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Steffen's Flexihedron

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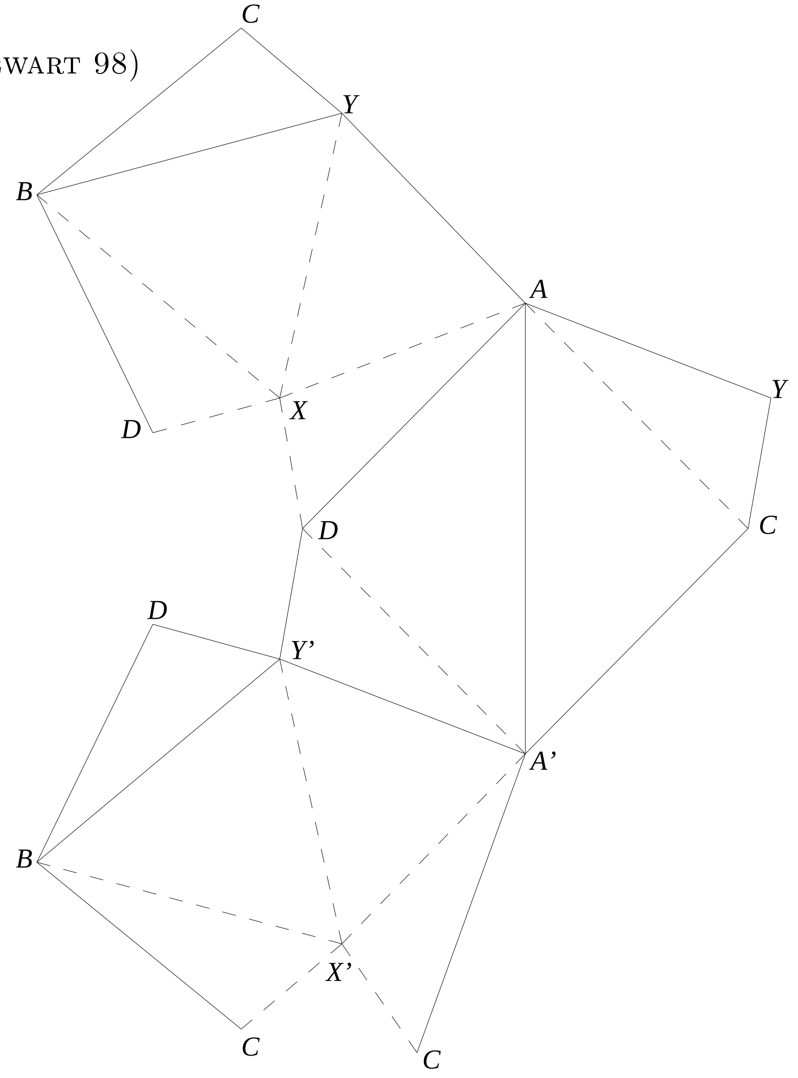
This remarkable construction, due to K.Steffen, is the simplest example known of a *flexihedron*, that is, a polyhedron in three dimensions which every face is rigid but the edges are free to bend, and the polyhedron as a whole is consequently flexible. If you hold the model by its 'roof', that is, the two large triangles meeting on the longest edge AA', you will find that the 'spike' B at the bottom can move back and forth, parallel to the ridge of the roof.

Construction (Stewart 98)



Valley folds are indicated by dashed lines

The diagram shows a possible template for a cardboard model. If you take the edge lengths to be

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$$AD = AC = BX = BY = A'D = A'C = BX' = BY' = 12 \text{ cm},$$

$$AX = AY = BC = BD = A'X' = A'Y' = 10 \text{ cm},$$

$$DX = CY = DY' = CX' = 5 \text{ cm},$$

$$X'Y' = XY = 11 \text{ cm}$$

you get a manageable object.

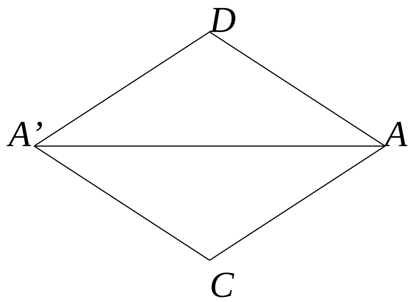
How do we know it really works?

However carefully you make the model, with each of the faces as flat and as stiff as possible, and the folds at each edge with as little twist as you can manage, it is not quite obvious that some hidden flexing is not taking place, making the whole thing a cheat. (Many origami models depend for their effect precisely on small amounts of flexibility in unexpected places leading to dramatic movements of the construction as a whole.) Unless we have a proper geometric proof it is very hard to be sure that the movements we detect are not due to twisting or stretching somewhere.

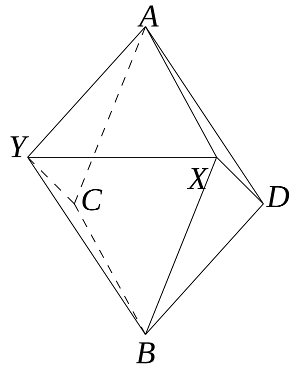
A possible argument runs as follows.

