Notes on the Control Measure Problem

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1. Definitions Let $\mathfrak{A}$ be a Boolean algebra. A **submeasure** on $\mathfrak{A}$ is a functional $\phi : \mathfrak{A} \to [0, \infty[$ such that (i) $\phi(\mathbf{0}) = 0$ (ii) $\phi(a) \le \phi(b)$ whenever $a \subseteq b \in \mathfrak{A}$ (iii) $\phi(a \cup b) \le \phi(a) + \phi(b)$ for all a, $b \in \mathfrak{A}$. A submeasure ϕ is **exhaustive** if $\lim_{n\to\infty}\phi(a_n) = 0$ whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$; it is **uniformly exhaustive** if for every $\epsilon > 0$ there is an $n \in \mathbb{N}$ such that $\min_{i\le n}\phi(a_i) \le \epsilon$ whenever $\langle a_i \rangle_{i\le n}$ is a disjoint family in $\mathfrak{A}$; it is **normalized** if $\phi(\mathbf{1}) = 1$; it is **pathological** if there is no non-zero additive functional μ on $\mathfrak{A}$ such that $0 \le \mu(a) \le \phi(a)$ for every $a \in \mathfrak{A}$; it is **minimal** if $\psi = \phi$ whenever ψ is a submeasure on $\mathfrak{A}$, $\psi \le \phi$ and $\psi(\mathbf{1}) = \phi(\mathbf{1})$.

2. Proposition Let ϕ be a pathological exhaustive submeasure on a Boolean algebra $\mathfrak{A}$ and $b \subseteq a \in \mathfrak{A}$. (i) For any β, γ with $\phi(b) \le \beta < \gamma \le \phi(a)$, there is a $c \in \mathfrak{A}$ such that $b \subseteq c \subseteq a$ and $\beta \le \phi(c) \le \gamma$. (ii) If $\mathfrak{A}$ is Dedekind σ-complete, then for any α with $\phi(b) \le \alpha \le \phi(a)$ there is a c with $b \subseteq c \subseteq a$ and $\phi(c) = \alpha$.

proof (a) I show first that if $a \in \mathfrak{A}$ and $0 < \delta < 2\delta \le \phi(a)$, there is a $c \subseteq a$ such that $\delta \le \phi(c) < 2\delta$. **P?** Suppose, if possible, otherwise. Set

$$\mathcal{I} = \{c : c \in \mathfrak{A},\ \phi(c \cap a) < \delta\}.$$

Then $\mathcal{I}$ is an ideal in $\mathfrak{A}$. So there is a maximal ideal $\mathcal{J}$ of $\mathfrak{A}$ including $\mathcal{I}$ and this can be used to define an additive functional $\mu : \mathfrak{A} \to \mathbb{R}$ by writing $\mu(c) = 0$ if $c \in \mathcal{J}$, $\mu(c) = \delta$ if $c \in \mathfrak{A} \setminus \mathcal{J}$. Now $0 \le \mu(c) \le \phi(c)$ for every $c \in \mathfrak{A}$, contradicting the hypothesis that ϕ is pathological. **XQ**

(b) Now suppose that $c \subseteq a \in \mathfrak{A}$ and that $\phi(c) \le \beta < \gamma \le \phi(a)$. Then there is a d such that $c \subseteq d \subseteq a$ and $\beta < \phi(d) < \gamma$. **P?** Suppose, if possible, otherwise. Construct an increasing sequence $\langle d_n \rangle_{n\in\mathbb{N}}$, as follows. $d_0 = c$. Given that $c \subseteq d_n \subseteq a$ and that $\phi(d_n) \le \beta$, then $\phi(a \setminus d_n) \ge \phi(a) - \phi(d_n) \ge \gamma - \beta$. By (a), applied with $a \setminus d_n$ in the place of a, there is a $d'_n \subseteq a \setminus d_n$ such that $\frac{1}{2}(\gamma - \beta) \le \phi(d'_n) < \gamma - \beta$. Set $d_{n+1} = d_n \cup d'_n$; then $c \subseteq d_n \subseteq d_{n+1} \subseteq a$, and $\phi(d_{n+1}) \le \phi(d_n) + \phi(d'_n) < \beta + (\gamma - \beta) = \gamma$; by hypothesis, $\phi(d_{n+1}) \le \beta$, and the induction proceeds.

But now observe that $\langle d'_n \rangle_{n\in\mathbb{N}}$, as constructed above, is a disjoint sequence with $\phi(d'_n) \ge \frac{1}{2}(\gamma - \beta)$ for every $n \in \mathbb{N}$; contradicting the hypothesis that ϕ is exhaustive. **XQ**

This proves (i).

(c) Now consider (ii), assuming that $\mathfrak{A}$ is Dedekind σ-complete. By repeated applications of (b), we may find sequences $\langle b_n \rangle_{n\in\mathbb{N}}$, $\langle a_n \rangle_{n\in\mathbb{N}}$ such that

$$b = b_0 \subseteq b_n \subseteq b_{n+1} \subseteq a_{n+1} \subseteq a_n \subseteq a_0 = a \ \ \forall\ \ n \in \mathbb{N},$$

$$\alpha - 2^{-n} \le \phi(b_{n+1}) \le \alpha \le \phi(a_{n+1}) \le \alpha + 2^{-n} \ \ \forall\ \ n \in \mathbb{N}.$$

Now set $c = \sup_{n\in\mathbb{N}} b_n$ and see that $\phi(c)$ must be exactly α.

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3. Proposition If ϕ is a normalized submeasure on a Boolean algebra $\mathfrak{A}$, then there is a submeasure $\phi' \le \phi$ such that $\phi'(a) = \phi(a)$ whenever $\phi'(a) < \frac{1}{2}$ and $\phi'(a) + \phi'(\mathbf{1} \setminus a) = 1$ for every $a \in \mathfrak{A}$.

proof Consider the set Ψ of submeasures ψ on $\mathfrak{A}$ such that $\psi \le \phi$, $\psi(a) = \phi(a)$ whenever $\psi(a) < \frac{1}{2}$ and $\psi(\mathbf{1}) = 1$. By Zorn's Lemma there is a $\phi' \in \Psi$ which is minimal in Ψ.

