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Invariant measures on metric spaces

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Definition Let (X,ρ) be a metric space. Its **isometry group** is the group of isometric bijections from X to itself (the group of invertible elements in the semigroup of all isometries from X to itself).

2. Lemma Let (X,ρ) be a metric space and G^* its isometry group. Let G be any subgroup of G^*.

(a) We have a pseudometric σ on X defined by the formula

$$\sigma(x,y)=\inf_{g\in G}\rho(g(x),y)$$

for all x, $y\in X$.

(b) Write $(Z,\bar\sigma)$ for the corresponding quotient metric space and $\pi:X\to Z$ for the canonical map, so that

$$\pi(x)=\{y:\sigma(x,y)=0\}=\overline{\{g(x):x\in X\}}$$

for $x\in X$. Every $F\in Z$ is invariant under the action of G, so that we have a subgroup $G_F=\{g\restriction F:g\in G\}$ of the isometry group of F, and $\{g(x):g\in G_F\}=\{g(x):g\in G\}$ is dense in F for every $x\in F$.

(c) In particular, if there is any $x\in X$ such that $\{g(x):g\in G\}$ is dense in X, so that Z is a singleton, then $\{g(y):g\in G\}$ is dense in X for any $y\in X$.

(d) $\pi:X\to Z$ is continuous and open; so that Z is locally compact or σ-compact if X is.

proof (a) If x, y, $z\in X$ then

$$\sigma(y,x)=\inf_{g\in G}\rho(g(y),x)=\inf_{g\in G}\rho(y,g^{-1}(x))=\sigma(x,y),$$

$$\begin{aligned}\sigma(x,y)+\sigma(y,z)&=\inf_{g,h\in G}\rho(g(x),y)+\rho(h(y),z)\\&=\inf_{g,h\in G}\rho(hg(x),h(y))+\rho(h(y),z)\le\inf_{g,h\in G}\rho(hg(x),z)=\sigma(x,z).\end{aligned}$$

So σ is a pseudometric on X.

(b) Since $\sigma(y,g(y))=0$ for every $y\in X$, $g\in G$, we see that

$$\sigma(x,y)=0\Rightarrow\sigma(x,g(y))=0,$$

so that the sets

$$\pi(x)=\{y:\sigma(x,y)=0\}=\overline{\{g(x):x\in X\}}$$

are G-invariant.

(c) is now obvious.

(d) As

$$\bar\sigma(\pi(x),\pi(y))=\sigma(x,y)\le\rho(x,y)$$

for all x, $y\in X$, π is continuous. If $U\subseteq X$ is open and $F\in\pi[U]$, take $x\in F$ and let $\delta>0$ be such that U includes $B(x,\delta)=\{y:\rho(y,x)\le\delta\}$. If $F'\in Z$ and $\bar\sigma(F',F)<\delta$, take any $x'\in F'$; then $\sigma(x',x)<\delta$, so there is a $g\in G$ such that $\rho(x,g(x'))<\delta$, and $g(x')\in U$. But this means that $F'=\pi(g(x'))\in\pi[U]$. So $\pi[U]$ includes $\{F':\sigma(F',F)<\delta\}$; as F is arbitrary, $\pi[U]$ is open; as U is arbitrary, π is an open mapping.

Now a continuous image of a σ-compact space is σ-compact, and an open continuous image of a locally compact space is locally compact.

3. Lemma Let (X,ρ) be a metric space, and G^* its isometry group. Give G^* the topology generated by the functions $g\mapsto g(x)$, $g\mapsto g^{-1}(x)$ as x runs over X.

(a) G^* is a Hausdorff topological group.

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(b) If X is separable, G^* is separable and metrizable.
(c) If X is complete, G^* is complete in its bilateral uniformity.
(d) If $Y \subseteq X$ is G-invariant, then $g \mapsto g \restriction Y$ is a continuous homomorphism from G^* to the isometry group of Y.

proof (a)(i) Multiplication is continuous. **P** Let $x_0 \in X$, g_0, $h_0 \in G^*$ and $\epsilon > 0$. Suppose that g, $h \in G$ are such that $\rho(h(x_0), h_0(x_0)) \le \frac{1}{2}\epsilon$, $\rho(gh_0(x_0), g_0h_0(x_0)) \le \frac{1}{2}\epsilon$. Then

$$\begin{aligned}\rho(gh(x_0), g_0h_0(x_0)) &= \rho(h(x_0), g^{-1}g_0h_0(x_0)) \\ &\le \rho(h(x_0), h_0(x_0)) + \rho(h_0(x_0), g^{-1}g_0h_0(x_0)) \\ &= \rho(h(x_0), h_0(x_0)) + \rho(gh_0(x_0), g_0h_0(x_0)) \le \epsilon.\end{aligned}$$

As g_0, h_0 and ϵ are arbitrary, the function $(g, h) \mapsto gh(x_0)$ is continuous. Similarly, the function $(g, h) \mapsto h^{-1}g^{-1}(x_0)$ is continuous. As x_0 is arbitrary, multiplication is continuous. **Q**

(ii) Of course inversion is continuous. So G^* is a topological group. Also, of course, it is Hausdorff.

(b) If $D \subseteq X$ is a countable dense subset, let $\mathfrak{S}$ be the metrizable topology on G^* generated by the functions $g \mapsto g(x)$, $g \mapsto g^{-1}(x)$ as x runs over D. Then, given $z \in X$, $\epsilon > 0$ there is an $x \in D$ such that $\rho(z, x) \le \epsilon$, in which case $\rho(g(z), g(x)) \le \epsilon$ for every $g \in G^*$. But this means that the map $g \mapsto g(z)$ is the uniform limit of $\mathfrak{S}$-continuous functions and is $\mathfrak{S}$-continuous. Similarly, $g \mapsto g^{-1}(z)$ is $\mathfrak{S}$-continuous, and $\mathfrak{S}$ is equal to the topology under consideration, which is therefore metrizable.

At the same time, since $\mathfrak{S}$ is the topology corresponding to an identification of G^* with a subset of $X^D \times X^D$, it is separable.

(c) Now suppose that X is complete. Let $\mathcal{F}$ be a filter on G^* which is Cauchy for the bilateral uniformity on G^*. Then $\lim_{g\to\mathcal{F}} g(x)$ is defined for every $x \in X$. **P** Let $\mathcal{F}_x$ be the image filter on X corresponding to $\mathcal{F}$ under the map $g \mapsto g(x)$, that is,

$$\mathcal{F}_x = \{A : A \subseteq X,\ \{g : g(x) \in A\} \in \mathcal{F}\}.$$

If $\epsilon > 0$, then $U = \{g : \rho(g(x), x) \le \frac{1}{2}\epsilon\}$ is a neighbourhood of the identity in G^*, so there is a $g_0 \in G^*$ such that $g_0U \in \mathcal{F}$. If g, $h \in g_0U$, then

$$\rho(g(x), h(x)) = \rho(g_0^{-1}g(x), g_0^{-1}h(x)) \le \rho(g_0^{-1}g(x), x) + \rho(x, g_0^{-1}h(x)) \le \epsilon$$

because $g_0^{-1}g$ and $g_0^{-1}h$ belong to U. So the diameter of $C = \{g(x) : g \in g_0U\}$ is at most ϵ. But $C \in \mathcal{F}_x$. As ϵ is arbitrary, $\mathcal{F}_x$ is a Cauchy filter and converges in X; and its limit is $\lim_{g\to\mathcal{F}} g(x)$. **Q**

Set $\tilde{g}(x) = \lim_{g\to\mathcal{F}} g(x)$ for every $x \in X$. Then $\tilde{g} : X \to X$ is an isometry.

