcal{P}k$ and $f : J \to I$ such that

$\langle I_{p_\theta}\rangle_{\theta\in D}$ is a constant-size Δ-system with root I,

$\langle J_{p_\theta}\rangle_{\theta\in D}$ is a constant-size Δ-system with root J,

$f_{p_\theta}(\xi) = f(\xi)$ for every $\xi \in J$, $\theta \in D$,

$k_{p_\theta} = k$ for every $\theta \in D$,

$\sigma_{p_\theta}(\xi) = \sigma(\xi)$ for every $\xi \in I$, $\theta \in D$.

Let n be the constant value of $\#(J_{p_\theta} \setminus J)$ for $\theta \in D$. For $\theta \in D$, enumerate $J_{p_\theta} \setminus J$ as $\langle \xi(\theta, i) \rangle_{i<n}$, and let $\langle (i_r, j_r) \rangle_{r<n^2}$ enumerate $n \times n$. Using Lemma 5D n^2 times, we can find uncountable sets C', $D' \subseteq D$ and $k_0 < \ldots < k_{n^2}$ and $s_r \subseteq k_{r+1} \setminus k_r$ such that

$k_0 = k$,

whenever $\theta \in C'$, $\theta' \in D'$, $r < n^2$ then $s_r \subseteq a_{\xi(\theta,i_r)} \cap a_{\xi(\theta',j_r)}$ and $\nu(H_{\xi(\theta,i_r)}(s_r) \triangle H_{\xi(\theta',j_r)}(s_r)) > \epsilon$.

Take any distinct $\theta \in C'$, $\theta' \in D'$. Set

$$I^* = I_{p_\theta} \cup I_{p_{\theta'}}, \quad k^* = k_{n^2},$$

$$J^* = J_{p_\theta} \cup J_{p_{\theta'}}, \quad f^* = f_{p_\theta} \cup f_{p_{\theta'}},$$

and for $\alpha \in I^*$ define $\sigma^*(\alpha)$ by setting

$$\begin{aligned}
\sigma_0^*(\alpha) &= \bigcup \{s_r : r < n^2,\ \alpha = f_{p_\theta}(\xi(\theta, i_r)) = f_{p_{\theta'}}(\xi(\theta', j_r))\} \\
\sigma^*(\alpha) &= \sigma_{p_\theta}(\alpha) \cup \sigma_0^*(\alpha) \text{ if } \alpha \in I_{p_\theta}, \\
&= \sigma_{p_{\theta'}}(\alpha) \cup \sigma_0^*(\alpha) \text{ if } \alpha \in I_{p_{\theta'}}.
\end{aligned}$$

Then, because the s_r are disjoint, $(I^*, k^*, \sigma^*, J^*, f^*) \in P$ and is an upper bound of $\{p_\theta, p_{\theta'}\}$. As $\langle p_\xi \rangle_{\xi<\omega_1}$ is arbitrary, P is upwards-ccc. **Q**

(d) For α, $\beta < \omega_1$ set

$$Q_{\alpha\beta} = \{p : p \in P,\ \exists \xi \in J_p \setminus \beta,\ f_p(\xi) = \alpha\}.$$

Then $Q_{\alpha\beta}$ is cofinal with P. **P** Let $p \in P$. **case 1** If $\alpha \notin I_p$, take any $\eta \in \omega_1 \setminus (\gamma \cup \alpha \cup \beta \cup J_p)$ and set

$I = I_p \cup \{\alpha\}$, $k = k_p$,

$\sigma(\alpha) = \emptyset$, $\sigma(\xi) = \sigma_p(\xi)$ for $\xi \in I_p$,

$J = J_p \cup \{\eta\}$,

$f(\eta) = \alpha$, $f(\xi) = f_p(\xi)$ for $\xi \in J_p$,

and see that $p \leq (I, k, \sigma, J, f) \in Q_{\alpha\beta}$. **case 2** If $\alpha \in I_p \setminus f_p[J_p]$, take any $\eta \in \omega_1 \setminus (\gamma \cup \beta \cup \alpha \cup J_p)$ and set