If $a \in \mathfrak{A}$ then $\phi'(a) + \phi'(\mathbf{1} \setminus a) = 1$. **P** Take α, β such that $\min(\frac{1}{2}, \phi'(a)) \le \alpha \le \phi'(a)$, $\min(\frac{1}{2}, \phi'(\mathbf{1} \setminus a)) \le \beta \le \phi'(\mathbf{1} \setminus a)$ and $\alpha + \beta = 1$. Define $\psi_0 : \mathfrak{A} \to \mathbb{R}$ by setting

$$\psi_0(b) = \min(\alpha, \phi'(b \cap a)) + \min(\beta, \phi'(b \setminus a))$$

for all $b \in \mathfrak{A}$; then ψ_0 is a submeasure. Define $\psi : \mathfrak{A} \to \mathbb{R}$ by setting $\psi(b) = \min(\psi_0(b), \phi'(b))$ for all $b \in \mathfrak{A}$; I claim that ψ is also a submeasure. Of course $\psi(\mathbf{0}) = 0$ and $\psi(b) \le \psi(b')$ whenever $b \subseteq b'$. Now suppose that b, $c \in \mathfrak{A}$. (i) If $\psi(b) = \psi_0(b)$ and $\psi(c) = \psi_0(c)$ then $\psi(b \cup c) \le \psi_0(b \cup c) \le \psi_0(b) + \psi_0(c) = \psi(b) + \psi(c)$. (ii) If $\psi(b) = \psi_0(b) < \phi'(b)$ and $\psi(c) = \phi'(c)$ we must consider various cases. (α) Perhaps $\alpha < \phi'(b \cap a)$. Because $\psi_0(b) < 1$, we must have $\psi_0(b) = \alpha + \phi'(b \setminus a)$. Now $\psi(b \cup c) \le \psi_0(b \cup c) \le \alpha + \phi'((b \cup c) \setminus a) \le \alpha + \phi'(b \setminus a) + \phi'(c) = \psi(b) + \psi(c)$. ($\beta$) Similarly $\psi(b \cup c) \le \psi(b) + \psi(c)$ if $\beta < \phi'(b \setminus a)$. ($\gamma$) Otherwise, $\psi(b) = \phi'(b \cap a) + \phi'(b \setminus a)$ so $\psi(b \cup c) \le \phi'(b \cup c) \le \phi'(b \cap a) + \phi'(b \setminus a) + \phi'(c) = \psi(b) + \psi(c)$. This deals with case (ii). (iii) Now the cases $\psi(b) = \phi'(b)$, $\psi(c) = \psi_0(c) < \phi'(c)$ and $\psi(b) = \phi'(b)$, $\psi(c) = \phi'(c)$ give no further difficulty.

So ψ is a submeasure. Of course $\psi \le \phi$. Also $\psi(\mathbf{1}) = 1$, because $\alpha \le \phi'(a)$ and $\beta \le \phi'(1 \setminus a)$. Now suppose that $\psi(b) < \frac{1}{2}$. I claim that $\psi(b) = \phi'(b) = \phi(b)$. For either $\psi_0(b) = \phi'(b \cap a) + \phi'(b \setminus a) \ge \phi'(b)$, so that $\psi(b) = \phi'(b)$, or $\psi_0(b) \ge \alpha \ge \frac{1}{2} > \psi(b)$, so $\psi(b) = \phi'(b)$, or $\psi_0(b) \ge \beta \ge \frac{1}{2} > \psi(b)$, so $\psi(b) = \phi'(b)$. Thus we have $\phi'(b) = \psi(b) < \frac{1}{2}$, and $\psi(b) = \phi'(b) = \phi(b)$, as claimed.

This shows that $\psi \in \Psi$. But by the minimality of ϕ', this means that $\psi = \phi'$. Consequently

$$1 = \alpha + \beta \ge \psi_0(a) + \psi_0(\mathbf{1} \setminus a) \ge \psi(a) + \psi(\mathbf{1} \setminus a) = \phi'(a) + \phi'(\mathbf{1} \setminus a). \ \mathbf{Q}$$

4. Corollary If ϕ is a submeasure on a Boolean algebra $\mathfrak{A}$, then ϕ is minimal iff $\phi(a) + \phi(\mathbf{1} \setminus a) = \phi(\mathbf{1})$ for every $a \in \mathfrak{A}$.

proof It is enough to consider the case in which $\phi(\mathbf{1}) = 1$. If ϕ is minimal, then take the ϕ' from Prop. 3; surely $\phi' \le \phi$ and $\phi'(\mathbf{1}) = 1 - \phi'(\mathbf{0}) = 1$, so $\phi' = \phi$ and we have the given condition on ϕ. On the other hand, if ϕ satisfies the additivity condition, and $\psi \le \phi$, $\psi(\mathbf{1}) = 1$ then we have

$$\phi(a) = 1 - \phi(\mathbf{1} \setminus a) \le 1 - \psi(\mathbf{1} \setminus a) \le \psi(a) \le \phi(a)$$

for every $a \in \mathfrak{A}$, so $\psi = \phi$; as ψ is arbitrary, ϕ is minimal.

5. Definition Let ϕ be a submeasure on a Boolean algebra $\mathfrak{A}$. Define $U\phi : \mathfrak{A} \to \mathbb{R}$ by writing

$$(U\phi)(a) = \inf_{n \in \mathbb{N}} \sup\{\min_{i \le n} \phi(a_i) : a_0, \dots, a_n \text{ are disjoint, } a_i \subseteq a \ \forall \ i \le n\}.$$

6. Proposition For any submeasure ϕ, $U\phi$ is also a submeasure, and $U\phi \le \phi$.

proof Evidently $U\phi \le \phi$, and, in particular, $(U\phi)(\mathbf{0}) = 0$; and also $(U\phi)(b) \le (U\phi)(b')$ whenever $b \subseteq b'$. Now take any a, $b \in \mathfrak{A}$ and $\eta > 0$. Then there are n_0, $n_1 \in \mathbb{N}$ such that whenever $\langle c_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}$, $\#(\{i : \phi(c_i \cap a) \ge (U\phi)(a) + \eta\}) \le n_0$ and $\#(\{i : \phi(c_i \cap b) \ge (U\phi)(b) + \eta\}) \le n_1$. Now evidently, for such families $\langle c_i\rangle_{i\in I}$,

$$\#(\{i : \phi(c_i \cap (a \cup b)) \ge (U\phi)(a) + (U\phi)(b) + 2\eta\}) \le n_0 + n_1,$$

which is finite. Accordingly $(U\phi)(a \cup b) \le (U\phi)(a) + (U\phi)(b) + 2\eta$. As a, b and η are arbitrary, $U\phi$ is a submeasure.

7. Theorem Let ϕ be a normalized pathological submeasure on a Boolean algebra $\mathfrak{A}$. Then for any $\epsilon > 0$ there is a finite family $\langle b_i\rangle_{i<m}$ in $\mathfrak{A}$ such that $\phi(b_i) \le \epsilon$ for every $i < m$ and $\inf_{i\in I}(\mathbf{1} \setminus b_i) = \mathbf{0}$ whenever $I \subseteq m$ and $\#(I) \ge \epsilon m$.

proof ? Suppose, if possible, otherwise. Set

$$C = \{\mathbf{1} \setminus b : \phi(b) \le \epsilon\}.$$

By Kelley's theorem (FREMLIN 89, 5.10), there is an additive functional $\mu : \mathfrak{A} \to \mathbb{R}$ such that $\mu(\mathbf{1}) = 1$ and $\mu(c) \ge \epsilon$ whenever $c \in C$.