Similarly, $\tilde{h}(x) = \lim_{g\to\mathcal{F}} g^{-1}(x)$ is defined for every $x \in X$, and $\tilde{h} : X \to X$ is an isometry.

Consider $\tilde{h}\tilde{g} : X \to X$. Take $x \in X$ and $\epsilon > 0$. By the arguments just above, there are A, $B \in \mathcal{F}$ such that

$$\rho(g(x), h(x)) \le \tfrac{1}{2}\epsilon \text{ for all } g,\ h \in A,$$

$$\rho(g^{-1}\tilde{g}(x), h^{-1}\tilde{g}(x)) \le \tfrac{1}{2}\epsilon \text{ for all } g,\ h \in B.$$

Consequently

$$\rho(g(x), \tilde{g}(x)) \le \tfrac{1}{2}\epsilon \text{ for all } g \in A,$$

$$\rho(g^{-1}\tilde{g}(x), \tilde{h}\tilde{g}(x)) \le \tfrac{1}{2}\epsilon \text{ for all } g \in B.$$

Take any $g \in A \cap B$; then

$$\rho(\tilde{h}\tilde{g}(x), x) \le \rho(\tilde{h}\tilde{g}(x), g^{-1}\tilde{g}(x)) + \rho(g^{-1}\tilde{g}(x), x) \le \frac{1}{2}\epsilon + \rho(\tilde{g}(x), g(x)) \le \epsilon.$$

As x and ϵ are arbitrary, $\tilde{h}\tilde{g}$ is the identity on X. Similarly, $\tilde{g}\tilde{h}$ is the identity, and $\tilde{g} \in G^*$, with $\tilde{h} = \tilde{g}^{-1}$.

Now, of course, $\mathcal{F} \to \tilde{g}$ in G^*. As $\mathcal{F}$ is arbitrary, G^* is complete.

(d) This is elementary.

4. Lemma Let X be a non-empty locally compact metric space and G^* its isometry group. Suppose that for some (equivalently, any) $x_0 \in X$, the orbit $\{g(x) : g \in G^*\}$ is dense in X.

(a) There is a $\delta > 0$ such that all the balls $B(x, 2\delta)$, for $x \in X$, are compact.

(b) Define $K_0 = \{x_0\}$, $K_{n+1} = \{x : \rho(x, K_n) \le \delta\}$ for $n \in \mathbb{N}$. Then every K_n is compact, and if $g \in G^*$, $g(x_0) \in K_m$ then $g[K_n] \subseteq K_{m+n}$ for every n.

(c) Setting $X_0 = \bigcup_{n\in\mathbb{N}} K_n$, X_0 is an open-and-closed subset of X (indeed, $\rho(x, y) > \delta$ whenever $x \in X_0$ and $y \in X \setminus X_0$), and for any $g \in G^*$ we have either $g[X_0] = X_0$ or $g[X_0] \cap X_0 = \emptyset$. The family $\langle g[X_0]\rangle_{g\in G^*}$ is a partition of X into open-and-closed sets; these are the equivalence classes in X under the transitive closure of the relation '$\rho(x, y) \le \delta$'.

(d) Writing G_0^* for the isometry group of X_0, $\{g(x_0) : g \in G_0^*\}$ is dense in X_0.

proof (a) Let $\epsilon > 0$ be such that $B(x_0, \epsilon)$ is compact, and take any $\delta < \frac{1}{2}\epsilon$. If $x \in X$, there is a $g \in G^*$ such that $\rho(g(x_0), x) \le \epsilon - 2\delta$, so that $B(x, 2\delta)$ is a closed subset of the compact set $B(g(x_0), \epsilon) = g[B(x_0, \epsilon)]$ and is therefore compact.

(b) Induce on n. Of course K_0 is compact. Given that K_n is compact, then for any $x \in K_{n+1}$ the continuous function $y \mapsto \rho(x, y)$ attains its bounds on K_n, and there is a $y \in K_n$ such that $\rho(x, y) \le \delta$. Now there is a finite $I \subseteq K_n$ such that $K_n \subseteq \bigcup_{y\in I} B(y, \delta)$; so that $K_{n+1} \subseteq \bigcup_{y\in I} B(y, 2\delta)$ is relatively compact. Also the function $x \mapsto \rho(x, K_n)$ is continuous, so K_{n+1} is closed, therefore compact.

If $g \in G^*$, $g(x_0) \in K_m$ then $g[K_0] \subseteq K_m$. If $g[K_n] \subseteq K_{m+n}$ then

$$g[K_{n+1}] = \{x : \rho(x, g[K_n]) \le \delta\} \subseteq K_{m+n+1}.$$

So $g[K_n] \subseteq K_{m+n}$ for every $n \in \mathbb{N}$.

(c) From the construction of the K_n, we surely have $\rho(x, y) > \delta$ whenever $x \in \bigcup_{n\in\mathbb{N}} K_n$, $y \in X \setminus \bigcup_{n\in\mathbb{N}} K_n$, so X_0 is open-and-closed. From the second part of (b), we have $g[X_0] \subseteq X_0$ whenever $g \in G^*$, $g(x_0) \in X_0$. But in this case we also have $g^{-1}(x_0) \in X_0$. **P** There are $x_1, \dots, x_n \in X$ such that $g(x_0) = x_n$ and $\rho(x_{i+1}, x_i) \le \delta$ for $i < n$. But now $\rho(g^{-1}(x_{i+1}), g^{-1}(x_i)) \le \delta$ for $i < n$, while $g^{-1}(x_n) = x_0$, so that $g^{-1}(x_0) \in K_n \subseteq X_0$. **Q** So $g^{-1}[X_0] \subseteq X_0$ and $g[X_0] = X_0$.

More generally, if $g \in G^*$ is such that $g[X_0]$ meets X_0, then $g[X_0] = X_0$. **P** Let $y \in X_0$ be such that $g(y) \in X_0$. Set $K'_0 = \{y\}$, $K'_{n+1} = \{x : \rho(x, K'_n) \le \delta\}$ for each n. If $y \in K_m$, then $x_0 \in K'_m$ and $K'_n \subseteq K_{m+n}$, $K_n \subseteq K'_{m+n}$ for every n. So the argument just above shows that

$$g[X_0] = g[\textstyle\bigcup_{n\in\mathbb{N}} K_n] = \bigcup_{n\in\mathbb{N}} K'_n = X_0. \quad \mathbf{Q}$$

For any $g \in G^*$, $g[X_0]$ is an open-and-closed set, and $\rho(x, y) > \delta$ whenever $x \in g[X_0]$, $y \in X \setminus g[X_0]$. Moreover, if g, $h \in G$ and $g[X_0] \cap h[X_0] \ne \emptyset$, then $X_0 \cap g^{-1}h[X_0]$ is not empty, so $X_0 = g^{-1}h[X_0]$ and $g[X_0] = h[X_0]$. Thus $\langle g[X_0]\rangle_{g\in G^*}$ is disjoint.

Finally, if $x \in X$, there is some $g \in G^*$ such that $\rho(x, g(x_0)) < \delta$, so that $g^{-1}(x) \in X_0$ and $x \in g[X_0]$. So $\langle g[X_0]\rangle_{g\in G^*}$ is a partition of X.