$I = I_p$, $k = k_p$, $\sigma = \sigma_p$,

$J = J_p \cup \{\eta\}$,

$f(\eta) = \alpha$, $f(\xi) = f_p(\xi)$ for $\xi \in J_p$,

and see that $p \leq (I, k, \sigma, J, f) \in Q_{\alpha\beta}$. **case 3** If $\alpha = f_p(\zeta)$ for some $\zeta \in J_p$, let $K \supseteq k_p$ be a finite set such that $H_\zeta(d) = H_\zeta(d \cap K) \cup H_\zeta(d \setminus K)$ for every $d \subseteq \mathbb{N}$; such exists because H_ζ is asymptotically additive. Set $K' = K \cap a_\zeta$, $F = H_\zeta \restriction K$. Since ζ belongs to $D_{KK'F} \setminus \gamma$, this is uncountable, and there is an $\eta \in D_{KK'F} \setminus (\gamma \cup \beta \cup \alpha \cup J_p)$. Now there is some $d \subseteq a_\zeta \cap a_\eta$ such that $\nu(H_\zeta(d) \triangle H_\eta(d)) > \epsilon$; because ζ, η both belong to $D_{KK'F}$, $\nu(H_\zeta(d') \triangle H_\eta(d')) > \epsilon$, where $d' = d \setminus K$, and there is some $k \geq k_p$ such that $\nu(H_\zeta(s_0) \triangle H_\eta(s_0)) \geq \epsilon$, where $s_0 = d' \cap k$. Now set

$\sigma(\alpha) = \sigma_p(\alpha) \cup (k \setminus k_p)$, $\sigma(\xi) = \sigma_p(\xi)$ if $\xi \in I_p \setminus \{\alpha\}$,

$J = J_p \cup \{\eta\}$,

$f(\eta) = \alpha$, $f(\xi) = f_p(\xi)$ for other $\xi \in I_p$.

Working through the conditions for membership of P, we see that $q = (I_p, k, \sigma, J, f)$ passes every test easily, except that we must check that

whenever $\xi \in J \setminus \{\eta\}$ and $f(\xi) = \alpha$, there is an $s \subseteq \sigma(\alpha) \cap a_\xi \cap a_\eta$ such that $\nu(H_\xi(s) \triangle H_\eta(s)) \geq \epsilon$.

But if $\xi = \zeta$ then s_0 above serves for s, while otherwise we have $f_p(\xi) = f_p(\zeta) = \alpha$, so there is an $s \subseteq \sigma_p(\alpha) \cap a_\xi \cap a_\zeta$ such that $\nu(H_\xi(s) \triangle H_\zeta(s)) \geq \epsilon$; since $H_\zeta \restriction \mathcal{P}K = H_\eta \restriction \mathcal{P}K$, $a_\zeta \cap K = a_\eta \cap K$ and $s \subseteq k_p \subseteq K$, we have $s \subseteq \sigma(\alpha) \cap a_\xi \cap a_\eta$ and $\nu(H_\xi(s) \triangle H_\eta(s)) \geq \epsilon$.

Thus in this case also we have a member of P, and of course $p \leq q \in Q_{\alpha\beta}$. **Q**

Note also that if $p \in Q_{\alpha\beta}$ and $q \geq p$ in P, then $q \in Q_{\alpha\beta}$.

(e) Because $\mathfrak{m} > \omega_1$, there is an upwards-directed set $R \subseteq P$ meeting $Q_{\alpha\beta}$ for every α, $\beta < \omega_1$. Set

$\tilde{\sigma}(\alpha) = \bigcup\{\sigma_p(\alpha) : p \in R,\ \alpha \in I_p\}$ for $\alpha < \omega_1$,

$\tilde{J} = \bigcup_{p \in R} J_p$, $\tilde{f} = \bigcup_{p \in J} f_p$.

If $\alpha < \alpha' < \omega_1$, there is a $p \in Q_{\alpha 0} \cap Q_{\alpha' 0} \cap R$. Now, for any q, $q' \in R$ such that $\alpha \in I_q$, $\alpha' \in I_{q'}$, there is a common upper bound r of p, q and q' in P, so that

$$\sigma_q(\alpha) \cap \sigma_{q'}(\alpha') \subseteq \sigma_r(\alpha) \cap \sigma_r(\alpha') \subseteq k_p.$$

As q and q' are arbitrary, $\tilde{\sigma}(\alpha) \cap \tilde{\sigma}(\alpha') \subseteq k_p$.

(f) This shows that $\langle \tilde{\sigma}(\alpha) \rangle_{\alpha < \omega_1}$ is almost disjoint. By Theorem 5B, there is some $\alpha < \omega_1$ such that $a = \tilde{\sigma}(\alpha) \in \mathcal{K}$. Let $H : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ be an asymptotically additive function such representing π on $\mathcal{P}a$.