Choose $\langle b_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. Given $\langle b_i\rangle_{i<n}$, set

$$\delta_n = \sup\{\mu(b) : b \cap b_i = \mathbf{0} \ \ \forall \ \ i < n,\ \phi(b) \le \epsilon\mu(b)\}.$$

Take b_n such that $b_n \cap b_i = \mathbf{0}$ for every $i < n$, $\phi(b_n) \le \epsilon\mu(b_n)$ and $\mu(b_n) \ge \frac{1}{2}\delta_n$. Note that $\langle b_n\rangle_{n\in\mathbb{N}}$ is disjoint; set $b'_n = \sup_{i<n} b_i$ for each $n \in \mathbb{N}$. Then $\phi(b'_n) \le \sum_{i<n}\phi(b_i) \le \epsilon\sum_{i<n}\mu(b_i) \le \epsilon$, so $\mu(\mathbf{1} \setminus b'_n) \ge \epsilon$ for every $n \in \mathbb{N}$.

Set $\nu(a) = \lim_{n\to\infty}\mu(a \setminus b'_n)$ for $a \in \mathfrak{A}$. Then ν is a finitely additive measure on $\mathfrak{A}$ and $\nu(\mathbf{1}) \ge \epsilon > 0$. Because ϕ is pathological, $\phi \not\ge \epsilon\nu$, and there is an $a \in \mathfrak{A}$ such that $\phi(a) < \epsilon\nu(a)$. Take any $n \in \mathbb{N}$ and set $a_n = a \setminus b'_n$. Then $\phi(a_n) \le \phi(a) < \epsilon\nu(a) \le \epsilon\mu(a \setminus b'_n) = \epsilon\mu(a_n)$, and $a_n \cap b_i = \mathbf{0}$ for $i < n$, so $\mu(a_n) \le \delta_n$ and $\mu(b_n) \ge \frac{1}{2}\mu(a_n) \ge \frac{1}{2}\nu(a) > 0$. But this means that $\sum_{n\in\mathbb{N}}\mu(b_n)$ must be infinite, which is impossible. **X**

Remark This result is due to CHRISTENSEN 78, Theorem 5.

8. Theorem Let ϕ be a pathological submeasure on a Boolean algebra $\mathfrak{A}$. Then $U\phi \ge \frac{1}{3}\phi$.

proof It is enough to show that if $\phi(\mathbf{1}) = 1$ then $(U\phi)(\mathbf{1}) \ge \frac{1}{3}$.

? Suppose, if possible, otherwise. Take γ such that $(U\phi)(\mathbf{1}) < \gamma < \frac{1}{3}$. Let n be such that there is no disjoint family $\langle a_i\rangle_{i\le n}$ in $\mathfrak{A}$ with $\phi(a_i) \ge \gamma$ for every $i \le n$. Then we see that

$$\textstyle\sum_{i<m}\phi(a_i) \le n + m\gamma$$

for every disjoint sequence $\langle a_i\rangle_{i<m}$ in $\mathfrak{A}$.

Set

$$\epsilon = \frac{1 - 3\gamma}{n + 3} > 0,\ \delta = \min(\frac{\epsilon^2}{18}, \frac{\epsilon}{n}) > 0.$$

By Christensen's theorem, there is a family $\langle b_i\rangle_{i<m}$ in $\mathfrak{A}$ such that $\phi(b_i) \le \delta$ for every $i < m$ and $\inf_{i\in I}(\mathbf{1} \setminus b_i) = \mathbf{0}$ whenever $I \subseteq m$ and $\#(I) \ge \delta m$. Note that we must have $\sup_{i\in I} b_i = \mathbf{1}$ so $m\delta \ge \sum_{i<m}\phi(b_i) \ge 1$ and $n \le \epsilon m$.

Set $l = \lceil \epsilon m \rceil + 1$, $k = \lceil \delta m \rceil$, writing $\lceil \alpha \rceil$ for the integral part of α. Then $9 \le 18km \le \epsilon^2 m^2 \le l^2$, so there is an $R \subseteq m \times l$ such that $\#(R) = 3m$ (in fact, $\#(R[\{i\}]) = 3$ for every $i < m$) and $\#(R[E]) \ge \#(E)$ for every $E \in [m]^{\le k}$ (Fremlin 89, 5.8). For $E \subseteq m$ set

$$c_E = \inf_{i \in E}(\mathbf{1} \setminus b_i) \cap \inf_{i \in m \setminus E} b_i;$$

observe that $c_E = \mathbf{0}$ when $\#(E) > k$, so that

$$\sup\{c_E : E \in [m]^{\le k}\} = \mathbf{1}.$$

For $E \in [m]^{\le k}$ take an injective function $f_E : E \to l$ such that $f_E \subseteq R$. Set

$$a_{ij} = \sup\{c_E : i \in E \in [m]^{\le k},\ f_E(i) = j\}$$

for $i < m$, $j < l$. Then, for any particular $j < l$, $\langle a_{ij} \rangle_{i<m}$ is disjoint (because every f_E is injective), so

$$\begin{aligned} \sum_{i<m} \phi(a_{ij}) &\le n + \gamma\#(\{i : a_{ij} \ne \mathbf{0}\}) \\ &\le n + \gamma\#(R^{-1}[\{j\}]). \end{aligned}$$

Accordingly

$$\begin{aligned} \sum_{i<m, j<l} \phi(a_{ij}) &\le nl + \gamma \sum_{j<l} \#(R^{-1}[\{j\}]) \\ &\le n(\epsilon m + 1) + \gamma\#(R) \\ &\le n\epsilon m + \epsilon m + 3\gamma m \\ &= m(3\gamma + (n+1)\epsilon). \end{aligned}$$

On the other hand, for each $i < m$,

$$\begin{aligned} \mathbf{1} \setminus b_i &= \sup\{c_E : i \in E \subseteq m\} \\ &= \sup\{c_E : i \in E \in [m]^{\le k}\} \\ &= \sup_{j<l} a_{ij}, \end{aligned}$$

so

$$1 = \phi(\mathbf{1}) \le \phi(b_i) + \phi(\mathbf{1} \setminus b_i) \le \delta + \textstyle\sum_{j<l} \phi(a_{ij}).$$

Now, summing over $i < m$,

$$\begin{aligned} m &\le m\delta + \sum_{i<m, j<l} \phi(a_{ij}) \\ &\le m(\delta + 3\gamma + (n+1)\epsilon) \\ &\le m(3\gamma + (n+2)\epsilon) \\ &< m(3\gamma + 1 - 3\gamma) = m, \end{aligned}$$

which is impossible. **X**

Remark Compare FREMLIN 89, 5.11; the ideas here are largely derived from KALTON & ROBERTS 83.

9. Problem Is $\frac{1}{3}$ the best possible constant in the above result?

From Proposition 3 we see that $(U\phi)(\mathbf{1}) \le \frac{1}{2}$ whenever ϕ is a minimal submeasure; since there do exist pathological minimal submeasures, we shall not be able to get a better general constant than $\frac{1}{2}$.

The constant $\frac{1}{3}$ is derived from the fact that $\#(R) = 3m$ in the proof above. To see that we cannot improve on the factor 3 there, we have the following.