(d) From (c), we see that $g\restriction X_0 \in G_0^*$ for every $g \in G^*$, so that $\{g(x_0) : g \in G_0^*\}$ includes $X_0 \cap \{g(x) : g \in G^*\}$, which is dense in X_0 (because X_0 is open in X).

5. Lemma Let (X, ρ) be a metric space, and suppose that $\delta > 0$ is such that (i) $B(x, 2\delta)$ is compact for every $x \in X$ (ii) any two points in X can be joined by a finite chain in which each link is of length at most δ. Then

(a) X is σ-compact;

(b) the isometry group G^* is metrizable, σ-compact and locally compact (in the topology described in Lemma 3);

(c) for any $x \in X$, the map $g \mapsto g(x) : G^* \to X$ is perfect (that is, $\{g : g(x) = y\}$ is compact for every y, and $\{g(x) : g \in F\}$ is closed in X whenever F is a closed subset of G^*).

proof If X is empty, this is all trivial, so let us take it for granted henceforth that X is non-empty.

(a) For any $x \in X$, set $K_0(x) = \{x\}$, $K_{n+1}(x) = \{y : \rho(y, K_n(x)) \le \delta\}$ for each $n \in \mathbb{N}$. Then $X = \bigcup_{n\in\mathbb{N}} K_n$, by the hypothesis (ii), and every $K_n(x)$ is compact, as in Lemma 4. Accordingly X is σ-compact. Also $x \in K_n(y)$ iff $y \in K_n(x)$, and $K_n(x) \subseteq K_{m+n}(y)$ if $x \in K_m(y)$; finally, $g[K_n(x)] = K_n(g(x))$ for any $x \in X$, $g \in G^*$. In particular, if $g \in G^*$ and $g(x) \in K_n(x)$, then $x \in K_n(g^{-1}(x))$ so $g^{-1}(x) \in K_n(x)$.

(b)(i) By Lemma 3b, G^* is metrizable.

(ii) If $x \in X$ and $K \subseteq X$ is compact, then $L = \{g : g \in G^*,\ g(x) \in K\}$ is compact. **P** (α) Since $K_n(x) \subseteq \operatorname{int} K_{n+1}(x)$ for every n, $X = \bigcup_{n\in\mathbb{N}} \operatorname{int} K_n(x)$ and there is some n such that $K \subseteq K_n(x)$. (β) If $g \in L$ and $y \in K_m(x)$, then $g(y) \in K_m(g(x)) \subseteq K_{m+n}(x)$. But this means that $\{g(y) : g \in L\}$ is relatively compact. At the same time, $g^{-1}(x) \in K_n(x)$ for every $g \in L$, so $\{g^{-1}(y) : g \in L\} \subseteq K_{m+n}(x)$ is also relatively compact. (γ) It follows that if $\mathcal{F}$ is any ultrafilter on L, $\tilde{g}(y) = \lim_{g\to\mathcal{F}} g(y)$ and $\tilde{h}(y) = \lim_{g\to\mathcal{F}} g^{-1}(y)$ are defined in X. Of course both $\tilde{g}$ and $\tilde{h}$ are isometries. (δ) As in the proof of Lemma 3, $\tilde{g}\tilde{h}$ and $\tilde{h}\tilde{g}$ are both the identity function on X, so that $\tilde{g} \in G^*$. (ϵ) Now, of course, $\tilde{g} = \lim \mathcal{F}$ in G^*, and $\tilde{g} \in L$. (ζ) As $\mathcal{F}$ is arbitrary, L is compact. **Q**

(iii) Taking any $x \in X$, $g_0 \in G^*$ we see that

$$\{g : \rho(g(x), g_0(x)) \le \delta,\ \rho(g^{-1}(x), g_0^{-1}(x)) \le \delta\}$$

is a closed neighbourhood of g_0 included in the compact set $\{g : g(x),\ g^{-1}(x) \in B(g_0(x), \delta) \cup B(g_0^{-1}(x), \delta)\}$, so is compact. Thus G^* is locally compact. Also

$$G^* = \textstyle\bigcup_{n\in\mathbb{N}} \{g : g(x) \in K_n(x)\}$$

is σ-compact.

(c) Writing $\phi(g) = g(x)$, we see from (b) that the inverse image of any compact set is compact. Because X is locally compact and ϕ is continuous, it follows that ϕ is perfect.

Remark Compare Theorem I.47 of [KoNoFC]: if X is connected and locally compact then G^* is locally compact.

6. Lemma Let X be a metric space and G^* its isometry group; give G^* the topology of Lemma 3. Let μ be any Radon measure on X. Then

(a) for any open sets U, $V \subseteq X$, $C_{VU} = \{g : g \in G^*,\ \mu(V \cap g^{-1}[U]) \le \mu U\}$ is closed in G^*;

(b) $\{g : g \in G^*,\ \mu g^{-1} = \mu\}$ is closed in G^*.

proof (a) ? Suppose, if possible, that $g \in \overline{C_{VU}}$ but $\mu(V \cap g^{-1}[U]) > \mu U$.. Then there is a compact set $K \subseteq V \cap g^{-1}[U]$ such that $\mu U < \mu K$. $g[K]$ is a compact set included in U, so there is an $\eta > 0$ such that $B(g(x), 3\eta) \subseteq U$ for every $x \in K$. Let $x_0, \dots, x_n \in K$ be such that $K \subseteq \bigcup_{i\le n} B(x_i, \eta)$. Then there is an $h \in C_{VU}$ such that $\rho(h(x_i), g(x_i)) \le \eta$ for every $i \le n$. In this case, if $x \in K$, there is some $i \le n$ such that $\rho(x, x_i) \le \eta$, so that

$$\begin{aligned}\rho(h(x), g(x)) &\le \rho(h(x), h(x_i)) + \rho(h(x_i), g(x_i)) + \rho(g(x_i), g(x))\\ &\le \rho(x, x_i) + \eta + \rho(x_i, x) \le 3\eta.\end{aligned}$$

So $h[K] \subseteq U$ and $K \subseteq V \cap h^{-1}[U]$ and

$$\mu U < \mu K \le \mu(V \cap h^{-1}[U]);$$

which is impossible, as $h \in C_{VU}$. **X**

(b) Note that as $g : X \to X$ is a homeomorphism for every $g \in G^*$, μg^{-1} is always a Radon measure on X. So

$$\begin{aligned}\{g : \mu g^{-1} = \mu\} &= \bigcap_{U\subseteq X \text{ is open}} \{g : \mu g^{-1}[U] = \mu U\}\\ &= \bigcap_{U\subseteq X \text{ is open}} \{g : g,\ g^{-1} \in C_{XU}\}\end{aligned}$$

is closed.

7. Theorem Let X be a non-empty locally compact metric space and G a group of isometries of X such that $\{g(x) : x \in X\}$ is dense in X for some (equivalently, any) $x \in X$. Then there is a non-zero Radon measure on X which is invariant under the action of G, and any two such measures are multiples of each other.

proof (a) We know from [Frp00*], 441G that there is a non-zero Radon measure μ on X which is invariant under the action of the whole isometry group G^*. The rest of the proof is therefore devoted to showing that μ is, up to a scalar multiple, unique.