Because R meets $Q_{\alpha\beta}$ for every $\beta < \omega_1$, $D = \{\xi : \xi \in \tilde{J},\ \tilde{f}(\xi) = \alpha\}$ is uncountable. By 3G, there is for each $\xi \in D$ an $m_\xi \in \mathbb{N}$ such that $\nu(H(d) \triangle H_\xi(d)) \leq \frac{1}{3}\epsilon$ for every $d \subseteq a \cap a_\xi \setminus m_\xi$. Next, there is a finite set $K_\xi \supseteq m_\xi$ such that $H_\xi(d) = H_\xi(d \cap K_\xi) \cup H_\xi(d \setminus K_\xi)$ for every $d \subseteq \mathbb{N}$. Because D is uncountable, we have distinct ξ, $\eta \in D$ such that $m_\xi = m_\eta = m$ say, $K_\xi = K_\eta = K$ say and $H_\xi(s) = H_\eta(s)$ for every $s \subseteq K$.

At this point observe that $\tilde{f}(\xi) = \tilde{f}(\eta) = \alpha$, so there must be a $p \in R$ such that $f_p(\xi) = f_p(\eta) = \alpha$, and (by the definition of P) there is an $s \subseteq \sigma_p(\alpha) \cap a_\xi \cap a_\eta$ such that $\nu(H_\xi(s) \triangle H_\eta(s)) \geq \epsilon$. In this case, $\nu(H_\xi(s') \triangle H_\eta(s')) \geq \epsilon$, where $s' = s \setminus K$. But now $s' \subseteq a \cap a_\xi \cap a_\eta \setminus m$, so $\nu(H_\xi(s') \triangle H(s))$ and $\nu(H_\xi(s') \triangle H(s))$ are both at most $\frac{1}{3}\epsilon$, which is impossible. **X**

This contradiction proves the lemma.

5F. Theorem (Farah 00, 3.3.5/3.12.1) [OCA + $\mathfrak{m} > \omega_1$] Let $\mathcal{I}$ be an ideal of $\mathcal{P}\mathbb{N}$ including $[\mathbb{N}]^{<\omega}$, and $\mathcal{J}$ an analytic P-ideal. Let $\pi : \mathcal{P}\mathbb{N}/\mathcal{I} \to \mathcal{P}\mathbb{N}/\mathcal{J}$ be a Boolean homomorphism. Write $\mathcal{K}$ for the ideal of those $a \subseteq \mathbb{N}$ such that there is a continuous function from $\mathcal{P}\mathbb{N}$ to itself representing π on $\mathcal{P}a$. Then there is a continuous $H^* : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ which represents π on $\mathcal{K}$.

proof Let ν be a lower semi-continuous submeasure on $\mathcal{P}\mathbb{N}$ such that $\mathcal{J} = \mathrm{Exh}(\nu)$ (Theorem 1F).

(a) Let X be the set of pairs (H, a) such that $a \subseteq \mathbb{N}$ and $H : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ is an asymptotically additive function representing π on $\mathcal{P}a$ such that $H(b) \subseteq H(a)$ for every $b \subseteq a$. Then every set in $\mathcal{K}$ is the second member of an element of X, by Corollary 1K.

(b) Let $\epsilon > 0$. Let $S \subseteq [X]^2$ be the set of pairs $\{(H, a), (H', a')\}$ such that $\sup_{d \subseteq a \cap a'} \nu(H(d) \triangle H'(d)) > \epsilon$. Just as in part (d) of the proof of 5C, S is a countable union of rectangles. Now there is no uncountable $Y \subseteq X$ such that $[Y]^2 \subseteq X$.

