A theorem of Gitik and Shelah on disjoint refinements of sequences of subsets of the real line

I rewrite and amplify the material of BURKE N96, itself drawn from GITIK & SHELAH 89, GITIK & SHELAH 93, GITIK & SHELAH P95 and conversations with S.Shelah. My thanks to M.R.Burke for his explanations, and for reading this note.

1. The ideal $\mathcal{J}$ and the algebra $\mathfrak{C}$

1A Notation Throughout the paper, Z will be $\{0,1\}^{\mathbb{N}}$ and S will be $\bigcup_{n\in\mathbb{N}}\{0,1\}^n$. For $\sigma\in S$, set $Z_\sigma=\{x:x\in Z,\ x\supseteq\sigma\}$. Write $\mathfrak{G}$ for the regular open algebra of Z.

Throughout §1, (X,Σ,μ) will be an arbitrary probability space.

1B Definition Write $\mathcal{B}$ for the σ-algebra of Borel subsets of Z, $\Sigma\widehat{\otimes}\mathcal{B}$ for the σ-algebra of subsets of $X\times Z$ generated by $\{E\times F:E\in\Sigma,\ F\in\mathcal{B}\}$. For $W\subseteq X\times Z$, $x\in X$ write $W[\{x\}]=\{\alpha:(x,\alpha)\in W\}$ for the corresponding vertical section of W. Now write

$$\mathcal{J}=\{W:W\in\Sigma\widehat{\otimes}\mathcal{B},\ W[\{x\}]\text{ is meager for almost every }x\in X\}.$$

It is easy to see that $\mathcal{J}$ is a σ-ideal of $\Sigma\widehat{\otimes}\mathcal{B}$. Let $\mathfrak{C}$ be the quotient algebra $\Sigma\widehat{\otimes}\mathcal{B}/\mathcal{J}$.

1C We have the following straightforward facts about $\mathcal{J}$.

Proposition (a) Write $\mathcal{V}$ for the family of subsets of $X\times Z$ expressible in the form $\bigcup_{\sigma\in S}E_\sigma\times Z_\sigma$, where every $E_\sigma\in\Sigma$. Then

(i) $\mathcal{V}$ is closed under countable unions and finite intersections;

(ii) for every $V\in\mathcal{V}$ there is a $V'\in\mathcal{V}$ such that $V'\cap V=\emptyset$ and $V[\{x\}]\cup V'[\{x\}]$ is dense for every $x\in X$, so that $(X\times Z)\setminus(V\cup V')\in\mathcal{J}$;

(iii) for every $W\in\Sigma\widehat{\otimes}\mathcal{B}$ there is a $V\in\mathcal{V}$ such that $W\triangle V\in\mathcal{J}$;

(iv) $\mathcal{J}$ is the σ-ideal of $\Sigma\widehat{\otimes}\mathcal{B}$ generated by $\mathcal{J}_0$, where $\mathcal{J}_0=\{W:W\in\mathcal{J},\ (X\times Z)\setminus W\in\mathcal{V}\}$. If $W\in\mathcal{J}_0$ all its vertical sections are closed and almost all of them are nowhere dense.

(b) If $W\in\Sigma\widehat{\otimes}\mathcal{B}\setminus\mathcal{J}$, there are a non-negligible set $E\in\Sigma$ and a non-empty open set $G\subseteq Z$ such that $(E\times G)\setminus W\in\mathcal{J}$.

(c) $\mathcal{J}$ is ω_1-saturated in $\Sigma\widehat{\otimes}\mathcal{B}$, so that $\mathfrak{C}$ is ccc and Dedekind complete.

proof (a)(i)

$$\begin{aligned}\bigcup_{\sigma\in S}E_\sigma\times Z_\sigma\cap\bigcup_{\sigma\in S}E'_\sigma\times Z_\sigma&=\bigcup_{\sigma,\tau\in S}(E_\sigma\cap E'_\tau)\times(Z_\sigma\cap Z_\tau)\\&=\bigcup_{\sigma\in S}(\bigcup_{\upsilon,\tau\subseteq\sigma}E_\upsilon\cap E'_\tau)\times Z_\sigma,\end{aligned}$$

$$\bigcup_{n\in\mathbb{N}}\bigcup_{\sigma\in S}E_{n\sigma}\times Z_\sigma=\bigcup_{\sigma\in S}(\bigcup_{n\in\mathbb{N}}E_{n\sigma})\times Z_\sigma.$$

(ii) Express V as $\bigcup_{\sigma\in S}E_\sigma\times Z_\sigma$ where each E_σ belongs to Σ. For each $\sigma\in S$ set $E'_\sigma=X\setminus\bigcup\{E_\tau:\tau\in S,\ Z_\tau\cap Z_\sigma\neq\emptyset\}$, and write $V'=\bigcup_{\sigma\in S}E'_\sigma\times Z_\sigma$.

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(iii)-(iv) Now write $\mathcal{J}_1$ for the σ-ideal generated by $\mathcal{J}_0$, and consider the set

$$\Lambda = \{W : W \in \Sigma\widehat{\otimes}\mathcal{B} \text{ and there is some } V \in \mathcal{V} \text{ such that } W \triangle V \in \mathcal{J}_1\}.$$

We have just seen that if $V \in \mathcal{V}$ then $(X \times Z) \setminus V \in \Lambda$. So if $W \in \Lambda$ then $(X \times Z) \setminus W \in \Lambda$. Also $\mathcal{V}$ and $\mathcal{J}_1$ are closed under countable unions so Λ also is. So Λ is a σ-subalgebra of $\Sigma\widehat{\otimes}\mathcal{B}$ containing all sets of the form $E \times G$ where $E \in \Sigma$ and $G \subseteq Z$ is open, and must be the whole of $\Sigma\widehat{\otimes}\mathcal{B}$.

$\mathcal{J}_1 \subseteq \mathcal{J}$ because $\mathcal{J}_0 \subseteq \mathcal{J}$. If $W \in \mathcal{J}$, there is a $V \in \mathcal{V}$ such that $((X \times Z) \setminus W) \triangle V \in \mathcal{J}_1$. Set $V' = (X \times Z) \setminus V$, so that $V' \triangle W \in \mathcal{J}$ and $V' \in \mathcal{J}$. As $(X \times Z) \setminus V' = V \in \mathcal{V}$, $V' \in \mathcal{J}_0 \subseteq \mathcal{J}_1$ and $W \in \mathcal{J}_1$. Thus $\mathcal{J} = \mathcal{J}_1$, as claimed.

(b) Let $V \in \mathcal{V}$ be such that $W \triangle V \in \mathcal{J}$. Because $W \notin \mathcal{J}$, $V \notin \mathcal{J}$ and there must be a non-negligible $E \in \Sigma$ and a $\sigma \in S$ such that $E \times Z_\sigma \subseteq V$; now $(E \times Z_\sigma) \setminus W \in \mathcal{J}$.

(c) ? If $\langle W_\xi \rangle_{\xi<\omega_1}$ were a disjoint family in $\Sigma\widehat{\otimes}\mathcal{B} \setminus \mathcal{J}$, then we could choose for each $\xi < \omega_1$ a non-negligible set $E_\xi \in \Sigma$ and a $\sigma_\xi \in S$ such that $(E_\xi \times Z_{\sigma_\xi}) \setminus W_\xi \in \mathcal{J}$. Let $\sigma \in S$ be such that $D = \{\xi : \sigma_\xi = \sigma\}$ is uncountable. Then there are distinct ξ, $\eta \in D$ such that $E = E_\xi \cap E_\eta$ is non-negligible, in which case $E \times Z_\sigma$ does not belong to $\mathcal{J}$, while $(E \times Z_\sigma) \setminus (W_\xi \cap W_\eta) \in \mathcal{J}$. **X**

1D Lemma If $W \in \Sigma\widehat{\otimes}\mathcal{B}$ and $W[\{x\}]$ is non-meager for every $x \in X$, there is a measurable function $f : X \to Z$ such that $(x, f(x)) \in W$ for every $x \in X$.

proof (a) Let Λ be the set of those $W \in \Sigma\widehat{\otimes}\mathcal{B}$ such that there are V, $\langle V_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{V}$ such that $(W \triangle V) \cap \bigcap_{n\in\mathbb{N}} V_n = \emptyset$ and $V_n[\{x\}]$ is dense for every $x \in X$, $n \in \mathbb{N}$. Then Λ is closed under countable unions and contains V, $(X \times Z) \setminus V$ for every $V \in \mathcal{V}$, so is closed under complementation and is the whole of $\Sigma\widehat{\otimes}\mathcal{B}$.

(b) So if $W \in \Sigma\widehat{\otimes}\mathcal{B}$ and every vertical section of W is non-meager, we have V, $\langle V_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{V}$ such that $V \cap \bigcap_{n\in\mathbb{N}} V_n \subseteq W$, every vertical section of V is non-empty, and every vertical section of every V_n is dense. So we may choose a sequence $\langle W_n \rangle_{n\in\mathbb{N}}$ inductively in such a way that (i) every W_n is of the form $\bigcup_{\sigma\in S} E_\sigma \times Z_\sigma$ where $\langle E_\sigma \rangle_{\sigma\in S}$ is a disjoint cover of X by measurable sets and $E_\sigma = \emptyset$ if $\#(\sigma) < n$ (ii) $W_0 \subseteq V$, $W_{n+1} \subseteq W_n \cap V_n$ for every n. The intersection $\bigcap_{n\in\mathbb{N}} W_n$ is now the graph of a measurable function covered by W.