(b) To begin with, let us suppose that $\delta > 0$ is such that X satisfies the conditions of Lemma 5. In this case, we know that G^* is locally compact, etc. Let $\overline{G}$ be the closure of G in G^*; then $\overline{G}$ is also a locally compact Hausdorff group.

Fix any $x_0 \in X$. By Lemma 5, $\{g(x_0) : g \in \overline{G}\}$ is closed, so must be the whole of X. Setting $H = \{g : g \in \overline{G},\, g(x_0) = x_0\}$, H is a compact subgroup of $\overline{G}$, and we have a natural bijection θ between the set $\overline{G}/H$ of left cosets of H in $\overline{G}$, and X, given by setting $\theta(gH) = g(x_0)$ for every $g \in \overline{G}$; for

$$g_1H = g_2H \iff g_1^{-1}g_2 \in H \iff g_1^{-1}g_2(x_0) = x_0 \iff g_1(x_0) = g_2(x_0).$$

Let ν be the Haar probability measure on H.

Because the map $g \mapsto g(x_0) : \overline{G} \to X$ is continuous, the topology $\mathfrak{S}$ on X corresponding to the quotient topology on $\overline{G}/H$ is finer than the original metric topology $\mathfrak{T}$ on X. Because $\overline{G}$, like G^*, is metrizable and σ-compact, $\mathfrak{S}$ has a countable network consisting of $\mathfrak{S}$-compact sets. But this means that both $\mathfrak{S}$ and $\mathfrak{T}$ are analytic, so every $\mathfrak{T}$-Radon measure is $\mathfrak{S}$-Radon ([Frp00*], 432H).

Now any G-invariant $\mathfrak{T}$-Radon measure μ on X is also $\overline{G}$-invariant (Lemma 6), and we have just seen that it is $\mathfrak{S}$-Radon. By [Frp00*], 444P, we have a corresponding left Haar measure λ_μ on $\overline{G}$ defined by the formulae

$$\lambda_\mu(E) = \textstyle\int \nu_x(E)\mu(dx), \qquad \nu_x(E) = \nu(H \cap h^{-1}[E]) \text{ whenever } h(x_0) = x$$

for Haar measurable sets $E \subseteq \overline{G}$. Now if μ, μ' are two G-invariant measures on X, then $\lambda_{\mu'} = \alpha\lambda_\mu$ for some α, so $\mu' = \alpha\mu$.

Thus in this case we have the result.

(c) For the general case, observe that by Lemma 4 there is a non-empty open-and-closed set $X_0 \subseteq X$ satisfying the conditions of Lemma 5, and that $g[X_0] = X_0$ for any $g \in G^*$ such that $g[X_0] \cap X_0 \neq \emptyset$. So $G_0 = \{g\restriction X_0 : g \in G\}$ satisfies the conditions of this theorem. If μ, μ' are G-invariant Radon measures on X, then the corresponding subspace measures μ_0, μ_0' are G_0-invariant Radon measures on X_0, and $\mu_0' = \alpha\mu_0$ for some α.

Now $\{g[X_0] : g \in G^*\}$ is a disjoint family of open sets in X with union X; since $\bigcup_{g\in G} g[X_0]$ is dense, it must be the whole of X. For any $g \in G$, $\mu_0 g^{-1}$ and $\mu_0' g^{-1}$ are the subspace measures on $g[X_0]$ induced by μ, μ' respectively, and $\mu_0' g^{-1} = \alpha\mu_0 g^{-1}$. But this means that $\mu' = \alpha\mu$. As μ and μ' are arbitrary, we have the result in the general case.

8. Corollary Let X be a locally compact metric space and G a group of isometries of X such that $\{g(x) : g \in G\}$ is dense in X for every $x \in X$. Let μ be any G-invariant measure on X. If E, $A \subseteq X$ are such that $\mu E > 0$ and $\mu^* A > 0$, then there is a $g \in G$ such that $\mu^*(g[E] \cap A) > 0$.

proof There is surely some compact set K such that $\mu^*(A \cap K) > 0$, so we may suppose that A is relatively compact, and there is a Borel set $F_0 \supseteq A$ which is a measurable envelops oe A. For Borel sets $F \subseteq X$, set

$$\nu F = \sup\{\mu(F \cap \textstyle\bigcup_{g\in I} g[E]) : I \subseteq G \text{ is finite}\}.$$

It is easy to check that ν is a measure, inner regular with respect to the compact sets and locally finite; so that its complete locally determined version $\tilde{\nu}$ is a Radon measure ([Frp00*], 416F). Also ν is G-invariant, so $\tilde{\nu}$ also is, and ν is non-zero, since it agrees with μ on (Borel) subsets of E. But this means that $\tilde{\nu}$ is a non-zero multiple of μ and has the same negligible sets. Now $\mu F_0 > 0$, so $\nu F_0 = \tilde{\nu} F_0 > 0$, and there must be some $g \in G$ such that $0 < \mu(F_0 \cap g[E]) = \mu^*(A \cap g[E])$.

9. Lemma Let X be a metric space and G a group of isometries of X such that $\{g(x) : x \in X\}$ is dense in X for some (equivalently, any) $x \in X$. Suppose that $U \subseteq X$ is an open set and μ_0 a Radon measure on U such that $\mu_0(U \cap g[V]) \leq \mu_0 V$ whenever $V \subseteq U$ is open and $g \in G$. Then there is a G-invariant Radon measure μ on X, extending μ_0, such that $\mu(X \setminus \bigcup_{g\in G} g[U]) = 0$.

proof (a) If $\mu_0 U = 0$ the result is trivial. Otherwise, take an open set $\tilde{U} \subseteq U$ such that $0 < \mu_0\tilde{U} < \infty$. Note that if $V \subseteq U$ is open, $g \in G$ and $g[V] \subseteq U$, then

$$\mu_0 g[V] = \mu_0(U \cap g[V]) \le \mu_0 V, \quad \mu_0 V = \mu_0(U \cap g^{-1}[g[V]]) \le \mu_0 g[V],$$

so $\mu_0 g[V] = \mu_0 V$. Every G-orbit in X meets U, so $X = \bigcup_{g \in G} g[\tilde{U}]$. Let $\mathcal{V}$ be a locally finite open cover of X refining the open cover $\{g[\tilde{U}] : g \in G\}$, and enumerate $\mathcal{V}$ as $\langle V_\xi \rangle_{\xi<\kappa}$. For each ξ choose $g_\xi \in G$ such that $V_\xi \subseteq g_\xi[U]$, and set $\tilde{V}_\xi = \bigcup_{\eta<\xi} V_\eta$, $V'_\xi = V_\xi \setminus \tilde{V}_\xi$.

Set $\Sigma_0 = \operatorname{dom} \mu_0$,

$$\Sigma = \{E : E \subseteq X,\ g_\xi^{-1}[E \cap V'_\xi] \in \Sigma_0 \text{ for every } \xi < \kappa\},$$

$$\mu E = \textstyle\sum_{\xi<\kappa} \mu_0 g_\xi^{-1}[E \cap V'_\xi]$$

for $E \in \Sigma$. It is easy to see that μ is a complete locally determined topological measure, inner regular with respect to the compact sets, because μ_0 is. To see that μ is locally finite, note that if $x \in X$ then there is an open set V containing X such that $I = \{\xi : V \cap V_\xi \ne \emptyset\}$ is finite, so that $V \subseteq \bigcup_{\xi \in I} V'_\xi$ and $\mu V \le \#(I)\mu\tilde{U}$ is finite.