P Let $\langle (H_\xi, a_\xi) \rangle_{\xi < \omega_1}$ be any family of distinct elements of X. As in Lemma 5C, write $\mathcal{K}^*$ for the set of those $b \subseteq \mathbb{N}$ such that there is a continuous function representing π on $\mathcal{K} \cap \mathcal{P}b$. Then $\mathcal{K}^*$ is a P-ideal including $\mathcal{K}$ (Lemma 5C), so we can choose $\langle b_\xi \rangle_{\xi < \omega_1}$ in $\mathcal{K}^*$ inductively such that $b_\eta \setminus b_\xi$ is finite whenever $\eta < \xi$ and $a_\xi \subseteq b_\xi$ for every $\xi < \omega_1$. For each $\xi < \omega_1$, there is a continuous function G_ξ representing π on $\mathcal{K} \cap \mathcal{P}b_\xi$. Since $\mathcal{K} \cap \mathcal{P}b_\xi$ is ccc over $[b_\xi]^{<\omega}$, therefore non-meager in $\mathcal{P}b_\xi$, we may suppose that G_ξ is asymptotically additive (Lemma 1M); and replacing G_ξ by the function $d \mapsto H_\xi(d \cap a_\xi) \cup (G_\xi(d \setminus a_\xi) \setminus H_\xi(a_\xi))$ if necessary, we may suppose that $G_\xi(d) = H_\xi(d)$ for every $d \subseteq a_\xi$. (This alteration leaves G_ξ asymptotically additive because $H(d) \subseteq H(a_\xi)$ for every $d \subseteq a_\xi$.)

By Lemma 5E, there are distinct ξ, $\eta < \omega_1$ such that $\nu(G_\xi(d) \triangle G_\eta(d)) \le \epsilon$ whenever $d \subseteq b_\xi \cap b_\eta$. So $\{(H_\xi, a_\xi), (H_\eta, a_\eta)\} \notin S$. **Q**

By OCA, we can express X as $\bigcup_{n \in \mathbb{N}} X_n$ where $[X_n]^2 \cap S = \emptyset$ for every $n \in \mathbb{N}$. By Lemma 3F, there is for each $n \in \mathbb{N}$ a function $G_n : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$, with the restricted Baire property, such that $\nu(G_n(d) \triangle H(d)) \le \epsilon$ whenever $(H, a) \in X_n$ and $d \subseteq a$. By Lemma 3D, there is a continuous $G : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ which 12ϵ-approximates π on $\mathcal{K}$.

(c) This is true for every $\epsilon > 0$. By Lemma 3E, there is a continuous function $H^* : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ which represents π on $\mathcal{K}$, as claimed.

5G. Corollary (Farah 00, 3.4.1) [OCA + $\mathfrak{m} > \omega_1$] Let $\mathcal{I}$, $\mathcal{J} \triangleleft \mathcal{P}\mathbb{N}$ be analytic P-ideals, and $\pi : \mathcal{P}\mathbb{N}/\mathcal{I} \to \mathcal{P}\mathbb{N}/\mathcal{J}$ a Boolean isomorphism. Then π has a continuous representation.

proof By Theorem 5F, there is a continuous $H : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ which represents π on $\mathcal{K}$, where $\mathcal{K}$ is an ideal including $[\mathbb{N}]^{<\omega}$ and is ccc over $[\mathbb{N}]^{<\omega}$ (Theorem 5B). By Lemma 1M, we may suppose that H is asymptotically additive, so that $H(a)^\bullet = \pi(a^\bullet) \cap H(\mathbb{N})^\bullet$ for every $a \subseteq \mathbb{N}$, by Lemma 1P. Let $c \subseteq \mathbb{N}$ be such that $\pi(c^\bullet) = (\mathbb{N} \setminus H(\mathbb{N}))^\bullet$. Then $\{a : a \cap c \in \mathcal{I}\}$ is an analytic ideal including $\mathcal{K}$, so must be the whole of $\mathcal{P}\mathbb{N}$, and $c \in \mathcal{I}$, $\mathbb{N} \setminus H(\mathbb{N}) \in \mathcal{J}$. Accordingly $H(a)^\bullet = \pi(a^\bullet)$ for every $a \subseteq \mathbb{N}$, and H represents π.

5H. Corollary (Farah 00, 3.4.2) [OCA + $\mathfrak{m} > \omega_1$] Let $\mathcal{I} \triangleleft \mathcal{P}\mathbb{N}$ be an analytic P-ideal, and ν an entirely non-pathological submeasure on $\mathcal{P}\mathbb{N}$. Let $\pi : \mathcal{P}\mathbb{N}/\mathcal{I} \to \mathcal{P}\mathbb{N}/\operatorname{Exh}(\nu)$ be a Boolean isomorphism. Then there are cofinite sets I, $J \subseteq \mathbb{N}$ and a bijection $h : I \to J$ such that $\pi a^\bullet = h^{-1}[a]^\bullet$ for every $a \subseteq \mathbb{N}$.

proof By Corollary 5G, there is a continuous function $H : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ representing π. Now use Proposition 2Eb.

