Compact sets without convergent sequences in random real models

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1. Lemma Let (X, Σ, μ) be a probability space and $(Y, \mathfrak{S})$ a separable metric space. Suppose that we have a non-negative finitely additive functional ν defined on the Borel subsets of Y, with $\nu Y = 1$, and a sequence $\langle f_i \rangle_{i \in \mathbb{N}}$ of measurable functions, each defined on a set $X_i \in \Sigma$ and taking values in Y, such that $\{x : x \in X_i \cap X_j,\ f_i(x) = f_j(x)\}$ is negligible whenever $i \neq j$.

Set $Z = \bigcap_{j \in \mathbb{N}} \bigcup_{i \geq j} X_i$. Then for any $\epsilon > 0$ there is an open set $H \subseteq Y$ such that $\nu \overline{H} \leq \epsilon$ and $\mu(Z \setminus \bigcup_{i \in \mathbb{N}} f_i^{-1}[H]) \leq \epsilon$.

proof (a) Set $\eta = \frac{1}{4}\epsilon > 0$ and let $M \in \mathbb{N}$ be such that $(1-\eta)^M \leq \eta$. Let $N \in \mathbb{N}$ be such that $\mu(Z \setminus Z') \leq \eta$, where

$$Z' = \{x : x \in Z,\ \#(\{i : i < N,\ x \in X_i\}) \geq M\}.$$

(b) Let $\mathcal{A}$ be the ideal of subsets $A \subseteq Y$ such that $\inf\{\nu G : A \subseteq G \in \mathfrak{S}\} = 0$. Let $\mathcal{E}$ be the algebra $\{E : E \subseteq Y,\ \partial E \in \mathcal{A}\}$, where $\partial E = \overline{E} \setminus \operatorname{int} E$ is the boundary of E. Then there is a partition $\mathcal{D}$ of Y into members of $\mathcal{E}$ such that $\mu B \leq \eta$, where

$$B = \bigcup_{i<j<N} \bigcup_{D \in \mathcal{D}} f_i^{-1}[D] \cap f_j^{-1}[D].$$

P Note that if ρ is any metric on Y inducing its topology, and $y_0 \in Y$, then $\alpha \mapsto \nu B(y_0, \alpha)$ is non-decreasing, therefore continuous at all but countably many points, so that $B(y_0, \alpha) \in \mathcal{E}$ for all but countably many α, and $\mathfrak{S} \cap \mathcal{E}$ is a base for $\mathfrak{S}$. There is therefore a sequence $\langle U_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{E}$ running over a base for $\mathfrak{S}$. For each n, let $\mathcal{D}_n$ be the partition of Y generated by $\{U_i : i \leq n\}$, and set

$$B_n = \bigcup_{i<j<N} \bigcup_{D \in \mathcal{D}_n} f_i^{-1}[D] \cap f_j^{-1}[D].$$

Then $\langle B_n \rangle_{n \in \mathbb{N}}$ is non-increasing and

$$\bigcap_{n \in \mathbb{N}} B_n = \bigcup_{i<j<n} \{x : x \in X_i \cap X_j,\ f_i(x) = f_j(x)\}$$

is negligible, so there is some n such that $\mu B_n \leq \eta$, and we can take $\mathcal{D} = \mathcal{D}_n$. **Q**

(c) Enumerate $\mathcal{D}$ as $\{E_r : r < m\}$. Let $K \subseteq m$ be a random set, in which each $r < m$ has independently a probability η of appearing. Set $F_K = \bigcup_{r \in K} E_r$ for each K. Then, for any particular $x \in Z' \setminus B$,

$$\Pr(x \notin f_i^{-1}[F_K] \text{ for every } i < N) = \Pr(K_x \cap K = \emptyset)$$

(where $K_x = \{r : x \in f_i^{-1}[E_r] \text{ for some } i < N\}$)

$$= (1-\eta)^{\#(K_x)} \leq (1-\eta)^M \leq \eta,$$

because if $x \in Z' \setminus B$ then

$$\#(K_x) = \#(\{i : i < N,\ x \in X_i\}) \geq M.$$

So

$$\Pr\big(\mu((Z' \setminus B) \setminus \bigcup_{i<N} f_i^{-1}[F_K]) \leq 2\eta\mu(Z' \setminus B)\big) \geq \frac{1}{2}.$$