10. Proposition Suppose that $\lambda < 3$. Take $\epsilon = (3-\lambda)/9 > 0$. Let $\delta > 0$. Then there is an m_0 such that whenever $m \ge m_0$, $l = \lceil \epsilon m \rceil + 1$, $k = \lceil \delta m \rceil$, $R \subseteq m \times l$ and $\#(R) \le \lambda m$, there is an $E \in [m]^{\le k}$ such that $\#(R[E]) < \#(E)$.

proof (a) Take m_0 so large that

$$2^{\delta m} > 2^5(m+1)^4,\ \delta m \ge 2,\ \epsilon m > 2$$

for every $m \ge m_0$. Take $m \ge m_0$, $l = \lceil \epsilon m \rceil + 1$, $k = \lceil \delta m \rceil$, $R \subseteq m \times l$ such that $\#(R) \le \lambda m$. If there is an $i < m$ such that $R[\{i\}] = \emptyset$, we may take $E = \{i\}$; so let us suppose henceforth that $R[\{i\}] \ne \emptyset$ for every $i < m$. Write $k' = \lceil k/4 \rceil$; then $2^{k'} - 1 > m$.

(b) Write $I_0 = \{i : i < m,\ \#(R[\{i\}]) \le 2\}$. Then $\#(R) \ge 3(m - \#(I_0))$, so $\#(I_0) \ge (3-\lambda)m/3 = 3\epsilon m > 2l$. For non-empty $I \subseteq I_0$ set $\theta(I) = \#(R[I])/\#(I)$, and let $I_1 \subseteq I_0$ be a non-empty set minimising $\theta(I_1)$. Of course $\theta(I_1) < \frac{1}{2}$.

(c) If $j \in R[I_1]$ then $\#(I_1 \cap R^{-1}[\{j\}]) \ge 3$. **P** Set $I = I_1 \setminus R^{-1}[\{j\}]$. Then $\theta(I) \ge \theta(I_1)$, i.e.,

$$\#(R[I])\#(I_1) \ge \#(I)\#(R[I_1]),$$

so that

$$(\#(R[I_1]) - 1)\#(I_1) \ge (\#(I_1) - \#(I_1 \cap R^{-1}[\{j\}])\#(R[I_1])$$

and

$$\#(I_1 \cap R^{-1}[\{j\}])\#(R[I_1]) \ge \#(I_1)$$

and

$$\#(I_1 \cap R^{-1}[\{j\}]) \ge (\theta(I_1))^{-1} > 2.\ \mathbf{Q}$$

(d) Consider the undirected multigraph G with vertex set $R[I_1]$ and edge set I_1, counting the endpoints of the edge $i \in I_1$ to be j and j' if $R[\{i\}]$ is a doubleton set $\{j, j'\}$, and counting i to be a loop starting and finishing at j if $R[\{i\}]$ is a singleton $\{j\}$. Then every vertex of G has valency at least 3.

(e) There is a cycle in G of length at most $2k'$ (we count a loop as a cycle of length 1, and a pair of vertices linked by a double edge as a cycle of length 2.) **P?** If not, take any vertex j_0 of G, and examine the vertices of G within distance k' of j_0 in G. Because the minimal valency of G is at least 3, these must form a tree in which every node divides, and it must have at least $3(2^{k'} - 1)$ vertices; but this is not possible, because $3(2^{k'} - 1) > l$. **XQ**

(f) So take such a cycle C; let I be the set of its edges and $R[I]$ the set of its vertices; then $\#(R[I]) = \#(I)$. Note that if $i \in I_1 \setminus I$ and $R[\{i\}] \subseteq R[I]$, we could take $E = I \cup \{i\}$. So let us suppose that this doesn't happen; then every vertex of C must have a neighbour in G which is not in C. Now consider the vertices of G which are within distance k' of the nearest vertex of C. There are strictly fewer than $\#(R[I])(2^{k'} - 1)$ of these. So again there must be a cycle present. This is represented *either* by a set $I' \subseteq I_0 \setminus I$ such that $\#(R[I']) = \#(I') \le 2k'$ and $R[I'] \cap R[I] \ne \emptyset$ *or* by a set $I' \subseteq I_0 \setminus I$ such that $\#(R[I']) = \#(I')+1 \le 2k'$ and $\#(R[I] \cap R[I']) = 2$. In either case we may take $E = I \cup I'$.

11. Proposition Let ϕ be a submeasure on a Boolean algebra $\mathfrak{A}$.

(a) ϕ is uniformly exhaustive iff $U\phi = 0$.

(b) If ϕ is exhaustive then $U\phi$ is pathological.

(c) If ϕ is exhaustive and $U^n\phi = 0$ for some $n \in \mathbb{N}$ then ϕ is uniformly exhaustive.

proof (a) Immediate from the definitions.

(b) ? Suppose, if possible, that there is an additive functional ψ such that $0 < \psi \le U\phi$. Set $\delta = \psi(\mathbf{1}) > 0$, and let $\langle \delta_k \rangle_{k\in\mathbb{N}}$ be a strictly decreasing sequence with $\delta_0 = \delta$, $\lim_{n\to\infty} \delta_n = \frac{1}{2}\delta$. Set $a_0 = \mathbf{1}$. Given $\delta_k \le \psi(a_k)$, take $n(= n(k))$ so large that $n(\delta_k - \delta_{k+1}) > 1$, and take a disjoint family $\langle b_i \rangle_{i<n}$ such that $b_i \subseteq a_k$, $\phi(b_i) \ge \frac{1}{2}\delta$ for every $i < n$; we can do this because $(U\phi)(a_k) \ge \psi(a_k) > \frac{1}{2}\delta$. There must be some $i < n$ such that $\psi(b_i) \le \delta_k - \delta_{k+1}$; set $a_{k+1} = a_k \setminus b_i$, so that $\psi(a_{k+1}) \ge \delta_{k+1}$ and the induction continues.

Now observe that $a_{k+1} \subseteq a_k$ for every $k \in \mathbb{N}$, so that $\langle a_k \setminus a_{k+1} \rangle_{k\in\mathbb{N}}$ is disjoint, and that $\phi(a_k \setminus a_{k+1}) \ge \frac{1}{2}\delta > 0$ for every $k \in \mathbb{N}$, so that ϕ is not exhaustive. **X**

(c) This is now immediate from (a), (b) and Theorem 8.

12. Proposition Let $\mathfrak{A}$ be an atomless countable Boolean algebra, not $\{\mathbf{0}\}$.

(a) Let ψ be any submeasure on $\mathfrak{A}$, $\alpha \ge \psi(\mathbf{1})$. Then there is a submeasure ϕ on $\mathfrak{A}$ such that $U\phi = \psi$ and $\phi(\mathbf{1}) = \alpha$.

(b) For any $\alpha \in [0, 1]$, there is a normalized submeasure ϕ on $\mathfrak{A}$ such that $U\phi = \alpha\phi$.