Appendix: Treelike families

A1. Definition A family $\langle a_i \rangle_{i \in I}$ of sets is **treelike** if there is an injective function $f : I \to \{0,1\}^{\mathbb{N}}$ such that $\bigcup\{a_i : i \in I,\ f(i)(j) = 0\} \cap \bigcup\{a_i : i \in I,\ f(i)(j) = 1\}$ is finite for every $j \in \mathbb{N}$.

Note that if $\langle a_i\rangle_{i\in I}$ is treelike so is $\langle b_i\rangle_{i\in J}$ whenever $J\subseteq I$ and $b_i\subseteq a_i$ for every $i\in I$.

A2. Lemma (Veličković 93) (a) Let $\langle a_t\rangle_{t\in T}$ be an almost-disjoint family of subsets of $\mathbb{N}$ such that $\#(T)<\mathfrak{m}$. Then there are treelike families $\langle a'_t\rangle_{t\in T}$, $\langle a''_t\rangle_{t\in T}$ such that $a'_t\cup a''_t$ is a cofinite subset of a_t for every $t\in T$.

(b) Suppose that $\mathfrak{m}>\omega_1$. Let $\langle a_\xi\rangle_{\xi<\omega_1}$ be an almost disjoint family of subsets of $\mathbb{N}$. Then there are an uncountable set $\Gamma\subseteq\omega_1$ and treelike families $\langle a'_\xi\rangle_{\xi\in\Gamma}$, $\langle a''_\xi\rangle_{\xi\in\Gamma}$ such that $a_\xi=a'_\xi\cup a''_\xi$ for every $\xi\in\Gamma$.

proof (a)(i) Define a partially ordered set P as follows. Members of P are objects of the form $p=(I_p,n_p,f_p,g_p,h_p)$ where

$$I_p \text{ is a finite subset of } T,\quad n_p\in\mathbb{N},$$

$$a_s\cap a_t\subseteq n_p \text{ for all distinct } s,\, t\in I_p,$$

$$f_p:I_p\times n_p\to\{0,1\},\quad g_p:n_p\times n_p\times\{0,1\}\to\{0,1\},\quad h_p:I_p\times n_p\to\{0,1\}$$

are functions. Say that $p\le q$ if

$$I_p\subseteq I_q,\quad n_p\le n_q,\quad f_q \text{ extends } f_p,\quad g_q \text{ extends } g_p,\quad h_q \text{ extends } h_p,$$

and moreover

whenever $n_p\le k<n_q$, $t\in I_p$, $k\in a_t$ then $g_q(k,j,h_q(t,k))=f_q(t,j)$ for every $j\le k$.

It is elementary to check that this is a partially ordered set.

(ii) P is upwards-ccc. **P** Let $\langle p_\xi\rangle_{\xi<\omega_1}$ be a family in P. By the Δ-system lemma (Fremlin 84, 12H) there is an uncountable set $\Gamma\subseteq\omega_1$ such that $\langle I_{p_\xi}\rangle_{\xi\in\Gamma}$ is a constant-size Δ-system with root I say. Now there are distinct ξ, $\eta\in\Gamma$ such that $n_{p_\xi}=n_{p_\eta}=n$ say, $f_{p_\xi}\restriction I\times n=f_{p_\eta}\restriction I\times n$, $g_{p_\xi}=g_{p_\eta}$ and $h_{p_\xi}\restriction I\times n=h_{p_\eta}\restriction I\times n$.

Let $m\ge n$ be such that $a_s\cap a_t\subseteq m$ for all distinct s, $t\in J=I_{p_\xi}\cup I_{p_\eta}$. Let f be any extension of $f_{p_\xi}\cup f_{p_\eta}$ to $J\times m$. For any $k\in m\setminus n$, $\{t:t\in J,\,k\in a_t\}$ has at most two members, so we can extend $h_{p_\xi}\cup h_{p_\eta}$ to a function $h:J\times m\to\{0,1\}$ in such a way that if s, $t\in J$ are distinct, $n\le j<m$ and $k\in a_s\cap a_t$, then $h(s,k)\ne h(t,k)$. We shall therefore be able to extend $g_{p_\xi}=g_{p_\eta}$ to a function $g:m\times m\times\{0,1\}\to\{0,1\}$ such that $g(k,j,h(t,k))=f(t,j)$ whenever $t\in J$, $n\le k<m$, $k\in a_t$ and $j\le k$.