At the same time,

$$\mathbb{E}(\nu F_K) = \sum_{r=0}^{m-1} \nu E_r \Pr(r \in K) = \eta,$$

so $\Pr(\nu F_K > 2\eta) < \frac{1}{2}$, and there must be some K such that

$$\nu F_K \leq 2\eta,$$

$$\mu((Z' \setminus B) \setminus \bigcup_{i<N} f_i^{-1}[F_K]) \leq 2\eta\mu(Z' \setminus B).$$

But this means that

$$\mu(Z \setminus \bigcup_{i\in\mathbb{N}} f_i^{-1}[F_K]) \leq 2\eta + \mu B + \mu(Z \setminus Z') \leq \epsilon.$$

(d) At this point, note that $F_K \in \mathcal{E}$, so there is an open set $G \supseteq \partial F_K$ such that $\nu G \leq \epsilon - \nu F_K$; now there is an open set H such that

$$F_K \subseteq H \subseteq \overline{H} \subseteq F_K \cup G,$$

and this serves.

Remark This result derives, we believe, from Shelah, via 1G of Fremlin n97 and §9 of Burke n96.

2. Lemma Let (X, Σ, μ) be a probability space, $(Y, \mathfrak{S})$ a separable metric space, and $\langle f_i\rangle_{i\in\mathbb{N}}$ a sequence of measurable functions from X to Y such that $\{x : f_i(x) = f_j(x)\}$ is negligible whenever $i \neq j$. Then for any $\epsilon > 0$ there is a disjoint sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ of open subsets of Y such that $\mu(\bigcap_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} f_i^{-1}[H_n]) \geq 1-\epsilon$.

proof Let $\langle \epsilon_n\rangle_{n\in\mathbb{N}}$ be a sequence of strictly positive numbers such that $\sum_{n=0}^{\infty} \epsilon_n \leq \frac{1}{2}\epsilon$. Fix on a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$, and set $\nu E = \lim_{i\to\mathcal{F}} \mu f_i^{-1}[E]$ for every Borel set $E \subseteq Y$.

Choose H_n, f_{ni}, X_{ni}, Z_n inductively, as follows. Each H_n will be an open subset of Y such that $\nu\overline{H}_n \leq \epsilon_n$. Given H_m for $m < n$, set

$$X_{ni} = X \setminus \bigcup_{m<n} f_i^{-1}[\overline{H}_m],$$

$$f_{ni} = f_i\restriction X_{ni}, \quad Z_n = \bigcap_{j\in\mathbb{N}} \bigcup_{i\geq j} X_{ni}.$$

Note that

$$\mu Z_n \geq \lim_{i\to\mathcal{F}} \mu X_{ni} = \nu(Y \setminus \bigcup_{m<n} \overline{H}_m) \geq 1 - \sum_{m=0}^{n-1} \epsilon_m \geq 1 - \frac{1}{2}\epsilon.$$

Also $X_{ni} \subseteq X_{mi}$ for every $m \leq n$, $i \in \mathbb{N}$ so $Z_n \subseteq Z_m$ for every $m \leq n$.

By Lemma 1, there is an open set $G_n \subseteq Y$ such that $\nu\overline{G}_n \leq \epsilon_n$, $\mu(Z_n \setminus \bigcup_{i\in\mathbb{N}} f_{ni}^{-1}[G_n]) \leq \epsilon_n$; set $H_n = G_n \setminus \bigcup_{m<n} \overline{H}_m$, and observe that $f_i^{-1}[H_n] = f_{ni}^{-1}[G_n]$ for every i. Continue.

At the end of the induction, set

$$Z = (\bigcap_{n\in\mathbb{N}} Z_n) \setminus \bigcup_{n\in\mathbb{N}}(Z_n \setminus \bigcup_{i\in\mathbb{N}} f_{ni}^{-1}[G_n]);$$

then $\mu(X \setminus Z) \leq \epsilon$, while for $x \in Z$ we must have $x \in \bigcup_{i\in\mathbb{N}} f_{ni}^{-1}[G_n] = \bigcup_{i\in\mathbb{N}} f_i^{-1}[H_n]$ for every n. So $\langle H_n\rangle_{n\in\mathbb{N}}$ is an appropriate sequence.