1E The subalgebras $\mathfrak{A}$, $\mathfrak{B}$ We have a natural map $E \mapsto (E \times Z)^\bullet$ from Σ to $\mathfrak{C}$ which has kernel $\{E : \mu E = 0\}$, so its image $\mathfrak{A}$ can be identified with the measure algebra of (X, Σ, μ); let $\bar\mu$ be the corresponding measure on $\mathfrak{A}$, that is, $\bar\mu(E \times Z)^\bullet = \mu E$ for every $E \in \Sigma$.

Similarly, we have a map $G \mapsto (X \times G)^\bullet$ from the algebra of open-and-closed subsets of X into $\mathfrak{C}$; write $\mathfrak{B} \subseteq \mathfrak{C}$ for its image.

1F Proposition (a) The map $(a, b) \mapsto a \cap b : \mathfrak{A} \times \mathfrak{B} \to \mathfrak{C}$ embeds $\mathfrak{A} \otimes \mathfrak{B}$ as an order-dense subalgebra of $\mathfrak{C}$, so $\mathfrak{C}$ may be regarded as the Dedekind completion of $\mathfrak{A} \otimes \mathfrak{B}$.

(b) If $\mathfrak{A}$ is homogeneous, $\mathfrak{C}$ is homogeneous.

proof (a) (Here $\mathfrak{A} \otimes \mathfrak{B}$ is the free product of $\mathfrak{A}$ and $\mathfrak{B}$. For the elementary theory of free products of Boolean algebras, see Koppelberg 89 or Fremlin p97*, §315.) Because

$a \cap b \neq 0$ for all non-zero $a \in \mathfrak{A}$, $b \in \mathfrak{B}$ we have an embedding of $\mathfrak{A} \otimes \mathfrak{B}$ in $\mathfrak{C}$; now 1Cb tells us that $\mathfrak{A} \otimes \mathfrak{B}$ is order-dense in $\mathfrak{C}$, while 1Cc tells us that $\mathfrak{C}$ is Dedekind complete, so $\mathfrak{C}$ may be identified with the Dedekind completion of $\mathfrak{A} \otimes \mathfrak{B}$.

(b) Because $\mathfrak{C}$ is Dedekind complete, it is enough to show that for every non-zero $c \in \mathfrak{C}$ there is a non-zero $d \subseteq c$ such that the principal ideal $\mathfrak{C}_d$ generated by d is isomorphic to $\mathfrak{C}$. But if $c \in \mathfrak{C} \setminus \{0\}$, take non-zero $a \in \mathfrak{A}$, $b \in \mathfrak{B}$ such that $d = a \cap b \subseteq c$; then $\mathfrak{C}_d$ will be the Dedekind completion of $\mathfrak{A}_a \otimes \mathfrak{B}_b \cong \mathfrak{A} \otimes \mathfrak{B}$, so $\mathfrak{C}_d \cong \mathfrak{C}$.

1G Lemma Let $g_0, \dots , g_n$ and $\langle f_{kj} \rangle_{k,j\in\mathbb{N}}$ be measurable functions from X to Z such that $\{x : f_{ki}(x) = f_{kj}(x)\}$ is negligible whenever $k \in \mathbb{N}$, $i \neq j$. Then for any $\epsilon > 0$ there is a compact set $F \subseteq Z$ such that $\mu g_i^{-1}[F] \geq 1 - \epsilon$ for every $i \leq n$ and $\bigcap_{j\in\mathbb{N}} f_{kj}^{-1}[F]$ is negligible for every $k \in \mathbb{N}$.

proof (a) Suppose that $\delta > 0$ and $k \in \mathbb{N}$. Then there is a compact set $F \subseteq Z$ such that

$$\mu g_i^{-1}[F] \geq 1 - \delta \text{ for every } i \leq n, \quad \mu(\textstyle\bigcap_{i\in\mathbb{N}} f_{ki}^{-1}[F]) \leq \delta.$$

P Take $\eta > 0$ and a finite set $I \subseteq \mathbb{N}$ such that

$$(1 - \eta)^{n+2} - (1 - \eta)^{\#(I)} \geq 1 - \delta + \eta.$$

Take $r \in \mathbb{N}$ such that

$$E_0 = \{x : f_{ki}(x){\restriction}r \neq f_{kj}(x){\restriction}r \text{ for all distinct } i,\, j \in I\}$$

has measure greater than $1 - \eta$. For $K \subseteq \{0, 1\}^r$ set $F_K = \bigcup_{\sigma\in K} Z_\sigma$. Let $K \subseteq \{0, 1\}^r$ be a random set, in which each member of $\{0, 1\}^r$ has independently a probability $1 - \eta$ of appearing. Then, for any particular $x \in E_0$,

$$\Pr(f_{ki}(x) \in F_K \text{ for every } i \in I) = \Pr(K \supseteq \{f_{ki}(x){\restriction}r : i \in I\}) = (1 - \eta)^{\#(I)},$$

while

$$\Pr(g_i(x) \in F_K \text{ for every } i \leq n) \geq (1 - \eta)^{n+1}.$$

So

$$\Pr(x \in \textstyle\bigcap_{i\leq n} g_i^{-1}[F_K] \setminus \bigcap_{i\in I} f_{ki}^{-1}[F_K]) \geq (1 - \eta)^{n+1} - (1 - \eta)^{\#(I)}.$$

There must therefore be some $K \subseteq \{0, 1\}^r$ such that

$$\begin{aligned}\mu(E_0 \cap \bigcap_{i\leq n} g_i^{-1}[F_K] \setminus \bigcap_{i\in I} f_{ki}^{-1}[F_K]) &\geq ((1 - \eta)^{n+1} - (1 - \eta)^{\#(I)})\mu E_0 \\ &\geq (1 - \eta)^{n+2} - (1 - \eta)^{\#(I)} \geq 1 - \delta + \eta.\end{aligned}$$

So we can take $F = F_K$. **Q**

(b) Now all we need to do is take a double sequence $\langle \delta_{mk} \rangle_{m,k\in\mathbb{N}}$ of strictly positive numbers such that $\sum_{m,k\in\mathbb{N}} \delta_{mk} \leq \epsilon$ and a corresponding family $\langle F_{mk} \rangle_{m,k\in\mathbb{N}}$ of compact subsets of Z such that, for each m, k,

$$\mu g_i^{-1}[F_{mk}] \geq 1 - \delta_{mk} \text{ for every } i \leq n, \quad \mu(\textstyle\bigcap_{i\in\mathbb{N}} f_{ki}^{-1}[F_{mk}]) \leq \delta_{mk},$$

and set $F = \bigcap_{m,k\in\mathbb{N}} F_{mk}$.

1H Outer approximations in $\mathfrak{A}$ For $c \in \mathfrak{C}$ I write

$$\operatorname{upr} c = \inf\{a : a \in \mathfrak{A},\ c \subseteq a\}.$$

Then $\operatorname{upr} 0 = 0$, $\operatorname{upr} 1 = 1$ and $\operatorname{upr}(\sup C) = \sup_{c\in C} \operatorname{upr} c$ for every $C \subseteq \mathfrak{C}$.

If $V \in \mathcal{V}$ then $\operatorname{upr} V^\bullet = (\pi_1[V] \times X)^\bullet$, where $\pi_1(x, y) = x$ for $x \in X$, $y \in Z$. **P** Express V as $\bigcup_{\sigma\in S} E_\sigma \times Z_\sigma$; then

$$\operatorname{upr} V^\bullet = \sup_{\sigma\in S}(E_\sigma \times X)^\bullet = (\pi_1[V] \times X)^\bullet. \text{ **Q**}$$

1I Lemma Let $\mathfrak{D} \subseteq \mathfrak{C}$ be an order-closed subalgebra such that no non-zero principal ideal of $\mathfrak{D}$ is isomorphic to any closed subalgebra of a principal ideal of $\mathfrak{A}$. Then

(a) If $a \in \mathfrak{A}$, $b \in \mathfrak{B}$ are non-zero, there is a $d \in \mathfrak{D}$ such that $a \cap \operatorname{upr}(b \cap d) \cap \operatorname{upr}(b \setminus d) \neq 0$.

(b) There is a sequence $\langle d_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{D}$ such that

$$\sup_{n\in\mathbb{N}}(\operatorname{upr}(b \cap d_n) \cap \operatorname{upr}(b \setminus d_n)) = 1$$

for every $b \in \mathfrak{B} \setminus \{0\}$.

(c) For any $d \in \mathfrak{D}$, $b_0, \dots , b_m \in \mathfrak{B}$, $\epsilon > 0$ there is a $d' \in \mathfrak{D}$ such that $d' \subseteq d$ and

$$\bar\mu(\operatorname{upr}(b_i \cap d')) \geq \bar\mu(\operatorname{upr}(b_i \cap d)) - \epsilon,$$

$$\bar\mu(\operatorname{upr}(b_i \cap d \setminus d')) \geq \bar\mu(\operatorname{upr}(b_i \cap d)) - \epsilon$$

for every $i \leq m$.