Now (J,m,f,g,h) is a common extension of p_ξ and p_η. **Q**

(iii) For any $t\in T$, $Q_t=\{p:p\in P,\,t\in I_p\}$ is cofinal with P. **P** If $p\in P$ and $t\notin I_p$, then take $n'\ge n_p$ such that $a_t\cap a_s\subseteq n'$ for every $s\in I_p$. Let $f':(I_p\cup\{t\})\times n'\to\{0,1\}$ be any extension of f_p and $h':(I_p\cup\{t\})\times n'\to\{0,1\}$ any extension of h_p; we can now find a function $g':n'\times n'\times\{0,1\}\to\{0,1\}$, extending g_p, such that $g'(k,j,h'(s,k))=f'(s,k)$ whenever $s\in J$ and $j\le k\in a_s\cap n'\setminus n$, and $p\le(I_p\cup\{t\},n',f',g',h')\in Q_t$. **Q**

(iv) For any distinct t, $t'\in T$, $Q'_{tt'}=\{p:t,\,t'\in I_p,\,\exists\, j<n,\,f_p(t,j)\ne f_p(t',j)\}$ is cofinal with P. **P** Given $p\in P$, then (iii) tells us that we can find $q\in P$ such that $p\le q$ and t, $t'\in I_q$. Let $f:I_q\times(n_q+1)\to\{0,1\}$ be any extension of f_q such that $f(t,n_q)\ne f(t',n_q)$. Let $h:I_q\times(n_q+1)\to\{0,1\}$ be any extension of h_q, and $g:(n_q+1)\times(n_q+1)\times\{0,1\}\to\{0,1\}$ any extension of g_q such that $g(n_q,j,h(s,n_q))=f(s,j)$

if $s \in I_q$ and $k \in a_s$ and $j \le n_q$ (by the definition of P, there can be at most one such s). Now $p \le (I_q, n_q+1, f, g, h) \in Q'_{tt'}$. **Q**

(v) For any $n \in \mathbb{N}$, $Q''_n = \{p : p \in P,\ n_p \ge n\}$ is cofinal with P. **P** The argument in (iv) describes how, given $p \in P$, there is a $q \ge p$ such that $n_q = n_p + 1$; now repeat this as often as necessary. **Q**

(vi) Because $\#(T) < \mathfrak{m}$, there is an upwards-directed family $R \subseteq P$ meeting every Q_t and every $Q'_{tt'}$ and every Q''_n. Set $f = \bigcup_{p \in R} f_p$, $g = \bigcup_{p \in P} g_p$ and $h = \bigcup_{p \in P} h_p$; then f is a function from $T \times \mathbb{N}$ to $\{0,1\}$, g is a function from $\mathbb{N} \times \mathbb{N} \times \{0,1\}$ to $\{0,1\}$, and h is a function from $T \times \mathbb{N}$ to $\{0,1\}$. Define $\phi : T \to \{0,1\}^{\mathbb{N}}$ by setting $\phi(t)(j) = f(t,j)$ for all $t \in T$, $j \in \mathbb{N}$; because R meets every $Q_{tt'}$, ϕ is injective.

For each $t \in T$, choose $p_t \in R$ such that $t \in I_{p_t}$. Now set

$$a'_t = \{k : k \in a_t,\ k \ge n_{p_t},\ h(t,k) = 0\},$$

$$a''_t = \{k : k \in a_t,\ k \ge n_{p_t},\ h(t,k) = 1\},$$

so that $a'_t \cup a''_t = a_t \setminus n_{p_t}$ is cofinite in a_t.

For any $j \in \mathbb{N}$, if $f(s,j) \ne f(t,j)$ then $a'_s \cap a'_t \subseteq j$ and $a''_s \cap a''_t \subseteq j$. **P?** Otherwise, we must be able to find $k \ge j$ and s, t such that $n_{p_s} \le k$, $n_{p_t} \le k$, $k \in a_s \cap a_t$, $h(s,k) = h(t,k)$ and $f(s,k) \ne f(t,k)$. Let $p \in R$ be such that $p_s \le p$ and $p_t \le p$; in this case both s and t belong to I_p, so $k < n_p$. Accordingly we must have $g_p(k, j, h_p(s,k)) = f_p(s,j)$ and $g_p(k, j, h_p(t,k)) = f_p(t,j)$, that is, $g(k, j, h(s,k)) = f(s,j)$ and $g(k, j, h(t,k)) = f(t,j)$. But this is impossible, because $h(s,k) = h(t,k)$ and $f(s,j) \ne f(t,j)$. **XQ**

It follows at once that $\langle a'_t \rangle_{t \in T}$ and $\langle a''_t \rangle_{t \in T}$ are treelike, as required.