(c) Let ψ be any submeasure on $\mathfrak{A}$. Then there is a submeasure ϕ on $\mathfrak{A}$ such that $\phi(\mathbf{1}) = \psi(\mathbf{1})$ and $\psi = \lim_{n\to\infty} U^n\phi$.

proof We may suppose that $\mathfrak{A}$ is actually the algebra of open-and-closed subsets of $X = \prod_{n\in\mathbb{N}}(n + 2)$.

For $n \in \mathbb{N}$, $i \le n + 1$ write

$$e_{ni} = \{t : t \in X,\ t(n) = i\} \in \mathfrak{A}.$$

Let $\mathfrak{A}_n$ be the subalgebra of $\mathfrak{A}$ generated by $\{e_{mi} : m < n,\ i \le m + 1\}$, so that $\#(\mathfrak{A}_n) = 2^{(n+1)!}$ for each $n \in \mathbb{N}$; in particular, $\mathfrak{A}_0 = \{\emptyset, X\}$. For $n \in \mathbb{N}$, $i \le n + 1$, $a \in \mathfrak{A}_{n+1}$ let $\theta_{ni}(a)$ be that unique member of $\mathfrak{A}_n$ such that $a \cap e_{ni} = \theta_{ni}(a) \cap e_{ni}$. Then $\theta_{ni} : \mathfrak{A}_{n+1} \to \mathfrak{A}_n$ is a Boolean homomorphism. (Draw the picture!)

(a) Define $\langle \phi_n \rangle_{n\in\mathbb{N}}$ inductively, as follows. $\phi_0(\mathbf{0}) = 0$, $\phi_0(\mathbf{1}) = \alpha$. Given $\phi_n : \mathfrak{A}_n \to \mathbb{R}$, set

$$\phi_{n+1}(a) = \max(\psi(a), \phi_n(\theta_{n0}(a)), \max_{i\le n+1} \psi(\theta_{ni}(a)))$$

for every $a \in \mathfrak{A}_{n+1}$. An easy induction on n shows that ϕ_n is a submeasure on $\mathfrak{A}_n$, that $\phi_n \ge \psi \restriction \mathfrak{A}_n$ and that ϕ_{n+1} extends ϕ_n, for each $n \in \mathbb{N}$. Consequently $\phi = \bigcup_{n\in\mathbb{N}} \phi_n$ is a submeasure on $\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n = \mathfrak{A}$. Of course $\phi(X) = \phi_0(X) = \alpha$.

To see that $U\phi \ge \psi$, take $a \in \mathfrak{A}$. Let n be such that $a \in \mathfrak{A}_n$, and take any $m \ge n$. Then $\langle a \cap e_{mi}\rangle_{i\le m+1}$ is disjoint, and

$$\phi(a \cap e_{mi}) = \phi_{m+1}(a \cap e_{mi}) \ge \psi(\theta_{mi}(a \cap e_{mi})) = \psi(a) \ \forall\ i \le m+1.$$

As m is arbitrary, $U\phi(a) \ge \psi(a)$.

To see that $U\phi \le \psi$, take $a \in \mathfrak{A}$. Again suppose that n is such that $a \in \mathfrak{A}_n$. Now if $m \ge n$ and $\langle a_j\rangle_{j<k}$ is a disjoint family in $\mathfrak{A}_m$ with $a_j \subseteq a$ and $\phi(a_j) > \psi(a)$ for every $j < k$, then $k \le \#(\mathfrak{A}_n)$. **P** Induce on m. For $m = n$, this is trivial. For the inductive step to $m+1$, take a disjoint family $\langle a_j\rangle_{j<k}$ in $\mathfrak{A}_{m+1}$ with $\phi(a_j) > \psi(a)$ and $a_j \subseteq a$ for each $j < k$. Then we must have, for each $j < k$,

$$\begin{aligned}\psi(a) < \phi_{m+1}(a_j) &= \max(\psi(a_j), \phi(\theta_{m0}(a_j)), \max_{i\le m+1} \psi(\theta_{mi}(a_j))) \\ &\le \max(\psi(a), \phi(\theta_{m0}(a_j))).\end{aligned}$$

So if we write $a'_j = \theta_{m0}(a_j)$, we have $\phi(a'_j) > \psi(a)$. On the other hand, because $a_j \subseteq a \in \mathfrak{A}_n \subseteq \mathfrak{A}_m$, $a'_j \subseteq a$. Now $\langle a'_j\rangle_{j<k}$ is a disjoint sequence in $\mathfrak{A}_m$ with $a'_j \subseteq a$ and $\phi(a'_j) > \psi(a)$ for every $j < k$; by the inductive hypothesis, $k \le \#(\mathfrak{A}_n)$. **Q** It follows at once that if $a \in \mathfrak{A}_n$ and $\langle a_j\rangle_{j<k}$ is a disjoint sequence in $\mathfrak{A}$ with $a_j \subseteq a$ and $\phi(a_j) > \psi(a)$ for each $j < k$, then $k \le \#(\mathfrak{A}_n) = 2^{(n+1)!}$. This shows that $U\phi(a) \le \psi(a)$.

(b) This time start the induction by setting $\phi_0(X) = 1$, $\phi_0(\emptyset) = 0$. For the inductive step write

$$\phi_{n+1}(a) = \max(\phi_n(\theta_{n0}(a)), \alpha \max\nolimits_{i\le n+1} \phi_n(\theta_{ni}(a)))$$

for $a \in \mathfrak{A}_{n+1}$. The rest of the argument proceeds as in (a), writing $\alpha\phi$ in place of ψ whenever required.

(c) This time we seek to define a double sequence $\langle \phi_{rn}\rangle_{r,n\in\mathbb{N}}$ of submeasures, as follows. $\phi_{rn} = \psi \restriction \mathfrak{A}_n$ whenever $r \ge n$. For n greater than r, we use the formula

$$\phi_{r,n+1}(a) = \max(\phi_{r+1,n+1}(a), \phi_{rn}(\theta_{n0}(a)), \max\nolimits_{i\le n+1} \phi_{r+1,n+1}(\theta_{ni}(a)))$$

to define $\phi_{r,n+1}$ from ϕ_{rn} and $\phi_{r+1,n+1}$; for a fixed valued of n this is a downwards induction on r from $r = n+1$ to $r = 0$. This time the arguments of part (a) show that $\phi_r = \bigcup_{n\in\mathbb{N}} \phi_{rn}$ is a submeasure on $\mathfrak{A}$, with $\phi_r(X) = \psi(X)$, and that $U\phi_r = \phi_{r+1} \ge \psi$ for each $r \in \mathbb{N}$. So if we take $\phi = \phi_0$ we get $U^r\phi = \phi_r$ for each r. Moreover, because $\phi_r \restriction \mathfrak{A}_r = \phi_{rr} = \psi \restriction \mathfrak{A}_r$ for each r, $\lim_{r\to\infty} \phi_r = \psi$.