(d) For $\tau \in S$ write e_τ for $(X \times Z_\tau)^\bullet$. There is a family $\langle d_\sigma\rangle_{\sigma\in S}$ in $\mathfrak{D}$ such that

$$d_\emptyset = 1, \quad d_{\sigma^\frown 1} = d_\sigma \setminus d_{\sigma^\frown 0} \text{ for every } \sigma \in S,$$

$$\sum_{\tau,\sigma\in S,\#(\tau)\leq\#(\sigma)} \bar\mu(\operatorname{upr}(d_\sigma \cap e_\tau) \setminus (\operatorname{upr}(d_{\sigma^\frown 0} \cap e_\tau) \cap \operatorname{upr}(d_{\sigma^\frown 1} \cap e_\tau)) < 1.$$

(e) $\mathfrak{D}$ has a principal ideal with an order-closed subalgebra isomorphic to $\mathfrak{G}$.

(f) $\mathfrak{D}$ has an order-closed subalgebra isomorphic to $\mathfrak{G}$.

proof (a) ? Otherwise, $d \mapsto \pi d = a \cap \operatorname{upr}(b \cap d)$ is an order-continuous Boolean homomorphism from $\mathfrak{D}$ into the principal ideal $\mathfrak{A}_a$ of $\mathfrak{A}$ generated by a. Setting

$$d_0 = \inf\{d : a \cap b \subseteq d \in \mathfrak{D}\} = 1 \setminus \sup\{d : \pi d = 0\},$$

$\pi\restriction\mathfrak{D}_{d_0}$ is an injective order-continuous Boolean homomorphism from the principal ideal $\mathfrak{D}_{d_0}$ into $\mathfrak{A}_a$, so is an isomorphism between $\mathfrak{D}_{d_0}$ and a closed subalgebra of $\mathfrak{A}_a$. But this is supposed to be impossible. **X**

(b) For each non-zero $b \in \mathfrak{B}$, (a) tells us that $\sup_{d\in\mathfrak{D}} \operatorname{upr}(b \cap d) \cap \operatorname{upr}(b \setminus d) = 1$. Because $\mathfrak{A}$ is ccc, there is a countable $D_b \subseteq \mathfrak{D}$ such that $\sup_{d\in D_b} \operatorname{upr}(b \cap d) \cap \operatorname{upr}(b \setminus d) = 1$. Take $\langle d_n\rangle_{n\in\mathbb{N}}$ to be an enumeration of $\bigcup_{b\in\mathfrak{B}} D_b$.

(c)(i) For each $n \in \mathbb{N}$ choose V_n, $V_n' \in \mathcal{V}$ such that $V_n^\bullet = d_n$, $V_n \cap V_n' = \emptyset$ and $V_n[\{x\}] \cup V_n'[\{x\}]$ is dense for every $x \in X$; then $(V_n')^\bullet = 1 \setminus d_n$. If $G \subseteq Z$ is a non-empty open set, then

$$\bigcup_{n\in\mathbb{N}}\big(\pi_1[V_n \cap (X \times G)] \cap \pi_1[V_n' \cap (X \times G)]\big)$$

is conegligible. **P** Set $b=(X\times G)^\bullet\in\mathfrak{B}$. Then

$$\begin{aligned}\Bigl(\bigcup_{n\in\mathbb{N}}(\pi_1[V_n\cap(X\times G)]\cap\pi_1[V'_n\cap(X\times G)])\Bigr)^\bullet\\ &=\sup_{n\in\mathbb{N}}(\operatorname{upr}(b\cap d_n)\cap\operatorname{upr}(b\setminus d_n))\\ &=1.\ \mathbf{Q}\end{aligned}$$

Accordingly

$$X_0=\bigcap_{\sigma\in S}\bigcup_{n\in\mathbb{N}}\bigl(\pi_1[V_n\cap(X\times Z_\sigma)]\cap\pi_1[V'_n\cap(X\times Z_\sigma)]\bigr)$$

is conegligible.

(ii) Define $h:X\times Z\to Z$ by setting $h(x,y)(n)=1$ if $(x,y)\in V_n$, 0 otherwise. Then $h^{-1}[F]^\bullet\in\mathfrak{D}$ for every Borel set $F\subseteq Z$. If $W\in\mathcal{V}$ is such that $\pi_1[W]=X$, and $f:X\to Z$ is any measurable function, there is a $W'\in\mathcal{V}$ such that $\pi_1[W']=X$ and $h(x,z)\ne h(x,f(x))$ whenever $x\in X_0$, $(x,z)\in W'$. **P** Set

$$E_n=\pi_1[W\cap V_n]\cap\pi_1[W\cap V'_n]$$

for each $n\in\mathbb{N}$. Then E_n is measurable and for every $x\in X_0$ there must be some $\sigma\in S$ such that $Z_\sigma\subseteq W[\{x\}]$ and some n such that

$$x\in\pi_1[V_n\cap(X\times Z_\sigma)]\cap\pi_1[V'_n\cap(X\times Z_\sigma)],$$

in which case $x\in E_n$. Accordingly we can find a partition $\langle E'_n\rangle_{n\in\mathbb{N}}$ of X_0 into measurable sets such that $E'_n\subseteq E_n$ for each n. Next, set

$$F_n=\{x:x\in E_n,\ (x,f(x))\in V_n\}\text{ for each }n\in\mathbb{N},$$

$$\tilde W=((X\setminus X_0)\times Z)\cup\bigcup_{n\in\mathbb{N}}(((E'_n\cap F_n)\times Z)\cap V'_n)\cup(((E'_n\setminus F_n)\times Z)\cap V_n),$$

$$W'=W\cap\tilde W.$$

This works. **Q**

(iii) For any measurable function $f:X\to Z$, write $\tilde f(x)=h(x,f(x))$ for each x; then $\tilde f$ is again measurable.

Let $V\in\mathcal{V}$ be such that $V^\bullet=d$.

Write R for the family of measurable functions $f:X\to Z$ such that $(x,f(x))\in\bigcap_{n\in\mathbb{N}}V_n\cup V'_n$ for every $x\in X$.

For each $k\le m$, let $G_k\subseteq Z$ be the open-and-closed set corresponding to $b_k\in\mathfrak{B}$, and write R_k for the set of functions $f\in R$ such that $f(x)\in G_k$, $(x,f(x))\in V$ whenever $x\in\pi_1[V\cap(X\times G_k)]$.

(iv) Now there is a sequence $\langle f_{ki}\rangle_{i\in\mathbb{N}}$ in R_k such that $\tilde f_{ki}(x)\ne\tilde f_{kj}(x)$ whenever $i<j$, $x\in X_0$. **P** Choose the sequence inductively. Given f_{ki} for $i<j$, set

$$W_0=(V\cap(X\times G_k))\cup((X\setminus\pi_1[V\cap(X\times G_k)])\times Z).$$

Then $W_0\in\mathcal{V}$ and its vertical sections are non-empty. By (ii), we can choose $W_1,\dots,W_j$ such that, for each $i<j$, $W_{i+1}\in\mathcal{V}$, $W_{i+1}\subseteq W_i$, $\pi_1[W_{i+1}]=X$ and $h(x,z)\ne h(x,f_{ki}(x))$

whenever $x \in X_0$, $(x, z) \in W_{i+1}$. By 1D, we can find a measurable function f_{kj} such that $(x, f_{kj}(x)) \in W_j \cap \bigcap_{n\in\mathbb{N}}(V_n \cup V_n')$ for every $x \in X$; this works. **Q**

(v) By 1G, there is a compact set $F \subseteq Z$ such that

$$\mu(\tilde{f}_{k0}^{-1}[F]) \geq 1 - \epsilon, \quad \mu(\bigcap_{i\in\mathbb{N}} \tilde{f}_{ki}^{-1}[F]) = 0 \text{ for every } k \leq m.$$

Let $G \supseteq F$ be an open-and-closed set such that

$$\mu(\bigcap_{i\in\mathbb{N}} \tilde{f}_{ki}^{-1}[G]) \leq \epsilon$$

for every $k \leq m$. Let $r \in \mathbb{N}$ be such that G is expressible as $\bigcup_{\sigma\in I} Z_\sigma$ for some $I \subseteq \{0,1\}^r$; set

$$W_\sigma = (X \times Z) \cap \bigcap_{i<\#(\sigma),\sigma(i)=1} V_i \cap \bigcap_{i<\#(\sigma),\sigma(i)=0} V_i'$$

for $\sigma \in S$,

$$I' = \{0,1\}^r \setminus I, \quad W = \bigcup_{\sigma\in I} W_\sigma, \quad W' = \bigcup_{i\in I'} W_\sigma, \quad d' = (V \cap W)^\bullet,$$

so that $d \setminus d' = (V \cap W')^\bullet$.