(b) By (a), we can find treelike families $\langle a'_\xi \rangle_{\xi < \omega_1}$ and $\langle a''_\xi \rangle_{\xi < \omega_1}$ such that $a'_\xi \cup a''_\xi$ is cofinite in a_ξ for every $\xi < \omega_1$. Now there is an $n \in \mathbb{N}$ such that $\Gamma = \{\xi : \xi < \omega_1,\ a_\xi \subseteq a'_\xi \cup a''_\xi \cup n\}$ is uncountable. If we set $\tilde{a}'_\xi = a_\xi \cap (a'_\xi \cup n)$, $\tilde{a}''_\xi = a_\xi \cap (a''_\xi \cup n)$ we shall obtain treelike families $\langle \tilde{a}'_\xi \rangle_{\xi \in \Gamma}$ and $\langle \tilde{a}''_\xi \rangle_{\xi \in \Gamma}$ such that $\tilde{a}'_\xi \cup \tilde{a}''_\xi = a_\xi$ for every $\xi < \omega_1$.

A3. Lemma $[\mathfrak{m} > \omega_1]$ Let $\langle a_i \rangle_{i \in I}$ be any uncountable almost disjoint family of subsets of $\mathbb{N}$. Then there is an uncountable almost disjoint family $\langle b_j \rangle_{j \in J}$ such that $\{i : i \in I,\ a_i \setminus b_j$ is finite$\}$ is infinite for every $j \in J$.

proof We may suppose that $\#(I) = \omega_1$. Let $\langle I_\xi \rangle_{\xi < \omega_1}$ be a partition of I into countably infinite sets. Choose $\langle b_\xi \rangle_{\xi < \omega_1}$ inductively so that

$a_i \setminus b_\xi$ is finite whenever $i \in I_\xi$,

$b_\xi \cap b_\eta$ is finite whenever $\eta < \xi$,

$b_\xi \cap a_i$ is finite whenever $i \in I \setminus I_\xi$;

this will always be possible, by Fremlin 84, 21A.

Appendix B. A combinatorial fact

B1. Lemma Let X be any set, and $g : X \to X$ any function. Set $C = \{x : g(x) \ne x\}$. Then we can express C as $C_0 \cup C_1 \cup C_2$ where $C_j \cap g[C_j] = \emptyset$ for each j.

proof (a) Suppose first that X is finite. Set $X_k = g^k[X]$ for $k \in \mathbb{N}$. Then $\langle X_k \rangle_{k \in \mathbb{N}}$ is decreasing, so there is some k such that $g[X_k] = X_k$, that is, $g{\restriction}X_k : X_k \to X_k$ is a surjection; because X_k is finite, it is a bijection. For each orbit A of $g{\restriction}X_k$, choose a point $z_A \in A$. For each $x \in X$, let $A(x)$ be the orbit of $g{\restriction}X_k$ such that $g^k(x) \in A(x)$, and let $r(x)$ be the least integer such that $g^{r(x)}(x) = z_{A(x)}$. Set $C_0 = C \cap \{x : r(x) = 0\}$, $C_1 = \{x : r(x) > 0 \text{ is odd}\}$, $C_2 = \{x : r(x) > 0 \text{ is even}\}$; this works.

(b) For the general case, take any point $\infty \notin X$ and for $I \in [X]^{<\omega}$ define $g_I : I \cup \{\infty\} \to I \cup \{\infty\}$ by setting $g_I(x) = g(x)$ if $x \in I \cap g^{-1}[I]$, ∞ for other $x \in I \cup \{\infty\}$. By (a), we can express $C^{(I)} = \{x : x \in I \cup \{\infty\},\ g_I(x) \neq x\}$ as $\bigcup_{j \le 2} C_j^{(I)}$ where $g_I[C_j^{(I)}] \cap C_j^{(I)} = \emptyset$ for all j. Now let (C_0, C_1, C_2) be any cluster point of $\langle (C_0^{(I)}, C_1^{(I)}, C_2^{(I)}) \rangle_{I \in [X]^{<\omega}}$ as I increases; this works.

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