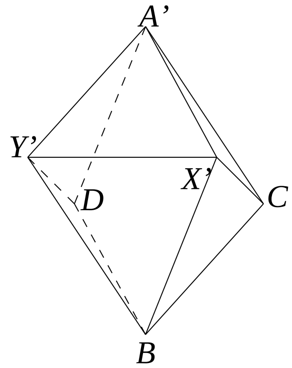
First, divide the model into three sections:

the *roof*



the *front* and the *back*





Next, identify the edges of equal length:

$$AD = AC = BX = BY = A'D = A'C = BX' = BY',$$

$$AX = AY = BC = BD = A'X' = A'Y',$$

$$DX = CY = DY' = CX',$$

$$X'Y' = XY.$$

(For the moment, we don't need to know what these lengths are. In fact, as long as you keep them fairly close to the proportions given, you will get a flexihedron whenever the equations above are satisfied.)

Fundamental Fact The reason the whole thing works is that

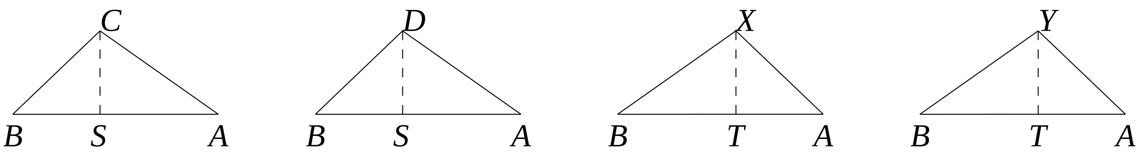
$$CD = XY.$$

proof Consider the line ℓ through A and B and

(i) the projections of C, D, X and Y on ℓ;

(ii) the projections of A, B, C, D, X and Y onto a plane π perpendicular to ℓ.

(i) If we look at the four triangles ACB, ADB, BXA and BYA, these are all congruent (because they all have sides of the same lengths):



So C and D project onto the same point of ℓ, call it S, while X and Y project onto the same point T.

(ii) At the same time we see that CS, DS, XT and YT are all equal, so if we write A_1, $C_1, \dots, Y_1$ for the projections of A, $C, \dots, Y$ on π, then $C_1A_1 = CS$, $D_1A_1 = DS$, $X_1A_1 = XT$ and $Y_1A_1 = YT$ will all be equal; that is, C_1, D_1, X_1 and Y_1 will be arranged in a circle around A_1.

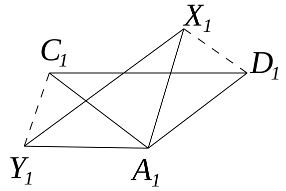


Fig. 4

Now look at the lengths Y_1C_1 and X_1D_1. By Pythagoras' theorem, we have

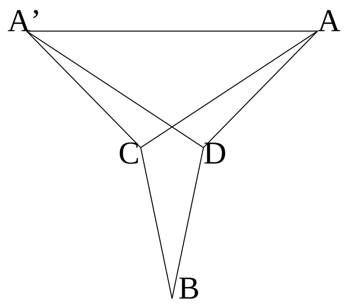
$$XD^2 = TS^2 + (X_1D_1)^2, \quad YC^2 = TS^2 + (Y_1C_1)^2;$$

since $XD = YC$, $X_1D_1 = Y_1C_1$. But this means that the angles $X_1\widehat{A_1}D_1$ and $Y_1\widehat{A_1}C_1$ must be equal, so that the angles $C_1\widehat{A_1}D_1$ and $Y_1\widehat{A_1}X_1$ are equal, and $Y_1X_1 = C_1D_1$. But as X and Y both project onto T, while C and D both project onto S, we get

$$XY = X_1Y_1 = C_1D_1 = CD,$$

as announced.

Armed with this fact, we can proceed to build the model as follows. Start by assembling the roof and the spike



fixing the length CD to be equal to the length required for XY and $X'Y'$, so that the tetrahedron $A'ADC$ is rigid, while the spike B can pivot in a circle with centre the midpoint of CD. Now build the front section by putting X the right distances from A, D and B, and Y the right distances from A, C and B; the 'fundamental fact' tells us that XY will have to be equal to CD, so will be the right length. Finally, build the back section by putting X' at the right distances from A, C and B, while Y' is the right distances from A, D and B; and again $X'Y'$ will be equal to CD, as required. Since this construction will work wherever B is (as long as it's sufficiently close to the central bottom point), the model will be able to flex without straining either the faces or the joints.

Of course the arguments here miss out any consideration of whether things can be fitted together as claimed; the lengths have to be such that, for instance, there actually is a point X at the right distances from A, D and B. A more subtle problem is that in fact there will be two such points, and X and Y must be the *right* points at the given distances for the arrangement of A_1, C_1, D_1, X_1 and Y_1 in fig. 4 above to be correct. Finally, it's not clear from the two-dimensional calculations that the model as a whole can be built without faces having to pass through each other. (Indeed, if B is too far away from the bottom point, or if you take the left-right mirror image of the front section without also reflecting the back section, it won't work.) All these questions are *combinatorial* rather than *metric*; you can build an approximate model to show that in fact we have room to spare.

There is a further remarkable fact about this object. If you move the spike backwards and forwards, you will see that the front and back sections move with it, so that the internal volume doesn't change much. And actually it does not change at all, and this is true of flexihedra in general: we have

The Bellows Theorem (Connelly Sabitov & Walz 97) For any flexihedron, the volume inside does not change as the polyhedron flexes.

References

Connelly R., Sabitov I. & Walz A. [97] 'The bellows conjecture', Beiträge zur Algebra und Geometrie 38 (1997) 1-10.

Stewart I. [98] 'The bellows conjecture', Scientific American 279(1) (July 1998) 90-92.