Remark In 12b we have an alternative description of ϕ, as follows. Set

$$Z = \{x : x \in X,\ \{n : x(n) \ne 0\} \text{ is finite }\}.$$

For $x \in Z$ set $k(x) = \#(\{n : x(n) \ne 0\})$. Then $\phi(a) = \sup_{x\in a\cap Z} \alpha^{k(x)}$ for every $a \in \mathfrak{A}$. (Prove this for $a \in \mathfrak{A}_n$ by induction on n.) If $\alpha > 0$, ϕ cannot be exhaustive; for setting

$$a_n = \{x : x(m) = 0 \ \forall\ m < n,\ x(n) \ne 0\},$$

we have $\phi(a_n)=\alpha$ for every $n\in\mathbb{N}$, while $\langle a_n\rangle_{n\in\mathbb{N}}$ is disjoint.

13. Definition Let $\mathfrak{A}$ be a Boolean algebra, and ϕ and ψ two submeasures on $\mathfrak{A}$. Set

$$\begin{aligned}(\phi\wedge\psi)(a)&=\inf\{\phi(b)+\psi(c):b,\,c\in\mathfrak{A},\,b\cup c\supseteq a\}\\&=\inf\{\phi(b)+\psi(a\setminus b):b\subseteq a\}\end{aligned}$$

for every $a\in\mathfrak{A}$. It is easy to see that $\phi\wedge\psi$ is a submeasure on $\mathfrak{A}$, the largest submeasure on $\mathfrak{A}$ less than or equal to both ϕ and ψ.

14. Example There are $\mathfrak{A}$, $\mathfrak{A}_0$, ϕ, ψ, ψ^* such that
$\mathfrak{A}$ is a Boolean algebra and $\mathfrak{A}_0$ is a finite subalgebra of $\mathfrak{A}$;
ϕ and ψ are submeasures on $\mathfrak{A}$, with $\psi\le U\phi$;
ψ^* is the submeasure defined on $\mathfrak{A}$ by writing

$$\psi^*(a)=\min\{\psi(a_0):a\subseteq a_0\in\mathfrak{A}_0\}\ \ \forall\ a\in\mathfrak{A};$$

$\psi\not\le U(\phi\wedge\psi^*)$;
there is no submeasure ϕ' on $\mathfrak{A}$ such that $\phi'\le\phi$ and $U\phi'=\psi$.
Moreover, $\mathfrak{A}$ may be taken to be either countable or Dedekind complete, according to taste.

proof (a) For a non-empty set C, any set A, write

$$\begin{aligned}\theta_C(A)&=0\text{ if }A\cap C=\emptyset\\&=\frac{1}{5}\text{ otherwise;}\\\theta'_C(A)&=0\text{ if }A\cap C=\emptyset\\&=\frac{2}{5}\text{ if }C\subseteq A\\&=\frac{1}{5}\text{ otherwise;}\\\theta''_C(A)&=\theta_C(A)+\theta'_C(A).\end{aligned}$$

Then, for any set X, $\theta_C\restriction\mathcal{P}X$, $\theta'_C\restriction\mathcal{P}X$ and $\theta''_C\restriction\mathcal{P}X$ are submeasures on $\mathcal{P}X$.

(b) Set $X=\{0,1\}\times\prod_{n\in\mathbb{N}}(n+1)\times\prod_{n\in\mathbb{N}}(n+1)$; express members of X as $x=(\epsilon_x,x',x'')$ where $\epsilon_x\in\{0,1\}$, $x'\in\prod_{n\in\mathbb{N}}(n+1)$, $x''\in\prod_{n\in\mathbb{N}}(n+1)$. Write

$$\begin{aligned}b&=\{x:x\in X,\,\epsilon_x=1\},\\b'&=X\setminus b;\\c_{ni}&=\{x:x\in X,\,x'(n)=i\},\\d_{ni}&=c_{ni}\cap b,\\e_{ni}&=c_{ni}\cap b',\\f_{ni}&=\{x:x\in b',\,x''(n)=i\}\end{aligned}$$

for $i\le n\in\mathbb{N}$.

(c) Take $\mathfrak{A}$ to be $\mathcal{P}X$ (if you want a Dedekind complete $\mathfrak{A}$) or the algebra of open-and-closed subsets of X (if you want a countable $\mathfrak{A}$); take $\mathfrak{A}_0$ to be $\{\emptyset, b, b', X\}$. For $a \in \mathfrak{A}$ set

$$\phi(a) = \max(\sup_{i\le n}(\theta''_{d_{ni}}(a) + \theta'_{e_{ni}}(a)), \sup_{i\le n}\theta''_{f_{ni}}(a)).$$

Then ϕ is a submeasure on $\mathfrak{A}$. Set

$$\psi_0 = (2\theta_b + 3\theta_{b'}) \restriction \mathfrak{A},$$

$$\psi = \psi_0 \wedge U\phi,$$

defining $\wedge$ as in §13; then ψ_0 and ψ are submeasures on $\mathfrak{A}$ and $\psi \le U\phi$.

(d) To compute the values of these submeasures, it is convenient to note first that the only values which can be taken by ϕ and ψ_0 are 0, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$ and 1; these are therefore also the only values which can be taken by $U\phi$,ψ_0, ψ, ψ^* and $\phi \wedge \psi^*$.

It is now straightforward to check that

$$\phi(X) = 1,\ \phi(b) = \phi(b') = \tfrac{3}{5},$$

$$\phi(c_{ni}) = 1,\ \phi(d_{ni}) = \phi(f_{ni}) = \tfrac{3}{5},\ \phi(e_{ni}) = \tfrac{2}{5}$$

for $i \le n \in \mathbb{N}$; consequently, because $\langle c_{ni}\rangle_{i\le n}$, $\langle d_{ni}\rangle_{i\le n}$ and $\langle f_{ni}\rangle_{i\le n}$ are disjoint for each $n \in \mathbb{N}$,

$$(U\phi)(X) = 1,\ (U\phi)(b) = (U\phi)(b') = \tfrac{3}{5}.$$

We also need to know a little about ψ. (i) Of course $\psi(b) \le \psi_0(b) = \frac{2}{5}$. (ii) Next, $\psi(X) = 1$. **P** Of course $\psi(X) \le \psi_0(X) = 1$. On the other hand, if $a \in \mathfrak{A}$, $a' = X \setminus a$ *either* $a = \emptyset$ and $\psi_0(a) + (U\phi)(a') = (U\phi)(X) = 1$ *or* $\emptyset \ne a \subseteq b$ and $\psi_0(a) + (U\phi)(a') \ge \frac{2}{5} + (U\phi)(b') = 1$ *or* $\emptyset \ne a \subseteq b'$ and $\psi_0(a) + (U\phi)(a') \ge \frac{3}{5} + (U\phi)(b) = \frac{6}{5}$ *or* $a \cap b \ne \emptyset$, $a \cap b' \ne \emptyset$ and $\psi_0(a) + (U\phi)(a') \ge \psi_0(a) = 1$. As a is arbitrary, $\psi(X) \ge 1$.**Q**

(e) We are now in a position to show that $U(\phi\wedge\psi^*)(X) \le \frac{4}{5}$. **P?** If not, there must be disjoint a, $a' \subseteq X$ such that $(\phi\wedge\psi^*)(a) = (\phi\wedge\psi^*)(a') = 1$. Because $1 = (\phi\wedge\psi^*)(a) \le \phi(a)$, there must be i, n such that $i \le n$ and $c_{ni} \subseteq a$. Because $1 = (\phi \wedge \psi^*)(a') \le \phi(a' \cap b') + \psi^*(b) \le \phi(a' \cap b') + \frac{2}{5}$, we have $\phi(a' \cap b') \ge \frac{3}{5}$ and there must be j, m such that $j \le m$ and $f_{mj} \subseteq a'$. But $c_{ni} \cap f_{mj} \ne \emptyset$, so this is impossible. **XQ**

But this means that $\psi \not\le U(\phi \wedge \psi^*)$, because we have seen that $\psi(X) = 1$.