(vi) For each $k \leq m$,

$$\bar{\mu}(\mathrm{upr}(b_k \cap d)) = \mu\pi_1[V \cap (X \times G_k)],$$

$$\bar{\mu}(\mathrm{upr}(b_k \cap d')) = \mu\pi_1[V \cap W \cap (X \times G_k)],$$

$$\bar{\mu}(\mathrm{upr}(b_k \cap d \setminus d')) = \mu\pi_1[V \cap W' \cap (X \times G_k)].$$

Now for any $x \in \tilde{f}_{k0}^{-1}[G]$,

$$\begin{aligned} x \in \pi_1[V \cap (X \times G_k)] &\Rightarrow (x, f_{k0}(x)) \in V \cap (X \times G_k) \cap W \\ &\Rightarrow x \in \pi_1[V \cap W \cap (X \times G_k)]; \end{aligned}$$

since the measure of the set of x for which this fails is at most ϵ,

$$\begin{aligned} \bar{\mu}(\mathrm{upr}(b_k \cap d)) - \bar{\mu}(\mathrm{upr}(b_k \cap d')) &= \mu\pi_1[V \cap (X \times G_k)] - \mu\pi_1[V \cap W \cap (X \times G_k)] \\ &\leq \epsilon. \end{aligned}$$

At the same time, if $x \notin \bigcap_{i\in\mathbb{N}} \tilde{f}_{ki}^{-1}[G]$, and $x \in \pi_1[V \cap (X \times G_k)]$, there must be some $i \in \mathbb{N}$ such that $(x, f_{ki}(x)) \in V \cap W' \cap (X \times G_k)$, so that $x \in \pi_1[V \cap W' \cap (X \times G_k)]$. Again, this happens for all x outside a set of measure at most ϵ, so

$$\begin{aligned} \bar{\mu}(\mathrm{upr}(b_k \cap d)) - \bar{\mu}(\mathrm{upr}(b_k \cap d \setminus d')) &= \mu\pi_1[V \cap (X \times G_k)] - \mu\pi_1[V \cap W' \cap (X \times G_k)] \\ &\leq \epsilon. \end{aligned}$$

So we have found an appropriate d'.

(d) Start by choosing a family $\langle\epsilon_{\sigma\tau}\rangle_{\#(\tau)\leq\#(\sigma)}$ of strictly positive numbers summing to something strictly less than 1. Construct the d_σ inductively, using (c), so that

$$\bar{\mu}(\mathrm{upr}(d_\sigma \cap e_\tau) \setminus (\mathrm{upr}(d_{\sigma^\frown 0} \cap e_\tau) \cap \mathrm{upr}(d_{\sigma^\frown 1} \cap e_\tau))) \le \epsilon_{\sigma\tau}$$

for all σ, τ such that $\#(\sigma) \ge \#(\tau)$.

(e) Take the d_σ from (d). Set

$$a_0 = \sup_{\#(\tau)\le\#(\sigma)} \mathrm{upr}(d_\sigma \cap e_\tau) \setminus (\mathrm{upr}(d_{\sigma^\frown 0} \cap e_\tau) \cap \mathrm{upr}(d_{\sigma^\frown 1} \cap e_\tau)),$$

$a = 1 \setminus a_0$. Then we have $a \ne 0$, and

$$a \cap \mathrm{upr}(e_\tau \cap d_\sigma) = a \cap \mathrm{upr}(e_\tau \cap d_{\sigma^\frown i})$$

whenever $\#(\tau) \le \#(\sigma)$ and $i \in \{0,1\}$. So

$$a \cap \mathrm{upr}(e_\tau \cap d_\sigma) = a \cap \mathrm{upr}(e_\tau \cap d_\upsilon)$$

whenever $\#(\tau) \le \#(\sigma)$ and $\sigma \subseteq \upsilon$. In particular, $a \subseteq \mathrm{upr}(d_\sigma)$ for every $\sigma \in S$.

Set $d^* = \inf\{d : d \in \mathfrak{D},\ d \supseteq a\}$. Then we have an injective Boolean homomorphism $\phi : \mathfrak{B} \to \mathfrak{D}_{d^*}$ defined by setting $\phi e_\sigma = d_\sigma \cap d^*$ for every $\sigma \in S$. Now ϕ is order-continuous. **P?** Otherwise, there is a set $S' \subseteq S$ such that $\sup_{\tau\in S'} e_\tau = 1$, that is, $\bigcup_{\tau\in S'} Z_\tau$ is dense in Z, but $d^* \not\subseteq \sup_{\tau\in S'} d_\tau$. In this case $a \not\subseteq \sup_{\tau\in S'} d_\tau$, so there must be a non-zero $a' \subseteq a$ and an $\upsilon \in S$ such that $a' \cap e_\upsilon \cap d_\tau = 0$ for every $\tau \in S'$. Now, however, there is surely a σ such that $\#(\sigma) = \#(\upsilon)$ and $a' \cap e_\upsilon \cap d_\sigma \ne 0$, and now there is a $\tau \in S'$ such that $\tau \supseteq \sigma$, so that

$$\begin{aligned} 0 &= a' \cap \mathrm{upr}(d_\tau \cap e_\upsilon) = a' \cap a \cap \mathrm{upr}(d_\tau \cap e_\upsilon) \\ &= a' \cap a \cap \mathrm{upr}(d_\sigma \cap e_\upsilon) = a' \cap \mathrm{upr}(d_\sigma \cap e_\upsilon) \ne 0, \end{aligned}$$

which is impossible. **XQ**

Accordingly ϕ has an extension to an embedding of the Dedekind completion of $\mathfrak{B}$, which is isomorphic to $\mathfrak{G}$, into $\mathfrak{D}_{d^*}$, and $\mathfrak{D}_{d^*}$ has a closed subalgebra isomorphic to $\mathfrak{G}$.

(f) It follows that if $d_0 \in \mathfrak{D}$ is non-zero, there is a non-zero $d_1 \in \mathfrak{D}$ such that $d_1 \subseteq d_0$ and the principal ideal $\mathfrak{D}_{d_1}$ has an order-closed subalgebra isomorphic to $\mathfrak{G}$. **P** There are non-zero $a \in \mathfrak{A}$, $\sigma \in S$ such that $a \cap e_\sigma \subseteq d_0$. Now $\mathfrak{D}' = \{d \cap a \cap e_\sigma : d \in \mathfrak{D}\}$ is an order-closed subalgebra of $\mathfrak{C}_{a\cap e_\sigma}$ isomorphic to the principal ideal of $\mathfrak{D}$ generated by $d^* = \inf\{d : d \in \mathfrak{D},\ a \cap e_\sigma \subseteq d\}$, so itself has no principal ideal isomorphic to any closed subalgebra of $\mathfrak{A}_a$. By (e), applied to $\mathfrak{C}_{a\cap e_\sigma}$ identified with the Dedekind completion of $\mathfrak{A}_a \otimes \mathfrak{G}$, there is a $d_1 \in \mathfrak{D}_{d^*}$ such that the principal ideal $\mathfrak{D}_{d_1}$ has an order-closed subalgebra isomorphic to $\mathfrak{G}$. **Q**

Accordingly there is a partition of unity $\langle d_i \rangle_{i\in I}$ in $\mathfrak{D}$ such that every principal ideal $\mathfrak{D}_{d_i}$ has an order-closed subalgebra $\mathfrak{G}_i$ isomorphic to $\mathfrak{G}$. But now

$$\{d : d \in \mathfrak{D},\ d \cap d_i \in \mathfrak{G}_i \text{ for every } i \in I\}$$

is an order-closed subalgebra of $\mathfrak{D}$ isomorphic to $\prod_{i\in I} \mathfrak{G}_i$; because $\mathfrak{D}$, like $\mathfrak{C}$, is ccc, I is countable and $\prod_{i\in I} \mathfrak{G}_i \cong \mathfrak{G}^I \cong \mathfrak{G}$. So we have an order-closed subalgebra of $\mathfrak{D}$ isomorphic to $\mathfrak{G}$, as required.

1J I rewrite 1If as a theorem.

Theorem Let $\mathfrak{D}$ be an order-closed subalgebra of $\mathfrak{C}$. Then *either* $\mathfrak{D}$ has a non-zero principal ideal isomorphic to a closed subalgebra of a principal ideal of $\mathfrak{A}$ *or* $\mathfrak{D}$ has a closed subalgebra isomorphic to $\mathfrak{G}$.

1K I conclude this section with another property of $\mathcal{J}$ which we shall need. Recall that if ρ is a metric on Z inducing its topology, then the space $C(Z;Z)$ of continuous functions from Z to itself is a Polish space under the metric $\tilde{\rho}$, where

$$\tilde{\rho}(f,g) = \sup_{x\in Z} \rho(f(x), g(x))$$

for f, $g \in C(Z;Z)$.

Proposition Write μ_Z for the restriction of the usual measure on Z to $\mathcal{B}$. Let $\mathcal{J}_Z$ be the corresponding σ-ideal of the algebra $\mathcal{B}\widehat{\otimes}\mathcal{B}$ of Borel sets in $Z\times Z$, taking $(X,\Sigma,\mu) = (Z,\mathcal{B},\mu_Z)$ in 1B. If $W\in\mathcal{J}_Z$ then the set

$$\{g : g\in C(Z;Z),\ \{x : (x,g(x))\in W\} \text{ is negligible}\}$$

is comeager in $C(Z;Z)$.