(b) μ extends μ_0. **P** If $U' \subseteq U$ is open, then for any $\xi < \kappa$

$$\mu_0 g_\xi^{-1}[U' \cap V_\xi] = \mu_0(U' \cap V_\xi),$$

$$\mu_0 g_\xi^{-1}[U' \cap V_\xi \cap \tilde{V}_\xi] = \mu_0(U' \cap V_\xi \cap \tilde{V}_\xi),$$

so

$$\mu_0 g_\xi^{-1}[U' \cap V'_\xi] = \mu_0(U' \cap V'_\xi).$$

Now it is easy to check that

$$\mu_0(U' \cap \tilde{V}_\xi) = \textstyle\sum_{\eta<\xi} \mu_0(U' \cap V'_\eta)$$

for every $\xi <\le \kappa$ (using the fact that μ_0 is τ-additive for the inductive step to limit ordinals), so that

$$\mu_0 U' = \textstyle\sum_{\eta<\kappa} \mu_0(U' \cap V'_\xi) = \sum_{\eta<\kappa} \mu_0 g_\xi^{-1}[U' \cap V'_\xi] = \mu U'.$$

What this means is that the subspace measure on U induced by μ agrees on open sets with μ_0, and must therefore be equal to μ_0, since both are Radon measures. So μ extends μ_0. **Q**

(c) μ is G-invariant. **P** Let $U' \subseteq X$ be open, and $g \in G$. If $\xi, \eta < \kappa$ and $W \subseteq V_\xi$, $W' \subseteq V_\eta$ are open, then

$$\begin{aligned}\mu_0(g_\xi^{-1}[g[U' \cap W'] \cap W]) &= \mu_0(g_\xi^{-1} g^{-1}[U' \cap W' \cap g[W]]) \\ &= \mu_0(g_\eta g g_\xi g_\xi^{-1} g^{-1}[U' \cap W' \cap g[W]]) \\ &= \mu_0(g_\eta^{-1}[U' \cap W' \cap g[W]]),\end{aligned}$$

noting that

$$g_\xi^{-1}[g^{-1}[U' \cap W'] \cap W] \subseteq g_\xi^{-1}[V_\xi] \subseteq U,$$

$$g_\eta^{-1}[U' \cap W' \cap g[W]] \subseteq g_\eta^{-1}[V_\eta] \subseteq U.$$

Applying this to each of the four cases

$$W = V_\xi \text{ or } \tilde{V}_\xi, \quad W' = V_\eta \text{ or } \tilde{V}_\eta$$

we see that

$$\mu_0(g_\xi^{-1}[g^{-1}[U' \cap V'_\eta] \cap V'_\xi]) = \mu_0(g_\eta^{-1}[U' \cap V'_\eta \cap g[V'_\xi]]).$$

Summing over ξ and η we have

$$\begin{aligned}\mu g^{-1}[U'] &= \sum_{\xi<\kappa} \mu_0(g_\xi^{-1}[g^{-1}[U'] \cap V'_\xi])\\ &= \sum_{\xi,\eta<\kappa} \mu_0(g_\xi^{-1}[g^{-1}[U' \cap V'_\eta] \cap V'_\xi])\\ &= \sum_{\xi,\eta<\kappa} \mu_0(g_\eta^{-1}[U' \cap V'_\eta \cap g^{-1}[V'_\xi]])\\ &= \sum_{\eta<\kappa} \mu_0(g_\eta^{-1}[U' \cap V'_\eta]) = \mu U'.\end{aligned}$$

But this means that the two Radon measures μ, μg^{-1} agree on the open sets, and must be equal; as g is arbitrary, μ is G-invariant. **Q**

(d) Thus we have the required extension.

Remark Of course the extension is unique (except in the trivial case $U = \emptyset$.)

10. Lemma Let (X, ρ) be a locally compact metric space, with isometry group G^*. Then there is a partition $\mathcal{U}$ of X into σ-compact open sets such that $g[U] \in \mathcal{U}$ for every $U \in \mathcal{U}$ and $g \in G^*$.

proof (a) For $x, y \in X$, say that $x \dot\sim y$ if there is a $\delta > \rho(x,y)$ such that $B(x,\delta) \cup B(y,\delta)$ is compact. Then $\dot\sim$ is reflexive and symmetric; let $\sim$ be its transitive closure, so that $\sim$ is an equivalence relation. Let $\mathcal{U}$ be the family of $\sim$-equivalence classes, so that $\mathcal{U}$ is a partition of X. If $g \in G^*$, then $g(x) \dot\sim g(y)$ iff $x \dot\sim y$, and $g(x) \sim g(y)$ iff $x \sim y$, so $g[U] \in \mathcal{U}$ for every $U \in \mathcal{U}$.

(b) Every $U \in \mathcal{U}$ is open. **P** If $x \in U$ there is a $\delta > 0$ such that $B(x,\delta)$ is compact, in which case $B(x, \frac{1}{3}\delta) \subseteq U$. **Q**

(c) If $K \subseteq X$ is compact then

$$K' = \{y : \text{there is some } x \in K \text{ such that } y \dot\sim x\}$$

is σ-compact. **P** For q, $q' \in \mathbb{Q}$ such that $0 \le q < q'$, set

$$\begin{aligned}K'_{qq'} = \{y :\ &\text{there is some } x \in K \text{ such that } \rho(y,x) \le q\\ &\text{and } B(x,q') \cup B(y,q') \text{ is compact}\}.\end{aligned}$$

(i) There is a finite set $I \subseteq K$ such that $K \subseteq \bigcup_{x\in I} B(x, q'-q)$, in which case $K'_{qq'} \subseteq \bigcup_{x\in I} B(x,q')$ and is relatively compact. (ii) If $y \in \overline{K'_{qq'}}$, then there is a $y' \in K'_{qq'}$ such that $\rho(y,y') \le \eta = \frac{1}{3}(q'-q)$. Let $x \in K$ be such that $\rho(y',x) \le q$ and $B(x,q') \cup B(y',q')$ is compact. In this case $\rho(y,x) \le q + \eta$ while

$$B(x, q+2\eta) \cup B(y, q+2\eta) \subseteq B(x,q') \cup B(y',q')$$

is compact, so $y \in K'$. (iii) This shows that

$$K = \bigcup\{K'_{qq'} : q,\, q' \in \mathbb{Q},\, 0 \le q < q'\} = \bigcup\{\overline{K'_{qq'}} : q,\, q' \in \mathbb{Q},\, 0 \le q < q'\}$$

is σ-compact. **Q**

(d) It follows that if $E \subseteq X$ is σ-compact, so is $E' = \{y : \text{there is some } x \in E \text{ such that } y \dot\sim x\}$. So if $U \in \mathcal{U}$ and $x \in U$, we can form a sequence $\langle E_n \rangle_{n\in\mathbb{N}}$ of σ-compact sets by setting $E_0 = \{x\}$, $E_{n+1} = E'_n$ for each n, and $U = \bigcup_{n\in\mathbb{N}} E_n$ will be σ-compact. Thus $\mathcal{U}$ is a G^*-invariant partition of X into open σ-compact sets, as claimed.

11. Theorem Let X be a σ-compact locally compact metric space, and G a subgroup of the isometry group G^* of X. Let $(Z, \bar\sigma)$ be the quotient space defined in Lemma 2, and for $F \in Z$ let $G_F = \{g{\restriction}F : g \in G\}$ be the corresponding group of automorphisms of F (2b).