3. Theorem If we add any number of random reals to a model of ZFC + CH, we obtain a model in which there is a compact Hausdorff space of weight ω_1 with no non-trivial convergent sequence.

proof (a) In the ground model, let $K \subseteq \{0,1\}^{\omega_1}$ be homeomorphic to $\beta\mathbb{N}$, so that K is extremally disconnected. Write $\tilde{K}$ for the corresponding compact set in the extended model, so that $\tilde{K}$ can again be regarded as a closed subset of $\{0,1\}^{\omega_1}$, and has weight ω_1.

(b) Let $\langle \dot{x}_i\rangle_{i\in\mathbb{N}}$ be a name for a sequence of distinct elements of $\tilde{K}$, so that $[\![\dot{x}_i = \dot{x}_j]\!] = 0$ whenever $i \neq j$. Because the forcing is ccc, there is a countable ordinal $\alpha < \omega_1$ such that $[\![\dot{x}_i(\xi) = \dot{x}_j(\xi)$ for every $\xi < \alpha]\!] = 0$ whenever $i \neq j$. Set $Y = \{0,1\}^{\alpha}$ (in the ground model), and let (X, Σ, μ) be a probability space representing the measure algebra $(\mathfrak{A}, \bar{\mu})$ of the forcing, so that each $\dot{x}_i$ can be represented by a Baire measurable function $f_i : X \to \{0,1\}^{\omega_1}$. Set $g_i = f_i\restriction\alpha : X \to Y$ for each i; we have $g_i \neq g_j$ a.e. whenever $i \neq j$.

(c) Take any non-zero $a \in \mathfrak{A}$. Then there are (in the ground model) disjoint open sets H, H' in Y such that $\mu Z > 1 - \bar{\mu}a$, where

$$Z = (\bigcap_{n\in\mathbb{N}} \bigcup_{i\geq n} g_i^{-1}[H]) \cap (\bigcap_{n\in\mathbb{N}} \bigcup_{i\geq n} g_i^{-1}[H']).$$

P By Lemma 2, we can find a disjoint sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ of open sets in Y such that $\mu Z' > 1 - \bar{\mu}a$, where $Z' = \bigcap_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} g_i^{-1}[H_n]$. Set $H = \bigcup_{n\in\mathbb{N}} H_{2n}$, $H' = \bigcup_{n\in\mathbb{N}} H_{2n+1}$, and observe that $Z' \subseteq Z$. **Q**

Let $b \in \mathfrak{A}$ be the equivalence class of Z; then $b \cap a \neq 0$.

Set $V = \{w : w \in \{0,1\}^{\omega_1},\ w{\restriction}\alpha \in H\}$, $V' = \{w : w \in \{0,1\}^{\omega_1},\ w{\restriction}\alpha \in H'\}$. Because K is homeomorphic to $\beta\mathbb{N}$, which is extremally disconnected, and V, V' are disjoint open sets, there is a relatively open-and-closed $L \subseteq K$ including $K \cap V$ and disjoint from $K \cap V'$; express L as $K \cap W$ where $W \subseteq \{0,1\}^{\omega_1}$ is open-and-closed.

Interpreting $[\![\dot{x} \in V]\!]$ as $\sup_{U \subseteq V \text{ is open-and-closed}} [\![\dot{x} \in V]\!]$ for any name $\dot{x}$ for a member of $\{0,1\}^{\omega_1}$, we see that

$$[\![\dot{x}_i \in W]\!] \supseteq [\![\dot{x}_i \in V]\!],$$

$$[\![\dot{x}_i \notin W]\!] \supseteq [\![\dot{x}_i \in V']\!]$$

for every $i \in \mathbb{N}$. But this means that, given $n \in \mathbb{N}$,

$$\begin{aligned}[\![\dot{x}_i \in W \text{ for some } i \geq n]\!] &\supseteq [\![\dot{x}_i \in V \text{ for some} i \geq n]\!] \\ &\supseteq b.\end{aligned}$$