proof (a) Suppose that $W\in\mathcal{J}_{Z0}$, as defined in 1C, and $\epsilon>0$. Then

$$G = \{g : g\in C(Z;Z),\ \mu_Z\{x : x\in Z,\ (x,g(x))\in W\}\le\epsilon\}$$

has dense interior in $C(Z;Z)$. **P** Take any $g_0\in C(Z;Z)$, $\delta>0$. Then there is an $n\in\mathbb{N}$ such that $\rho(x,y)\le\delta$ whenever $x{\restriction}n = y{\restriction}n$. There is an $m\in\mathbb{N}$ such that $g_0(x){\restriction}n = g_0(x){\restriction}n$ whenever $x{\restriction}m = y{\restriction}m$; for $\sigma\in\{0,1\}^m$ take $\tau(\sigma) = g(x){\restriction}n$ whenever $x\in Z_\sigma$. We can express $(Z\times Z)\setminus W$ as $\bigcup_{\sigma\in S'} E_\sigma\times Z_\sigma$, where every E_σ belongs to $\mathcal{B}$ and $S' = \bigcup_{i\ge n}\{0,1\}^i$. Setting $H = \{x : W[\{x\}] \text{ is nowhere dense}\}$, so that H is conegligible, we see that

$$H\cap Z_\sigma \subseteq \bigcup_{v\in S', v\supseteq\tau(\sigma)} E_v$$

for each σ. There is a compact set $K\subseteq Z\cap H$, of measure greater than $1-\epsilon$, such that $K\cap E_v$ and $K\setminus E_v$ are compact for every $v\in S'$; in this case there is an $r\in\mathbb{N}$ such that

$$K\cap Z_\sigma \subseteq \bigcup_{v\in S', \#(v)\le r, v\supseteq\tau(\sigma)} E_v$$

for every $\sigma\in\{0,1\}^m$. Set $T = \{v : v\in S',\ \#(v)\le r\}$. There is a $g\in C(Z;Z)$ such that, for every $x\in K$, there are some $\sigma\in\{0,1\}^m$, $v\in T$ such that $v\supseteq\tau(\sigma)$, $x\in E_v\cap Z_\sigma$ and $g(x)\in Z_v$. Let $\eta>0$ be such that $\rho(x,y)\le\eta\Rightarrow x{\restriction}r = y{\restriction}r$; then if $\tilde{\rho}(f,g)\le\eta$, we must have

$$(x,f(x))\in\bigcup_{v\in T} E_v\times Z_v\subseteq (Z\times Z)\setminus W$$

for every $x\in K$, so that $f\in G$. Thus we have a non-empty open set $G_1 = \{f : \tilde{\rho}(f,g)<\eta\}$ included in G; moreover, $\tilde{\rho}(g_0,G_1)\le\tilde{\rho}(g,g_0)\le\delta$. As g_0, δ are arbitrary, $\operatorname{int} G$ is dense. **Q**

(b) Now if $W\in\mathcal{J}_Z$, there is a sequence $\langle W_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{J}_{Z0}$ covering W. The set

$$\{g : g\in C(Z;Z),\ \{x : (x,g(x))\in W\} \text{ is negligible}\}$$

includes

$$\bigcap_{n\in\mathbb{N}}\{g : \mu\{x : (x,g(x))\in W_n\}\le 2^{-n}\},$$

which by (a) is conegligible.

2. The main theorem

2A Definition I will say that a **power set σ-quotient algebra** is a Boolean algebra which is isomorphic to a quotient algebra $\mathcal{P}A/\mathcal{I}$ for some set A and some σ-ideal $\mathcal{I}$ of subsets of A. Then it is easy to see (i) that any product of power set σ-quotient algebras is a power set σ-quotient algebra (ii) any principal ideal of a power set σ-quotient algebra is a power set σ-quotient algebra.

Recall that it is relatively consistent with ZFC that no non-trivial atomless ccc Boolean algebra is a power set σ-quotient algebra (SOLOVAY 71, FREMLIN 93, §9). I write κ_{aqmc} for the least cardinal, if there is one, of any set A which has a proper σ-ideal $\mathcal{I}$ of subsets such that $\mathcal{P}A/\mathcal{I}$ is atomless and ccc. (If there is no such set, I write $\kappa_{\text{aqmc}} = \infty$.) Note that κ_{aqmc}, if not ∞, is necessarily weakly inaccessible, and much more (SOLOVAY 71, FREMLIN 93). We need to know in addition that $\text{cf}([\kappa_{\text{aqmc}}]^{\leq\omega}) = \kappa_{\text{aqmc}}$ (SHELAH SH430, FREMLIN N94, S7B).

2B The object of this section is to show that if $\mathfrak{C}$ is defined as in 1B, it cannot be a power set σ-quotient algebra unless κ_{aqmc} is strictly less than the Maharam type $\tau(\mathfrak{A})$ of $\mathfrak{A}$. (See FREMLIN 89 or FREMLIN P97*, chap. 33, for the structure theory of measure algebras.) The proof is by contradiction, and this whole section, down to 2K, will be governed by the hypothesis

(*) $(\mathfrak{A}, \bar{\mu})$ is a probability algebra, the Dedekind completion $\mathfrak{C}$ of $\mathfrak{A} \otimes \mathfrak{G}$ is a power set σ-quotient algebra, and $\tau(\mathfrak{A}) \leq \kappa_{\text{aqmc}}$.

The eventual unavoidable catastrophe will prove the theorem.

2C I begin with a pair of simple reductions.

Proposition We may suppose that $\mathfrak{A}$ is homogeneous, with infinite Maharam type.

proof Take a non-zero homogeneous principal ideal $\mathfrak{A}_a$ of $\mathfrak{A}$. Then the corresponding principal ideal $\mathfrak{C}_a$ can be identified with the Dedekind completion of $\mathfrak{A}_a \otimes \mathfrak{G}$, and is still a power set σ-quotient algebra. Of course $\mathfrak{A}_a$, with a suitable normalization of the induced measure, is again a probability algebra, so that $\mathfrak{A}_a$, $\mathfrak{C}_a$ satisfy (*). $\mathfrak{A}_a$ cannot be $\{0, 1\}$ because $\mathfrak{G}$ itself is not a power set σ-quotient algebra (GITIK & SHELAH 89). So $\tau(\mathfrak{A}_a)$ is infinite; and of course $\tau(\mathfrak{A}_a) \leq \tau(\mathfrak{A}) \leq \kappa_{\text{aqmc}}$.

2D Construction For the rest of this section, Γ will be a set of cardinal $\tau(\mathfrak{A})$, where $\mathfrak{A}$ is taken from 2C, so that $\omega \leq \#(\Gamma) \leq \kappa_{\text{aqmc}}$; X will be $\{0,1\}^\Gamma$, Σ will be the Baire σ-algebra of X, and μ will be the restriction of the usual measure of X to Σ. The measure algebra of (X, Σ, μ) will be isomorphic to $\mathfrak{A}$ in 2C, so that if we now employ the construction and language of §1, the algebra $\mathfrak{C}$, the Dedekind completion of $\mathfrak{A} \otimes \mathfrak{G}$, will be a power set σ-quotient algebra.

2E Proposition There is a σ-ideal $\mathcal{I}$ of subsets of $X \times Z$ such that (i) for $W \in \Sigma\widehat{\otimes}\mathcal{B}$, $W \in \mathcal{I}$ iff $W \in \mathcal{J}$ (ii) for every $A \subseteq X \times Z$ there is a $W \in \Sigma\widehat{\otimes}\mathcal{B}$ such that $W \triangle A \in \mathcal{I}$ (iii) setting $\kappa = \text{add}(\mathcal{I})$, $X \times Z$ is covered by κ members of $\mathcal{I}$.

proof (a) For each $\gamma \in \Gamma$, $n \in \mathbb{N}$ let $a_\gamma \in \mathfrak{A}$, $b_n \in \mathfrak{G}$ correspond to $\{x : x(\gamma) = 1\} \in \Sigma$, $\{z : z(n) = 1\} \subseteq Z$. Fix an isomorphism $\pi : \mathfrak{C} \to \mathcal{P}A_0/\mathcal{I}_0$, where $\mathcal{I}_0$ is a σ-ideal of subsets of A_0. Choose C_γ, $\tilde{C}_n \subseteq A_0$, for $\gamma \in \Gamma$ and $n \in \mathbb{N}$, such that $\pi a_\gamma = C_\gamma^\bullet$, $\pi b_n = \tilde{C}_n^\bullet$ for all γ, n. Define $f : A_0 \to X \times Z$ by saying that $f_1(t)(\gamma) = 1$ iff $t \in C_\gamma$, $f_2(t)(n) = 1$ iff $t \in \tilde{C}_n$. Set $\mathcal{I}_1 = \{I : I \subseteq X \times Z,\ f^{-1}[I] \in \mathcal{I}_0\}$.

We have $f^{-1}[W]^\bullet = \pi W^\bullet$ whenever W is of the form $C_\gamma \times \tilde{C}'_n$ and therefore (α) whenever $W = E \times \tilde{C}'_n$ where $E \in \Sigma$, $n \in \mathbb{N}$ (β) for every $W \in \Sigma\widehat{\otimes}\mathcal{B}$. So, for $W \in \Sigma\widehat{\otimes}\mathcal{B}$,

$$W \in \mathcal{I}_1 \iff f^{-1}[W] \in \mathcal{I}_0 \iff f^{-1}[W]^\bullet = 0 \iff W^\bullet = 0 \iff W \in \mathcal{J}.$$

If A is any subset of $X \times Z$, then there is a $W \in \Sigma\widehat{\otimes}\mathcal{B}$ such that $\pi W^\bullet = f^{-1}[A]^\bullet$, so that $f^{-1}[W \triangle A]^\bullet = 0$ and $W \triangle A \in \mathcal{I}_1$.