(a) Fix a non-decreasing sequence $\langle W_n \rangle_{n\in\mathbb{N}}$ of relatively compact open sets with union X, starting from $W_0 = \emptyset$. For $F \in Z$, let $k(F) \in \mathbb{N}$ be such that $F \cap W_{k(F)}$ is empty and $F \cap W_{k(F)+1}$ is not empty. Then there is a unique G-invariant Radon measure λ_F on X such that $\lambda_F(W_{k(F)+1}) = 1$, $\lambda_F(X \setminus F) = 0$.

(b) For any G-invariant Radon measure μ on X, there is a unique Radon measure ν on Z such that $\mu E=\int\lambda_F(E)\nu(dF)$ for every $E\in\operatorname{dom}\mu$.

proof (a) By Lemma 2, Z is σ-compact and locally compact.

Let $F\in Z$. By Theorem 7, there is a non-zero G_F-invariant Radon measure λ on F. Now λ is strictly positive. **P** If $V\subseteq F$ is non-empty and (relatively) open, then for any $x\in F$ the set $\{g(x):g\in G_F\}$ must meet V; so $F=\bigcup_{g\in G_F}g^{-1}[V]$. Because λ is non-zero and τ-additive, there is a $g\in G_F$ such that $\lambda(g^{-1}[V])>0$; because λ is G_F-invariant, $\lambda V>0$. As V is arbitrary, λ is strictly positive. **Q**

$k(F)$ is chosen so that $U=F\cap W_{k(F)+1}$ is non-empty, so that $\lambda U>0$ and $\gamma=\lambda(F\cap W_{k(F)+1})>0$. Since $F\cap\overline{W_{k(F)+1}}$ is compact, γ is finite. We may therefore define a Radon measure λ_F on X by setting $\lambda_F E=\gamma^{-1}\lambda(F\cap E)$ whenever $E\subseteq X$ and $F\cap E\in\operatorname{dom}\lambda$. Of course $\lambda_F(X\setminus F)=0$ and $\lambda_F(W_{k(F)+1})=1$.

If $g\in G$ and $E\in\operatorname{dom}\lambda_F$, then (because F is G-invariant and λ is G_F-invariant)

$$\lambda_F(g^{-1}[E])=\gamma^{-1}\lambda(F\cap g^{-1}[E])=\gamma^{-1}\lambda(g^{-1}[E\cap F])=\gamma^{-1}\lambda(E\cap F)=\lambda_F E.$$

Thus λ_F is G-invariant.

To see that λ_F is unique, let λ'_F be any other measure with the same properties. Then we have a corresponding Radon measure λ' on F, defined as the subspace measure from λ'_F. Because F and λ'_F are G-invariant, λ' is G_F-invariant; now $\lambda'U=\lambda U=1$, so by Theorem 7 $\lambda=\lambda'$ and $\lambda'_F=\lambda_F$.

(b)(i) Now let μ be a G-invariant Radon measure on X. For $n\in\mathbb{N}$, $E\subseteq X$ set $\mu^{(n)}(E)=\mu(E\cap W_n)$ when this is defined; then $\mu^{(n)}$ is a totally finite Radon measure on X. Let $\nu_n=\mu^{(n)}\pi^{-1}$ be the image Radon measure on Z, and T_n its domain. By [Frp00*], 453Gb, ν_n has an almost strong lifting, so by [Frp00*], 453K, there is a disintegration $\langle\mu_F^{(n)}\rangle_{F\in Z}$ of $\mu^{(n)}$ over ν_n such that every $\mu_F^{(n)}$ is a Radon probability measure (on X) concentrated on F; we have

$$\mu(E\cap W_n)=\int\mu_F^{(n)}(E)\nu_n(dF)$$

for every $E\in\operatorname{dom}\mu$.

(ii)($\boldsymbol{\alpha}$) For any open set $U\subseteq W_n$ and any $g\in G$, write

$$Z_{ngU}=\{F:\mu_F^{(n)}(W_n\cap g^{-1}[U])\le\mu_F^{(n)}(U)\}.$$

Then Z_{ngU} is ν_n-conegligible. **P** If $Y\subseteq Z$ is Borel, then

$$\int_Y\mu_F^{(n)}(W_n\cap g^{-1}[U])\nu_n(dF)=\mu(\pi^{-1}[Y]\cap W_n\cap g^{-1}[U])$$

$$\le\mu(\pi^{-1}[Y]\cap g^{-1}[U])=\mu g^{-1}[\pi^{-1}[Y]\cap U]$$

(because $\pi g=\pi$)

$$=\mu(\pi^{-1}[Y]\cap U)$$

(because μ is G-invariant)

$$=\int_Y\mu_F^{(n)}(U)\nu_n(dF).$$

As Y is arbitrary, $\mu_F^{(n)}(W_n\cap g^{-1}[U]\le\mu_F^{(n)}(U)$ for ν_n-almost every F. **Q**

($\boldsymbol{\beta}$) Next, for $g\in G$, set

$$Z_{ng}=\{F:\mu_F^{(n)}(W_n\cap g^{-1}[U])\le\mu_F^{(n)}(U)\text{ for every open }U\subseteq W_n\}.$$

Then Z_{ng} is ν_n-conegligible. **P** W_n has a countable base $\mathcal{U}$ closed under finite unions, and $\bigcap_{U\in\mathcal{U}}Z_{ngU}$ is conegligible. But if $F\in\bigcap_{U\in\mathcal{U}}Z_{ngU}$, and $U\subseteq W_n$ is open, there is a non-decreasing sequence $\langle U_i\rangle_{i\in\mathbb{N}}$ in $\mathcal{U}$ with union U, so that

$$\mu_F^{(n)}(W_n\cap g^{-1}[U])=\sup_{i\in\mathbb{N}}\mu_F^{(n)}(W_n\cap g^{-1}[U_i])\le\sup_{i\in\mathbb{N}}\mu_F^{(n)}(U_i)=\mu_F^{(n)}(U).$$

This shows that $Z_{ng}\supseteq\bigcap_{U\in\mathcal{U}}Z_{ngU}$, so is conegligible. **Q**

($\boldsymbol{\gamma}$) Thirdly, set

$$Z_n = \{F : \mu_F^{(n)}(W_n \cap g^{-1}[U]) \le \mu_F^{(n)}(U) \text{ for every } g \in G, \text{ open } U \subseteq W_n\}.$$

Then Z_n is ν_n-conegligible. **P** Recall that G^* is separable and metrizable (Lemma 3b), so that G must have a countable dense set D, and $\bigcap_{g\in D} Z_{ng}$ is conegligible. If $F \in \bigcap_{g\in D} Z_{ng}$, then

$$\{g : g \in G^*,\ \mu_F^{(n)} g^{-1}(W_n \cap g^{-1}[U]) \le \mu_F^{(n)}(U) \text{ for every open } U \subseteq W_n\}$$

is a closed set (Lemma 6) including D, so includes G; thus $Z_n \supseteq \bigcap_{g\in D} Z_{ng}$ and is ν_n-conegligible. **Q**

(iii) If $F \in Z_n$, as defined in (ii) above, we can apply Lemma 9 to F, G_F and the subspace measure on $F \cap W_n$ induced by $\mu_F^{(n)}$ to see that there is a G_F-invariant Radon measure on F agreeing with $\mu_F^{(n)}$ on subsets of $W_n \cap F$. But Theorem 7 tells us that this must be a multiple of the subspace measure induced by λ_F on F. So we have some number $f_n(F) \in [0,\infty[$ such that $\mu_F^{(n)}(E) = f_n(F)\lambda_F(E)$ for every $E \subseteq F \cap W_n$ which is in the domain of $\mu_F^{(n)}$, that is,

$$\mu_F^{(n)}(E) = \mu_F^{(n)}(E \cap F) = f_n(F)\lambda_F(E \cap F \cap W_n) = f_n(F)\lambda_F(E \cap W_n)$$

whenever $E \in \operatorname{dom}\mu_F^{(n)}$.