Similarly,

$$[\![\dot{x}_i \notin W \text{ for some } i \geq n]\!] \supseteq b.$$

But this means that

$$\begin{aligned}b &\subseteq [\![\dot{x}_i \in W \text{ for infinitely many } i,\ \dot{x}_i \notin W \text{ for infinitely many } i]\!] \\ &\subseteq [\![\langle \dot{x}_i \rangle_{i \in \mathbb{N}} \text{ does not converge}]\!],\end{aligned}$$

and $[\![\langle \dot{x}_i \rangle_{i \in \mathbb{N}} \text{ does not converge}]\!]$ meets a. As a is arbitrary, $[\![\langle \dot{x}_i \rangle_{i \in \mathbb{N}} \text{ does not converge}]\!] = 1$; as $\langle \dot{x}_i \rangle_{i \in \mathbb{N}}$ is arbitrary, $\tilde{K}$ has no non-trivial convergent sequence.

4. Efimov's problem An old question due to Efimov is

> Is there an infinite compact Hausdorff space without non-trivial convergent sequences which does not include a copy of $\beta\mathbb{N}$?

Fedorchuk 77 gives a construction of such a space if $\mathfrak{s} = \omega_1$ and $2^{\omega_1} < 2^{\mathfrak{c}}$; his method gives a space of weight $\mathfrak{c}$ and cardinal 2^{ω_1}. ($\mathfrak{s}$ is the least cardinal of any set $A \subseteq \mathcal{P}\mathbb{N}$ such that for every infinite $b \subseteq \mathbb{N}$ there is an $a \in A$ such that $b \cap a$ and $b \setminus a$ are both infinite.) Talagrand 80 gives a construction dependent on the continuum hypothesis. Of course $\beta\mathbb{N}$ cannot be embedded into any space of weight less than $\mathfrak{c}$, so Theorem 3 shows that in random real models we can get examples of weight ω_1 while $\mathfrak{c}$ is large. The special position of ω_1 is just that, in the ground model, there is a zero-dimensional compact Hausdorff space X of weight ω_1 which has no non-trivial convergent sequences. In particular, if κ is an atomlessly-measurable cardinal created by Solovay's method, there will be an example of weight less than κ.

> **Problem** Can we deduce the existence of such a space just from the assumption that there is some atomlessly-measurable cardinal?

Note that if there is an atomlessly-measurable cardinal, then there is a subset of $[0,1]$ of full outer measure and cardinal ω_1 (Fremlin 93, 6F), so that $\mathfrak{s} = \omega_1$. (If $A \subseteq \mathcal{P}\mathbb{N}$ is of full outer measure for the usual measure of $\mathcal{P}\mathbb{N}$, then it meets all the conegligible sets $\{a : b \cap a \text{ and } b \setminus a \text{ are infinite}\}$ for infinite sets $b \subseteq \mathbb{N}$.) S.Shelah (October 2004) remarked that if we start with GCH and a two-valued-measurable cardinal κ, and add λ random reals where $\lambda > \kappa$ has cofinality ω_1, then in the extension we shall have $\mathfrak{c} = \lambda$ and $2^{\omega_1} = 2^{\lambda} = \lambda^+$, while κ is atomlessly-measurable. In such a model, the conditions of Fedorchuk's and Talagrand's theorems are not satisfied, but we still have a positive answer to Efimov's question.

References

Burke M.R. [n96] 'A theorem of Gitik and Shelah on disjoint refinements of sequences of subsets of the real line', Note of 23.8.96.

Fedorchuk V.V. [77] 'A compact space having the cardinality of the continuum with no convergent sequences', Math. Proc. Cambridge Phil. Soc. 81 (1977) 177-181.

Fremlin D.H. [93] 'Real-valued-measurable cardinals', pp. 151-304 in Judah 93.

Fremlin D.H. [n97] 'A theorem of Gitik and Shelah on disjoint refinements of sequences of subsets of the real line', Note of 4.4.97.

Judah H. [93] (ed.) *Proceedings of the Bar-Ilan Conference on Set Theory and the Reals, 1991.* Amer. Math. Soc. (Israel Mathematical Conference Proceedings 6), 1993.

Talagrand M. [80] 'Un nouveau $\mathcal{C}(K)$ qui possède la propriété de Grothendieck', Israel J. Math. 37 (1980) 181-191.