Set $\kappa = \operatorname{add}(\mathcal{I}_1)$. (Of course this is well-defined because singletons belong to $\mathcal{I}_1$.) There is an $A_1 \subseteq X \times Z$ such that $A_1 \notin \mathcal{I}_1$ but A_1 is expressible as the union of κ members of $\mathcal{I}_1$. Let $W_1 \in \Sigma\widehat{\otimes}\mathcal{B}$ be such that $W_1 \triangle A_1 \in \mathcal{I}_1$. Let $E \subseteq X$, $\sigma \in S$ be such that E is a self-supporting compact set and $(E \times Z_\sigma) \setminus W_1 \in \mathcal{J}$. Then $E \times Z_\sigma$ can be covered by κ members of $\mathcal{I}_1$. But $E \cong X$ as topological measure space (up to a scalar multiple of the measure) and Z_σ is homeomorphic to Z. If we copy $E \times Z_\sigma$ onto $X \times Z$, then $\mathcal{I}_1 \cap \mathcal{P}(E \times Z_\sigma)$ will be copied onto an ideal $\mathcal{I}$ with the required properties.

2F Definition Write κ for the additivity of $\mathcal{I}$, as in 2E. Because X can be covered by $\mathfrak{c}$ negligible sets, $\kappa \le \mathfrak{c}$. Let $\psi : X \times Z \to \kappa$ be a function such that $\psi^{-1}[\xi] \in \mathcal{I}$ for every $\xi < \kappa$. Let $\mathcal{I}^*$ be the ideal $\{C : C \subseteq \kappa,\ \psi^{-1}[C] \in \mathcal{I}\}$.

2G Proposition $\mathcal{P}\kappa/\mathcal{I}^*$ is an atomless ccc Dedekind complete Boolean algebra, and has an order-closed subalgebra isomorphic to $\mathfrak{G}$.

proof Of course $\mathcal{P}\kappa/\mathcal{I}^*$ is Dedekind σ-complete. Because $\kappa \le \mathfrak{c}$ and $\mathcal{I}^*$ contains singletons, $\mathcal{P}\kappa/\mathcal{I}^*$ is atomless. The canonical map $C^\bullet \mapsto \psi^{-1}[C]^\bullet$ is an injective sequentially order-continuous Boolean homomorphism embedding the Dedekind σ-complete algebra $\mathcal{P}\kappa/\mathcal{I}^*$ as a subalgebra of $\mathcal{P}(X \times Z)/\mathcal{I} \cong \mathfrak{C}$. Because $\mathfrak{C}$ is ccc, so is $\mathcal{P}\kappa/\mathcal{I}^*$, and the embedding is order-continuous; the quotient is therefore isomorphic to a closed subalgebra of $\mathfrak{C}$. But a principal ideal of $\mathcal{P}\kappa/\mathcal{I}^*$ cannot be isomorphic to any closed subalgebra of any principal ideal of $\mathfrak{A}$, by (*); so by 1J, $\mathcal{P}\kappa/\mathcal{I}^*$ has a closed subalgebra isomorphic to $\mathfrak{G}$.

2H Corollary $\kappa \ge \kappa_{\mathrm{aqmc}}$.

2I Lemma There is no covering of Z by fewer than κ negligible sets; consequently every subset of Z of cardinal less than κ is meager. More generally, if Y is any Polish space without isolated points, any subset of Y of cardinal less than κ is meager.

proof Let $\theta : \mathbb{N} \to \Gamma$ be an injection. Define $f : X \times Z \to Z$ by setting $f(x,z)(n) = x(\theta n)$ for every $x \in X$, $z \in Z$, $n \in \mathbb{N}$. Then $f^{-1}[E] \in \mathcal{J} \subseteq \mathcal{I}$ for every negligible Borel set $E \subseteq Z$; as $X \times Z$ cannot be covered by fewer than κ members of $\mathcal{I}$, Z cannot be covered by fewer than κ negligible sets.

The second assertion is just Rothberger's theorem: there is a comeager negligible set $H \subseteq Z$; if $D \subseteq Z$ has cardinal less than κ, $D + H$ cannot cover Z, where $+$ is the usual addition operation on Z identified with $\mathbb{Z}_2^{\mathbb{N}}$; if $z \in Z \setminus (D + H)$, $Z \setminus (z - H)$ is a meager set including D.

Finally, any Polish space Y without isolated points has a dense G_δ subset homeomorphic to a dense G_δ subset of Z; because any subset of Z with fewer than κ points is meager, the same is true of Y.

2J Lemma There is a function $\xi \mapsto g_\xi : \kappa \to C(Z;Z)$ such that $\{\xi : g_\xi \in M\} \in \mathcal{I}^*$ for every meager set $M \subseteq C(Z;Z)$.

proof $C(Z;Z)$, with the metric described in 1K, is a Polish space without isolated points, so has a dense G_δ set Y homeomorphic to $\mathbb{N}^\mathbb{N}$. Let $\theta : \mathbb{N}^\mathbb{N} \to Y$ be a homeomorphism. The regular open algebra $\mathfrak{G}_1$ of $\mathbb{N}^\mathbb{N}$ is isomorphic to $\mathfrak{G}$, so there is an order-continuous injection $\pi : \mathfrak{G}_1 \to \mathcal{P}\kappa/\mathcal{I}^*$, by 2G. Set $\tilde{S} = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ and for $\sigma \in \tilde{S}$ set $\tilde{Z}_\sigma = \{\alpha : \sigma \subseteq \alpha \in \mathbb{N}^\mathbb{N}\}$. Choose $C_\sigma \subseteq \kappa$ inductively, for $\sigma \in S^*$, by setting $C_\emptyset = \kappa$ and, for each σ, taking $\langle C_{\sigma^\frown i}\rangle_{i\in\mathbb{N}}$ to be a partition of C_σ such that $C^\bullet_{\sigma^\frown i} = \pi\tilde{Z}^\bullet_{\sigma^\frown i}$ for every i. Define $\phi : \kappa \to \mathbb{N}^\mathbb{N}$ by saying that $\phi(\xi) \in \tilde{Z}_\sigma$ iff $\kappa \in C_\sigma$. Then $\pi G = \phi^{-1}[G]^\bullet$ for every G of the form $\tilde{Z}_\sigma$; consequently $\phi^{-1}[M]^\bullet = 0$ for every nowhere dense $M \subseteq \mathbb{N}$ and $\phi^{-1}[M] \in \mathcal{I}^*$ for every meager $M \subseteq \mathbb{N}^\mathbb{N}$. Now take $g_\xi = \theta(\phi(\xi))$ for every $\xi < \kappa$.

2K The contradiction Take the family $\langle g_\xi\rangle_{\xi<\kappa}$ from 2J. Because $\#(\Gamma) \le \kappa_{\text{aqmc}} \le \kappa$, and $\mathrm{cf}([\kappa_{\text{aqmc}}]^{\le\omega}) = \kappa_{\text{aqmc}}$, there is a family $\langle I_\xi\rangle_{\xi<\kappa}$ of countably infinite subsets of Γ such that every countable subset of Γ is included in some I_ξ. For each $\xi < \kappa$, let $\theta_\xi : \mathbb{N} \to I_\xi$ be a bijection, so that $x \mapsto x\theta_\xi$ is a continuous surjection from X onto Z, and for ξ, $\eta < \kappa$ write

$$\tilde{g}_{\xi\eta}(x) = g_\eta(x\theta_\xi)$$

for $x \in X$; this makes $\tilde{g}_{\xi\eta}$ a continuous function from X to Z.

For each $\alpha < \kappa$, $x \in X$,

$$M_{x\alpha} = \{\tilde{g}_{\xi\eta}(x) : \xi,\, \eta < \alpha\}$$

is meager in Z, by 2I. For each $x \in X$, $\alpha < \kappa$ choose a sequence $\langle K(x,\alpha,n)\rangle_{n\in\mathbb{N}}$ of subsets of S such that $\bigcup_{\sigma\in K(x,\alpha,n)} Z_\sigma$ is dense in Z for each n and $M_{x\alpha} \cap \bigcap_{n\in\mathbb{N}} \bigcup_{\sigma\in K(x,\alpha,n)} Z_\sigma = \emptyset$.

Recall that we have a function $\psi : X \times Z \to \kappa$ such that $\psi^{-1}[\alpha] \in \mathcal{I}$ for every $\alpha < \kappa$. Write

$$Q = \{(x, y, \tilde{g}_{\xi\eta}(x)) : x,\, y \in X,\, \xi,\, \eta < \psi(x,y)\} \subseteq X \times Z \times Z.$$

For $n \in \mathbb{N}$, $\sigma \in S$ set

$$C_{n\sigma} = \{(x,y) : \sigma \in K(x, \psi(x,y), n)\} \subseteq X \times Z.$$

Then

$$Q \cap \bigcap_{n\in\mathbb{N}} \bigcup_{\sigma\in S} C_{n\sigma} \times Z_\sigma = \emptyset.$$

Take $W_{n\sigma} \in \Sigma\widehat{\otimes}\mathcal{B}$ such that $C_{n\sigma} \triangle W_{n\sigma} \in \mathcal{I}$ and set

$$R = \bigcap_{n\in\mathbb{N}} \bigcup_{\sigma\in S} W_{n\sigma} \times Z_\sigma, \qquad I = \bigcup_{n\in\mathbb{N},\sigma\in S} C_{n\sigma} \triangle W_{n\sigma}.$$

Then $R \subseteq X \times Z \times Z$ belongs to $\Sigma\widehat{\otimes}\mathcal{B}\widehat{\otimes}\mathcal{B}$, $I \in \mathcal{I}$ and if $(x,y) \in (X \times Z) \setminus I$, ξ, $\eta < \psi(x,y)$ then $(x, y, \tilde{g}_{\xi\eta}(x)) \notin R$. Moreover, for $(x,y) \in (X \times Z) \setminus I$,

$$\{z : (x,y,z) \in R\} = \{z : z \in \bigcap_{n\in\mathbb{N}} \bigcup_{\sigma\in K(x,\psi(x,y),n)} Z_\sigma\}$$

is comeager. Since

$$J = \{(x,y) : \{z : (x,y,z) \in R\} \text{ is not comeager}\}$$

belongs to $\Sigma\widehat{\otimes}\mathcal{B}$ and also to $\mathcal{I}$, it belongs to $\mathcal{J}$.