Also $f_n : Z_n \to [0,\infty[$ is T_n-measurable. **P** For each $m \in \mathbb{N}$, set $W_m = \{F : F \in Z,\ k(F) = m\}$. Now this is just $\pi[W_{m+1}] \setminus \pi[W_m]$, so is a Borel set. And, for $F \in W_m$, $\lambda_F(W_{m+1}) = 1$, so that $f_n(F) = \mu_F^{(n)}(W_{m+1})$. Since $F \mapsto \mu_F^{(n)}(W_{m+1})$ is always T_n-measurable, so is f_n. **Q**

We can therefore form the indefinite-integral measure $\nu_n' = f_n\nu_n$, writing $\nu_n' Y = \int_Y f_n d\nu_n$ whenever this is defined in $[0,\infty]$, and we shall have

$$\mu(E \cap W_n) = \mu^{(n)}(E) = \int_{Z_n} f_n(F)\lambda_F(E)\nu_n(dF) = \int \lambda_F(E \cap W_n)\nu_n'(dF)$$

for every $E \in \operatorname{dom}\mu$. Note that ν_n' is a complete locally determined topological measure and is inner regular with respect to the compact sets, because ν_n is ([Frp00*], 234Eb and 412Q). But ν_n' is also totally finite. **P** Let Y_n be the open set $\pi[W_n]$. Then

$$\nu_n'(Z \setminus Y_n) = \nu_n(Z \setminus Y_n) = \mu_n\pi^{-1}[Z \setminus Y_n] \le \mu_n(X \setminus Y_n) = 0.$$

For $F \in Y_n$, we have $k(F) < n$, and $\lambda_F(W_n) \ge \lambda_F(W_{k(F)+1}) = 1$; so that

$$\nu_n' Z = \nu_n' Y_n \le \int \lambda_F(W_n)\nu_n'(dF) = \mu W_n < \infty. \text{ **Q**}$$

So in fact ν_n' is a Radon measure.

(iv) As in (iii), set $Y_n = \pi[W_n]$ for each n, so that $\langle Y_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of open sets with union Z. Now if $m \le n$, $\nu_m' Y = \nu_n Y$ for every Borel set $Y \subseteq Y_m$. **P** The point is that $\lambda_F(U) > 0$ whenever U is an open set meeting F, as noted in the proof of (a) above, so that $\lambda_F(W_m) > 0$ for every $F \in Y_m$; while

$$\int_Y \lambda_F(W_m)\nu_m'(dF) = \mu(\pi^{-1}[Y] \cap W_m) = \nu_m Y = \int_Y \lambda_F(W_m)\nu_n'(dF)$$

for every Borel set $Y \subseteq Y_m$. I should like to say that the result is now 'obvious', but if you have difficulty with this, observe that both ν_m' and ν_n' agree, on the Borel subsets of W_m, with the indefinite-integral measure $g\nu_m$, where $g(F) = 1/\lambda_F(W_m)$ for $F \in Y_m$. **Q**

Since both ν_m' and ν_n' are totally finite Radon measures, $\nu_m' Y = \nu_n' Y$ whenever $Y \subseteq Y_m$ is in the domain of ν_m'.

(v) There is therefore a measure ν on Z such that $\nu Y = \nu_n' Y$ whenever $n \in \mathbb{N}$, $Y \subseteq Y_n$ and $Y \in \operatorname{dom}\nu_n'$. If $E \in \operatorname{dom}\mu$, then

$$\mu E = \sup_{n\in\mathbb{N}} \mu(E \cap W_n)$$

$$= \sup_{n\in\mathbb{N}} \int \lambda_F(E \cap W_n)\nu_n'(dF) = \sup_{n\in\mathbb{N}} \int \lambda_F(E \cap W_n)\nu(dF)$$

(because $\lambda_F(E \cap W_n) = 0$ if $F \notin Y_n$)

$$= \int \lambda_F(E)\nu(dF).$$

(vi) ν is a Radon measure. **P** It is a complete locally determined topological measure, inner regular with respect to the compact sets, because every ν'_n is. To see that it is locally finite, observe that $\langle Y_n\rangle_{n\in\mathbb{N}}$ is a cover of Z by open sets of finite measure. **Q**

(vii) Finally, to see that ν is unique, repeat the idea of (iv). If $\tilde{\nu}$ is another measure with the same property, then for any $n \in \mathbb{N}$, Borel set $Y \subseteq Y_n$, we must have

$$\int_Y \lambda_F(W_n)\nu(dF) = \mu(W_n \cap \pi^{-1}[Y]) = \int_Y \lambda_F(W_n)\tilde{\nu}(dF),$$

so that $\nu Y = \tilde{\nu} Y$ for every Borel subset Y of Y_n. Accordingly ν and $\tilde{\nu}$ agree on the Borel sets and are identical.

This completes the proof.

12. Lemma Let X be a locally compact Hausdorff space and G a group of autohomeomorphisms of X. Then the following are equivalent:

(i) the topology of X is generated by G-invariant pseudometrics;

(ii) whenever $F \subseteq X$ is closed and $K \subseteq X \setminus F$ is compact, then X is covered by the open sets $U \subseteq X$ such that no $g[U]$, for $g \in G$, meets both K and F.

proof (i)⇒(ii) If (i) is true, and F, K are as in (ii), then there are a G-invariant continuous pseudometric ρ such that $\rho(x,y) \geq 1$ for every $x \in K$, $y \in F$. **P** For each $x \in K$, there are G-invariant continuous pseudometrics $\sigma_0, \dots, \sigma_n$ and $\delta > 0$ such that $\{y : \sigma_i(x,y) \leq \delta$ for every $i \leq n\}$ does not meet F. Set $\rho_x = \frac{2}{\delta}\sum_{i=0}^n \sigma_i$; then ρ_x is a G-invariant continuous pseudometric and $\rho_x(x,F) \geq 2$. Write $V_x = \{y : \rho_x(x,y) < 1\}$. Then $\{V_x : x \in K\}$ is an open cover of K, so there is a finite $I \subseteq K$ such that $K \subseteq \bigcup_{x\in I} V_x$. Set $\rho = \sum_{x\in I}\rho_x$; this works. **Q**

Now if U is an open set and $\operatorname{diam}_\rho(U) < 1$, no $g[U]$ can meet both K and F; and such open sets cover X. As K and F are arbitrary, X satisfies (ii).