(f) Now let ϕ' be any submeasure on $\mathfrak{A}$ with $\phi' \le \phi$ and $(U\phi')(\mathbf{1}) = 1$. Then $(U\phi')(b) = \frac{3}{5}$. **P** Of course $(U\phi')(b) \le \phi'(b) \le \phi(b) = \frac{3}{5}$. On the other hand, take any $\delta \in]0, \frac{1}{5}[$, $r \ge 1$. Then there is a disjoint sequence $\langle a_k\rangle_{k\le r}$ such that $\phi'(a_k) \ge 1 - \delta$ for each $k \le r$. Every $\phi(a_k)$ must be greater than $\frac{4}{5}$, so every a_k must include some c_{ni}; because the a_k are disjoint, no a_k can include any f_{mj}, and $\phi(a_k \cap b') \le \frac{2}{5}$, for every k. It follows that $\phi'(a_k \cap b) \ge 1 - \delta - \phi'(a_k \cap b') \ge \frac{3}{5} - \delta$ for every k. Because δ and r are arbitrary, $(U\phi')(b) \ge \frac{3}{5}$. **Q**

But this means of course that $U\phi' \ne \psi$, as claimed.

15. Proposition Let $\mathfrak{A}$ be a Boolean algebra, and ϕ a strictly positive exhaustive submeasure on $\mathfrak{A}$. Let $\mathfrak{A}^{\wedge}$ be the completion of $\mathfrak{A}$ associated with ϕ, as in Fremlin 89, Proposition 5.6b, and $\hat{\phi}$ the canonical extension of ϕ to $\mathfrak{A}^{\wedge}$. Then $U\phi = U\hat{\phi} \restriction \mathfrak{A}$.

proof Let $a \in \mathfrak{A}$ and set $\gamma = (U\hat{\phi})(a)$. It is easy to see that $(U\phi)(a) \le \gamma$, just because $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}^{\wedge}$ and $\hat{\phi}$ extends ϕ.

Now take any $\delta > 0$ and any $n \in \mathbb{N}$. Then we can find a disjoint sequence $\langle b_i \rangle_{i \le n}$ in $\mathfrak{A}^{\wedge}$ such that $b_i \subseteq a$ and $\hat{\phi}(b_i) \ge \gamma - \delta$ for every $i \le n$. For each $i \le n$ we may take $c_i \in \mathfrak{A}$ such that $\hat{\rho}(c_i, b_i) \le \delta$, where $\hat{\rho}$ is the metric on $\mathfrak{A}^{\wedge}$ extending the metric ρ on $\mathfrak{A}$ given by the formula

$$\rho(a_1, a_2) = \phi(a_1 \setminus a_2) + \phi(a_2 \setminus a_1) \ \ \forall \ a_1,\, a_2 \in \mathfrak{A}$$

(see Fremlin 89, 5.5b). Set

$$a_i = a \cap c_i \setminus \sup_{j<i} c_j$$

for $i \le n$; then $\langle a_i \rangle_{i \le n}$ is a disjoint sequence in $\mathfrak{A}$ and $a_i \subseteq a$ for each $i \le n$. Also

$$\begin{aligned}
\phi(a_i) &\ge \phi(c_i) - \phi(c_i \setminus a) - \sum_{j<i} \phi(c_i \cap c_j) \\
&\ge \hat{\phi}(b_i) - \hat{\phi}(b_i \setminus c_i) - \hat{\phi}(c_i \setminus b_i) - \sum_{j<i} (\hat{\phi}(c_i \setminus b_i) + \hat{\phi}(b_j \setminus c_j)) \\
&\ge \gamma - \delta - \hat{\rho}(b_i, c_i) - \sum_{j<i} (\hat{\rho}(b_i, c_i) + \hat{\rho}(b_j, c_j)) \\
&\ge \gamma - 2(n+1)\delta.
\end{aligned}$$

As n and δ are arbitrary, $(U\phi)(a) \ge \gamma$, which is what we needed to know.

16. Definitions Let ϕ be a submeasure on a Boolean algebra $\mathfrak{A}$.

(a) For $a \in \mathfrak{A}$ set

$$(E\phi)(a) = \sup\{\inf_{n\in\mathbb{N}} \phi(a_n) : \langle a_n \rangle_{n\in\mathbb{N}} \text{ is a disjoint sequence, } a_n \subseteq a \ \ \forall \ \ n \in \mathbb{N}\}.$$

(b) For $\epsilon > 0$, ordinals ξ define $Z_\xi(\epsilon, \phi)$ inductively by setting

$$Z_\xi(\epsilon, \phi) = \{a : a \in \mathfrak{A},\ b \in \textstyle\bigcup_{\eta<\xi} Z_\eta(\epsilon, \phi) \text{ whenever } b \subseteq a \text{ and } \phi(b \setminus a) > \epsilon\}.$$

Thus $Z_0(\epsilon, \phi) = \{a : \phi(a) \le \epsilon\}$, and $Z_\eta(\epsilon, \phi) \subseteq Z_\xi(\epsilon, \phi)$ whenever $\eta < \xi$.

(c) For $\epsilon > 0$, $a \in \mathfrak{A}$ set

$$\zeta_{\epsilon,\phi}(a) = \min\{\xi : a \in Z_\xi(\epsilon, \phi)\}$$

if this is defined, ∞ if $a \notin Z_\xi(\epsilon, \phi)$ for every ordinal ξ. Observe that if $b \subseteq a$ and $\phi(a \setminus b) > \epsilon$ and $\zeta_{\epsilon,\phi}(a) \ne \infty$, then $\zeta_{\epsilon,\phi}(b) < \zeta_{\epsilon,\phi}(a)$.