Let $E \subseteq X$ be the conegligible set

$$\{x : \{y : (x, y) \in J\} \text{ is meager}\}.$$

Then for any $x \in E$, $\{(y, z) : (x, y, z) \in R\}$ is comeager. But this means that if we write

$$J' = \{(x, z) : \{y : (x, y, z) \in R\} \text{ is not comeager}\},$$

$\{z : (x, z) \in J'\}$ is meager for every $x \in E$, so that $J' \in \mathcal{J}$.

At this point, note that because $R \in \Sigma \widehat{\otimes} \mathcal{B} \widehat{\otimes} \mathcal{B}$ there is some countable subset I of Γ such that R belongs to $\Sigma_I \widehat{\otimes} \mathcal{B} \widehat{\otimes} \mathcal{B}$, where Σ_I is the σ-algebra generated by the sets $\{x : x(\gamma) = 1\}$ as γ runs over I. Let $\xi < \kappa$ be such that $I \subseteq I_\xi$. Now J' is of the form

$$\{(x, z) : (\tilde{x}, z) \in \tilde{J}'\},$$

where for $x \in X$ I define $\tilde{x} \in Z$ by setting

$$\tilde{x}(n) = x(\theta_\xi(n)) \text{ for every } n \in \mathbb{N};$$

$\tilde{J}'$ is a Borel set in $Z \times Z$. E is also of the form $\{x : \tilde{x} \in \tilde{E}\}$ for some $\tilde{E} \subseteq Z$, and $\mu_Z \tilde{E} = \mu E = 1$. For $w \in \tilde{E}$, $\{z : (w, z) \in \tilde{J}'\} = \{z : (x, z) \in J'\}$ where $w = \tilde{x}$, so is meager; thus $\tilde{J}' \in \mathcal{J}_Z$, writing $\mathcal{J}_Z$ for the ideal in $\mathcal{B} \widehat{\otimes} \mathcal{B}$ as in 1K.

Next, recall that $\{g_\eta : \eta < \kappa\}$ cannot be meager in $C(Z; Z)$, by 2J, since surely $\kappa \notin \mathcal{I}^*$. So by 1K there is an $\eta < \kappa$ such that $\{w : (w, g_\eta(w)) \in \tilde{J}'\}$ is negligible in Z. So

$$\{x : (x, \tilde{g}_{\xi\eta}(x)) \in J'\} = \{x : (x, g_\eta(\tilde{x})) \in J'\} = \{x : (\tilde{x}, g_\eta(\tilde{x})) \in \tilde{J}'\}$$

is negligible in X. Thus

$$J_1 = \{(x, y) : (x, y, \tilde{g}_{\xi\eta}(x)) \notin R\} \in \mathcal{J},$$

and there must be an $(x, y) \in (X \times Z) \setminus (I \cup J_1 \cup \psi^{-1}[\alpha])$, where $\alpha = \max(\xi, \eta) + 1$. In this case, however,

$$(x, y, \tilde{g}_{\xi\eta}(x))$$

belongs to both R and its complement, because ξ, $\eta < \psi(x, y)$, which is absurd.

2L Thus we have reached the anticipated contradiction, and $(*)$ must be false. Turning it round, we have the theorem:

Theorem If $\mathfrak{A}$ is a probability algebra of Maharam type less than or equal to κ_{aqmc}, then the Dedekind completion of $\mathfrak{A} \otimes \mathfrak{G}$ is not a power set σ-quotient algebra.

3. Disjoint refinements

In this section I use the result above to prove some results on disjoint refinements of sequences of subsets of Radon measure spaces.

3A Notation Throughout this section, $(X, \mathfrak{T}, \Sigma, \mu)$ will be an atomless complete locally determined measure space of Maharam type less than or equal to κ_{aqmc} (2A).

3B Lemma Let $A \subseteq X$ be any set. Then there is a set $D \subseteq A$ such that $\mu^* D = \mu^*(A \setminus D) = \mu^* A$.

proof (a) If $B \subseteq X$ and $0 < \mu B < \infty$, there are F, D such that $D \subseteq B \cap F$, $F \in \Sigma$ and $\mu^* D = \mu^*(B \setminus D) = \mu F$. **P** Let μ_B be the subspace measure on B. Then the measure algebra of μ_B is atomless and has Maharam type at most κ_{aqmc}, and therefore cannot be a power set σ-quotient algebra (GITIK & SHELAH 93 or FREMLIN 93, 3G, since there is surely no atomlessly-measurable cardinal less than κ_{aqmc}). But this means, in particular, that there is a set $C \subseteq B$ which is not μ_B-measurable, so that $\mu_B^* C + \mu_B^*(B \setminus C) > \mu_B B$, that is, $\mu^* C + \mu^*(B \setminus C) > \mu^* B$. Let F_1, F_2 be measurable envelopes of C and $B \setminus C$ respectively, and set $F = F_1 \cap F_2$, $D = C \cap F$; then

$$\mu^* C = \mu^*(F \setminus C) = \mu F = \mu^* C + \mu^*(B \setminus C) - \mu^* B > 0. \text{ **Q**}$$

(b) Let H be a set expressible as a countable union of sets of finite measure such that $\mu^*(A \cap H) = \mu^* A$. Let $\langle (F_i, D_i) \rangle_{i \in I}$ be a maximal family such that

$$F_i \in \Sigma, \quad D_i \subseteq A \cap H \cap F_i, \quad \mu^* D_i = \mu^*(A \cap F_i \setminus D_i) = \mu F_i$$

for every i and $\langle F_i \rangle_{i \in I}$ is disjoint. Because $\mu(H \cap F_i) > 0$ for every i, I must be countable. Set $E = \bigcup_{i \in I} F_i$. **?** If $A \cap H \setminus E$ is not negligible, then by (a) we can find an (F, D) which we should have added to $\langle (F_i, D_i) \rangle_{i \in I}$. **X** So, setting $D = \bigcup_{i \in I} D_i$, we must have

$$\mu^* A = \mu^*(A \cap H) = \sum_{i \in I} \mu^*(A \cap F_i) = \sum_{i \in I} \mu F_i$$
$$= \sum_{i \in I} \mu^* D_i = \sum_{i \in I} \mu^*(A \cap F_i \setminus D_i) = \mu^* D = \mu^*(A \setminus D).$$

3C Lemma Let $\langle A_n \rangle_{n \in \mathbb{N}}$ be a sequence of subsets of X, and suppose that $\mu^* A > 0$. Then there is a set $C \subseteq A$ such that $\mu^* C > 0$ and $\mu^*(A_n \setminus C) = \mu^* A_n$ for every $n \in \mathbb{N}$.

proof ? Suppose, if possible, otherwise.

(a) To begin with (down to the end of (e)), suppose that $\mu^* A = \mu X = 1$. For $\sigma \in S$ choose $D_\sigma \subseteq A$ inductively, as follows. Start with $D_\emptyset = A_0$. Given that $D_\sigma \subseteq A$ and $\mu^* D_\sigma = 1$, where $\sigma \in \{0,1\}^n$, let E_σ be a measurable envelope of $D_\sigma \cap A_n$, and set $D'_\sigma = (D_\sigma \cap A_n) \cup (D_\sigma \setminus E_\sigma)$; then $\mu^* D'_\sigma = 1$. Choose $D_{\sigma^\frown 0} \subseteq D'_\sigma$ such that

$$\mu^* D_{\sigma^\frown 0} = \mu^*(D'_\sigma \setminus D_{\sigma^\frown 0}) = 1,$$

and set $D_{\sigma^\frown 1} = D_\sigma \setminus D_{\sigma^\frown 0}$. Then

$$\mu^* D_{\sigma^\frown 0} = \mu^*(D_{\sigma^\frown 1}) = 1.$$

Continue.