(ii)⇒(i)(α) Let $f : X \to [0,1]$ be any continuous function with compact support, and set $\rho_f(x,y) = \sup_{g\in G}|f(g(x)) - f(g(y))|$ for x, $y \in X$. Then ρ_f is a G-invariant pseudometric; also it is continuous. **P** Let $x_0 \in X$, $n \geq 1$. Set $K_i = \{x : f(x) \geq \frac{i+1}{n}\}$, $F_i = \{x : f(x) \leq \frac{i}{n}\}$ for $i < n$. By (ii), we can find for each $i < n$ an open set U_i containing x_0 and such that no $g[U_i]$ meets both K_i and F_i. Set $U = \bigcap_{i<n} U_i$; then if $g \in G$ and x, $y \in g[U]$, there is no i such that $x \in K_i$ and $y \in F_i$, so that $f(x) - f(y) \leq \frac{2}{n}$. But this means that if x, $y \in U$ then $|f(g(x)) - f(g(y))| \leq \frac{2}{n}$ for every $g \in G$, so that $\rho_f(x,y) \leq \frac{2}{n}$. In particular, $\rho_f(x_0,x) \leq \frac{2}{n}$ for every $x \in U$. As x_0 and n are arbitrary, ρ_f is continuous. **Q**

Since the topology of X is certainly defined by the continuous functions of compact support, and $\rho_f(x,y)$ is always at least $|f(x) - f(y)|$, the topology of X is defined by the continuous G-invariant pseudometrics.

13 Theorem Let X be a locally compact Hausdorff space, and G a group of autohomeomorphisms of X such that

(i) the topology of X is defined by the G-invariant continuous pseudometrics,

(ii) $\{g(x) : g \in G\}$ is dense in X for every $x \in X$.

Then there is a non-zero G-invariant Radon measure on X, and any two such measures are scalar multiples of each other.

proof (a) If K and L are disjoint compact subsets of X, there is a non-empty open subset U of X such that, for every $g \in G$, $g[U]$ meets at most one of K and L. **P** We need consider only the case in which both K and L are non-empty. Write P for the set of G-invariant continuous pseudometrics on X. For every $x \in K$, there is a $\rho_x \in \mathrm{P}$ such that $\inf_{y\in L}\rho_x(x,y) > 0$, because L is closed and $x \notin L$; because K is compact and P is upwards-directed, there is a $\rho \in \mathrm{P}$ such that $\epsilon = \inf_{x\in K, y\in L}\rho(x,y) > 0$. Now any non-empty open set of ρ-diameter less than ϵ will do for U. **Q**

By [Frp00*], 441C, there is a non-zero G-invariant Radon measure on X.

(b) ? Suppose, if possible, that μ, ν are G-invariant Radon measures on X, neither a scalar multiple of the other. Then neither is zero, so there is a compact set K such that $\mu K > 0$; set $\alpha = \nu K/\mu K$; there is a compact set L such that $\nu L \neq \alpha\mu L$, that is, $\nu L \mu K \neq \nu K \mu L$.

Let K', L' be G_δ sets including K, L such that

$$\mu(K' \setminus K) = \nu(K' \setminus K) = \mu(L' \setminus L) = \nu(L' \setminus L) = 0.$$

Then there is a G-invariant continuous pseudometric ρ on X such that

$$\{x : \rho(x, K) = 0\} \subseteq K',\ \{x : \rho(x, L) = 0\} \subseteq L',$$

and moreover $\{x : \rho(x, x_0) \le 1\}$ is compact for some $x_0 \in X$. **P** Express K' as $\bigcap_{n\in\mathbb{N}} U_n$, L' as $\bigcap_{n\in\mathbb{N}} V_n$ where every U_n, V_n is open. As in (i)$\Rightarrow$(ii) in the proof of Lemma 12, there are continuous G-invariant pseudometrics ρ_n, σ_n such that $\{x : \rho_n(x, K) < 1\} \subseteq U_n$, $\{x : \sigma_n(x, L) < 1\} \subseteq V_n$ for every $n \in \mathbb{N}$. Let τ be any continuous G-invariant pseudometric such that $\{x : \tau(x, x_0) \le 1\}$ is compact. Set

$$\rho(x, y) = \tau(x, y) + \sum_{n=0}^{\infty} \min(2^{-n}, \rho_n(x, y) + \sigma_n(x, y))$$

for x, $y \in X$; this works. **Q**

Let $\sim$ be the equivalence relation associated with ρ, so that $x \sim y$ iff $\rho(x, y) = 0$; then $g(x) \sim g(y)$ whenever $g \in G$ and $x \sim y$. Let Z be the set of equivalence classes for $\sim$. Then we have a metric $\bar\rho$ on Z defined by setting $\bar\rho(x^\bullet, y^\bullet) = \rho(x, y)$ for x, $y \in X$. Moreover, we have for each $g \in G$ a function $\bar g : Z \to Z$ defined by setting $\bar g(x^\bullet) = g(x)^\bullet$ for every $x \in X$, and every $\bar g$ is an isometry for $\bar\rho$; also $H = \{\bar g : g \in G\}$ is a group of bijections on Z.

If $z \in Z$, then $\{h(z) : h \in H\}$ is dense in Z. **P** Take $y \in X$ such that $y^\bullet = z$. The map $x \mapsto x^\bullet : X \to Z$ is continuous, so $\{h(z) : h \in H\} = \{g(y)^\bullet : g \in G\}$ is dense. **Q**

Set $\pi(x) = x^\bullet$ for $x \in X$. If $z \in Z$, there is a neighbourhood V of z such that $\pi^{-1}[V]$ is compact in X. **P** Express z as $x^\bullet$ where $x \in X$. There is a $g \in G$ such that $\rho(g(x), x_0) < 1$. Set $M = \{y : \rho(y, g^{-1}(x_0)) \le 1$, $V = \pi[M] = \{w : \bar\rho(w, g^{-1}(x_0)^\bullet \le 1\}$. Then V is a neighbourhood of z and compact and $\pi^{-1}[V] = M$ is compact. **Q** In particular, Z is locally compact.

So $(Z, \bar\rho)$ satisfis the conditions of Theorem 7. Note next that, writing $\pi(x) = x^\bullet$ fo $x \in X$, $\mu\pi^{-1}$ and $\nu\pi^{-1}$ are Radon measures on Z. **P** Because π is continuous, we have only to check that they are locally finite. But this is true because every point of Z belongs to an open set with relatively compact inverse image in X. **Q**

Also $\mu\pi^{-1}$ and $\nu\pi^{-1}$ are H-invariant. **P** If $h \in H$, express h as $\bar g$ where $g \in G$; then $h\pi = \pi g$, so

$$\mu\pi^{-1}[h^{-1}[E]] = \mu(g^{-1}[\pi^{-1}[E]]) = \mu(\pi^{-1}[E])$$

whenever the right-hand-side is defined. As h is arbitrary, $\mu\pi^{-1}$ is G-invariant. Similarly, $\nu\pi^{-1}$ is G-invariant. **Q**

By Theorem 7, one of $\mu\pi^{-1}$ and $\nu\pi^{-1}$ must be a multiple of the other, and

$$\mu(\pi^{-1}[\pi[K]]) \cdot \nu(\pi^{-1}[\pi[L]]) = \mu(\pi^{-1}[\pi[L]]) \cdot \nu(\pi^{-1}[\pi[K]]).$$

However, because $K \subseteq \pi^{-1}[\pi[K]] \subseteq K'$, $\mu(\pi^{-1}[\pi[K]] = \mu K$; and as we have similar equalities for the other three terms, we get $\mu K \cdot \nu L = \mu L \cdot \nu K$, contrary to the choice of K and L. **X**

This contradiction proves the theorem.

Reference

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