(b) On completing the induction, define $f : A \to X \times Z$ by setting $f(x) = (x, \alpha)$, where α is that member of Z such that $x \in D_{\alpha \restriction n}$ for every n. Then for any non-negligible $C \subseteq A$ there are a non-negligible measurable set $E \subseteq X$ and a $\sigma \in S$ such that $f^{-1}[E \times Z_\sigma] \subseteq C$. **P** By hypothesis, there is some $n \in \mathbb{N}$ such that $\mu^*(A_n \setminus C) < \mu^* A_n$. Let F be a measurable envelope of $A_n \setminus C$, so that $A_n \setminus F \subseteq C$ and $\mu^*(A_n \setminus F) > 0$. Of course $A_n \setminus F \subseteq A$, so there is some $\sigma \in \{0,1\}^n$ such that $\mu^*(D_\sigma \cap A_n \setminus F) > 0$. Set $E = E_\sigma \setminus F$. Now $\mu E = \mu^*(D_\sigma \cap A_n \setminus F) > 0$ and

$$f^{-1}[E \times Z_{\sigma^\frown 0}] = D_{\sigma^\frown 0} \cap E \subseteq D'_\sigma \cap E_\sigma \setminus F \subseteq D_\sigma \cap A_n \setminus F \subseteq C. \text{ **Q**}$$

(c) Define $\Sigma\widehat{\otimes}\mathcal{B}$, $\mathcal{J}$ as in 1B. Write $\mathcal{N}$ for the ideal of negligible subsets of A. Then $f^{-1}[W] \in \mathcal{N}$ for every $W \in \mathcal{J}$. **P?** Otherwise, by 1Cb, there is a $W \in \mathcal{J}_0$ such that $f^{-1}[W]$ is not negligible. By (b), there are a non-negligible measurable set E and a $\sigma \in \bigcup_{n\in\mathbb{N}}\{0,1\}^n$ such that $f^{-1}[E \times Z_\sigma] \subseteq f^{-1}[W]$. But we are supposing that $W \in \mathcal{J}_0$, so that $(X \times Z) \setminus W$ is expressible as $\bigcup_{\sigma\in S}(E_\sigma \times Z_\sigma)$ where every E_σ is measurable. Setting

$$E' = \bigcup\{E \cap E_\sigma : \sigma \in S,\ E \cap E_\sigma \text{ is negligible}\},$$

there must be some $x \in E\setminus E'$ such that the vertical section $W[\{x\}]$ is nowhere dense. Take $\sigma \in S$ such that $x \in E_\sigma$. Then there is a $\tau \in S$ such that $\tau \supseteq \sigma$ and $(E \cap E_\sigma) \times Z_\tau$ does not meet W, while $E \cap E_\sigma$ is non-negligible (because $x \notin E'$). In this case $D_\tau \cap E \cap E_\sigma$ is non-empty, and for any $y \in D_\tau \cap E \cap E_\sigma$ we have $f(y) \in (E \times Z_\sigma) \setminus W$; which is impossible. **XQ**

(d) It follows that $\Sigma\widehat{\otimes}\mathcal{B}/\mathcal{J} \cong \mathcal{P}A/\mathcal{N}$. **P** Since $f^{-1}[W] \in \mathcal{N}$ for every $W \in \mathcal{J}$, we have a Boolean homomorphism $\pi : \Sigma\widehat{\otimes}\mathcal{B}/\mathcal{J} \to \mathcal{P}A/\mathcal{N}$ defined by setting $\pi W^\bullet = (f^{-1}[W])^\bullet$ for every $W \in \Sigma\widehat{\otimes}\mathcal{B}$. π is sequentially order-continuous, and because $\Sigma\widehat{\otimes}\mathcal{B}/\mathcal{J}$ is ccc (1Cd), π is order-continuous. If $W \in (\Sigma\widehat{\otimes}\mathcal{B}) \setminus \mathcal{J}$, there are a non-negligible measurable set $E \subseteq X$ and a $\sigma \in S$ such that $(E \times Z_\sigma) \setminus W \in \mathcal{J}$. Now $\mu^* f^{-1}[E \times Z_\sigma] = \mu^*(E \cap D_\sigma) > 0$, while (by (b) again) $f^{-1}[(E \times Z_\sigma) \setminus W]$ is negligible, so $f^{-1}[W]$ is not negligible. Thus $\pi W^\bullet \neq 0$ in $\mathcal{P}A/\mathcal{N}$; as W is arbitrary, π is injective. Finally, if $C \in \mathcal{P}A \setminus \mathcal{N}$, (b) tells us that $\pi(E \times Z_\sigma)^\bullet \subseteq C^\bullet$ for some non-negligible E, some σ; so that the range of π is order-dense. Because $\Sigma\widehat{\otimes}\mathcal{B}/\mathcal{J}$ is Dedekind complete (1Cd), π is a surjection (Fremlin p97*, 314F(a-i)), and $\Sigma\widehat{\otimes}\mathcal{B}/\mathcal{J} \cong \mathcal{P}A/\mathcal{N}$. **Q**

(e) But this contradicts Theorem 2L. **X**

So (on the assumption that $\mu^*A = \mu X = 1$) the result is proved. Clearly it will also be true whenever $0 < \mu^*A = \mu X < \infty$.

(f) For the general case, start by taking a set $A' \subseteq A$ such that $0 < \mu^*A' < \infty$. Let H be a measurable envelope of A', and set $A'_n = A_n \cap H$ for each n. Because H, with the induced measure, is a complete locally determined measure space, and its measure algebra cannot have greater Maharam type than the measure algebra of X (Fremlin p97*, 331Gc), (a)-(e) tell us that there must be a non-negligible $C \subseteq A'$ such that $\mu^*(A'_n \setminus C) = \mu^*A_n$ for every n. But now

$$\mu^*(A_n \setminus C) = \mu^*(A_n \setminus H) + \mu^*(A'_n \setminus C) = \mu^*(A_n \setminus H) + \mu^*A'_n = \mu^*A_n$$

for every n, and we're done.

3D Lemma Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a sequence of subsets of X. Then there is a set $A \subseteq A_0$ such that $\mu^*A = \mu^*A_0$ and $\mu^*(A_n \setminus A) = \mu^*A_n$ for every $n \in \mathbb{N}$.

proof Let $A' \subseteq A_0$ be a set of the same outer measure as A_0 which can be covered by a set H which is the union of a sequence of sets of finite measure. Let $\langle(C_i, E_i)\rangle_{i\in I}$ be a maximal family such that

$C_i \subseteq A'$, $\mu^*(A_n \setminus C_i) = \mu^*A_n$ for every $n \geq 1$, $\mu^*C_i > 0$, E_i is a measurable envelope of C_i

for every $i \in I$, and $E_i \cap E_j = \emptyset$ for $i \neq j$. Then I must be countable, because $\mu(E_i \cap H) > 0$ for every i. Set

$$F = X \setminus \bigcup_{i \in I} E_i, \quad A = \bigcup_{i \in I} C_i.$$

If $n \geq 1$, then

$$\begin{aligned}
\mu^*(A_n \setminus A) &= \mu^*(A_n \cap F) + \sum_{i \in I} \mu^*(A_n \cap E_i \setminus A) \\
&= \mu^*(A_n \cap F) + \sum_{i \in I} \mu^*(A_n \cap E_i \setminus C_i) \\
&= \mu^*(A_n \cap F) + \sum_{i \in I} \mu^*(A_n \cap E_i) = \mu^* A_n.
\end{aligned}$$

? Suppose, if possible, that $\mu^* A < \mu^* A'$. Then

$$\begin{aligned}
\mu^* A &< \mu^*(A \cap F) + \sum_{i \in I} \mu^*(A \cap E_i) \\
&\leq \mu^*(A \cap F) + \sum_{i \in I} \mu E_i \\
&\leq \mu^*(A \cap F) + \sum_{i \in I} \mu^*(A \cap E_i) \leq \mu^*(A \cap F) + \mu^* A,
\end{aligned}$$

so $\mu^*(A \cap F) > 0$. By Lemma 3B, there is a $C \subseteq A \cap F$ such that $\mu^* C > 0$ and $\mu^*(A_n \setminus C) = \mu^* A_n$ for every $n \geq 1$. Let $E \subseteq F$ be a measurable envelope for C. Then we ought to have added (C, E) to $\langle (C_i, E_i) \rangle_{i \in I}$. **X**

Accordingly $\mu^* A = \mu^* A' = \mu A_0$ and we're home.

3E Theorem Let $\langle A_n \rangle_{n \in \mathbb{N}}$ be a sequence of subsets of X. Then there is a disjoint sequence $\langle A'_n \rangle_{n \in \mathbb{N}}$ of sets such that $A'_n \subseteq A_n$ for every n and $\mu^* A'_n = \mu^* A_n$ for every n.

proof By Lemma 3D, we may choose inductively, for $n \in \mathbb{N}$, sets A'_n such that, for each n,

$$A'_n \subseteq A_n \setminus \bigcup_{i<n} A'_i,$$

$$\mu^* A'_n = \mu^*(A_n \setminus \bigcup_{i<n} A'_i),$$

$$\mu^*(A_m \setminus \bigcup_{i \leq n} A'_i) = \mu^*(A_m \setminus \bigcup_{i<n} A'_i) \text{ for every } m > n.$$

Then an easy induction shows that $\mu^*(A_m \setminus \bigcup_{i \leq n} A_i) = \mu^* A_m$ whenever $n < m$, so that $\mu^* A'_n = \mu^* A_n$ for each n.

3F Corollary Let $\langle A_n \rangle_{n \in \mathbb{N}}$ be a sequence of subsets of X. Then there is a set $D \subseteq X$ such that $\mu^*(A_n \cap D) = \mu^*(A_n \setminus D) = \mu^* A_n$ for every $n \in \mathbb{N}$.

proof Take $\langle A'_n \rangle_{n \in \mathbb{N}}$ from 3E. For each n, we can choose $D_n \subseteq A'_n$ such that $\mu^* D_n = \mu^*(A'_n \setminus D_n) = \mu^* A'_n$ (3B). Set $D = \bigcup_{n \in \mathbb{N}} D_n$; then $\mu^*(A_n \cap D) \geq \mu^* D_n = \mu^* A'_n = \mu^* A_n$, $\mu^*(A_n \setminus D) \geq \mu^*(A'_n \setminus D_n) = \mu^* A'_n = \mu^* A_n$ for every n, as required.

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