17. Proposition For any submeasure ϕ, $E\phi$ is also a submeasure, and $E\phi \le U\phi$.

proof Evidently $E\phi \le U\phi$, and, in particular, $(E\phi)(\mathbf{0}) = 0$; and also $(E\phi)(b) \le (E\phi)(b')$ whenever $b \subseteq b'$. Now take any a, $b \in \mathfrak{A}$ and $\delta > 0$. Let $\langle c_n \rangle_{n\in\mathbb{N}}$ be a disjoint sequence with $c_n \subseteq a \cup b$ for every $n \in \mathbb{N}$. Then $\{n : \phi(c_n \cap a) > (E\phi)(a) + \delta\}$, $\{n : \phi(c_n \cap b) > (E\phi)(b) + \delta\}$

are both finite; so there must be an n such that $\phi(c_n) \le (E\phi)(a)+(E\phi)(b)+2\delta$. As $\langle c_n\rangle_{n\in\mathbb{N}}$ and δ are arbitrary, $(E\phi)(a\cup b) \le (E\phi)(a) + (E\phi)(b)$.

18. Proposition If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra and ϕ is a submeasure on $\mathfrak{A}$, then $E^2\phi = E\phi$.

proof Of course $E^2\phi \le E\phi$. Now suppose that $a \in \mathfrak{A}$ and that $\delta > 0$. Then there is a disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ such that $a_n \subseteq a$ and $\phi(a_n) \ge (E\phi)(a) - \delta$ for every $n \in \mathbb{N}$. Let $\langle I(n)\rangle_{n\in\mathbb{N}}$ be a disjoint sequence of infinite subsets of $\mathbb{N}$, and set $b_n = \sup_{i\in I(n)} a_i$ for each $n \in \mathbb{N}$. Then $(E\phi)(b_n) \ge \inf_{i\in I(n)} \phi(a_i) \ge (E\phi)(a) - \delta$ for every n, so $(E^2\phi)(a) \ge \inf_{n\in\mathbb{N}}(E\phi)(b_n) \ge (E\phi)(a) - \delta$. As a and δ are arbitrary, $E^2\phi = E\phi$.

19. Proposition Let ϕ be a submeasure on a Boolean algebra $\mathfrak{A}$, $a \in \mathfrak{A}$, $\delta > 0$. Then the following are equivalent:

(i) $(E\phi)(a) < \delta$;

(ii) there is an $\epsilon < \delta$ such that $\zeta_{\epsilon\phi}(a) \ne \infty$;

(iii) there are an $\epsilon < \delta$ and a function $\zeta^* : \mathfrak{A}_a \to \mathrm{On}$, where On is the class of ordinals and $\mathfrak{A}_a$ is the principal ideal of $\mathfrak{A}$ generated by a, such that $\zeta^*(b') < \zeta^*(b)$ whenever $b' \subseteq b \subseteq a$ and $\phi(b \setminus b') > \epsilon$.

proof (i)⇒(ii) Take ϵ with $(E\phi)(a) < \epsilon < \delta$. **?** Suppose, if possible, that $\zeta_{\epsilon,\phi}(a) = \infty$. Let η be an ordinal such that $Z_\eta(\epsilon,\phi) = Z_{\eta+1}(\epsilon,\phi)$. Set $a_0 = a$. Given a_n such that $\zeta_{\epsilon,\phi}(a_n) = \infty$, we have $a_n \notin Z_{\eta+1}(\epsilon,\phi)$, so there is an $a_{n+1} \subseteq a_n$ such that $\phi(a_n \setminus a_{n+1}) > \epsilon$ and $a_{n+1} \notin Z_\eta(\epsilon,\phi)$, i.e., $\zeta_{\epsilon,\phi}(a_{n+1}) = \infty$. In this way we can construct a decreasing sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ such that $\phi(a_n \setminus a_{n+1}) > \epsilon$ for every $n \in \mathbb{N}$, and $(E\phi)(a) \ge \epsilon$, which is impossible. **X** So this ϵ witnesses (ii).

(ii)⇒(iii) Trivial; take $\zeta^* = \zeta_{\epsilon,\phi} \restriction \mathfrak{A}_a$.

(iii)⇒(i) ? Suppose, if possible, that $\epsilon < (E\phi)(a)$. Then there is a disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ such that $a_n \subseteq a$ and $\phi(a_n) > \epsilon$ for every $n \in \mathbb{N}$. Set $b_n = a \setminus \sup_{i<n} a_i$ for each $n \in \mathbb{N}$. Then $\phi(b_n \setminus b_{n+1}) > \epsilon$ for every $n \in \mathbb{N}$, so $\zeta^*(b_{n+1}) < \zeta^*(b_n)$ for every $n \in \mathbb{N}$, and we have an infinite strictly decreasing sequence of ordinals; which is impossible. **X** So $(E\phi)(a) \le \epsilon < \delta$.

20. Proposition Let $\mathfrak{A}$ be a Boolean algebra, ϕ a submeasure on $\mathfrak{A}$, $a \in \mathfrak{A}$ and $\delta > 0$. Then the following are equivalent:

(i) $(U\phi)(a) < \delta$;

(ii) there is an $\epsilon < \delta$ such that $\zeta_{\epsilon\phi}(a) < \omega$;

(iii) there are an $\epsilon < \delta$ and a function $\zeta^* : \mathfrak{A}_a \to \mathbb{N}$ such that $\zeta^*(b') < \zeta^*(b)$ whenever $b' \subseteq b \subseteq a$ and $\phi(b \setminus b') > \epsilon$.

proof (i)⇒(ii) Take ϵ with $(U\phi)(a) < \epsilon < \delta$. Then there is an $n \in \mathbb{N}$ such that $\inf_{i\le n} \phi(a_i) < \epsilon$ whenever $\langle a_i\rangle_{i\le n}$ is disjoint and $a_i \subseteq a$ for every $i \le n$. But it follows at once (the argument is really an induction on n) that $\zeta_{\epsilon\phi}(a) \le n$.

(ii)⇒(iii) Trivial.

(iii)⇒(i) Again, setting $n = \zeta^*(a)$, we find that if $\langle a_i\rangle_{i\le n}$ is a disjoint finite sequence with $a_i \subseteq a$ for each $i \le n$, then $\inf_{i\le n} \phi(a_i) < \epsilon$; so $(U\phi)(a) \le \epsilon$.

21. Lemma Let ϕ be a submeasure on a Boolean algebra $\mathfrak{A}$, and $\epsilon > 0$.

(a) For any a, $b \in \mathfrak{A}$ with $a \cap b = \mathbf{0}$, $\zeta_{\epsilon\phi}(a \cup b) \geq \zeta_{\epsilon\phi}(a) + \zeta_{\epsilon\phi}(b)$.

(b) If $a \in \mathfrak{A}$ and $\zeta_{\epsilon\phi}(a) = \xi + n$ for some $n \in \mathbb{N}$, ordinal ξ, then there is a $b \subseteq a$ such that $\zeta_{\epsilon\phi}(b) = \xi$, $\zeta_{\epsilon\phi}(a \setminus b) = n$.

proof Write ζ for $\zeta_{\epsilon\phi}$.

(a) Induce on $\eta = \zeta(b)$. If $\zeta(b) = 0$ this is elementary (strictly speaking, by induction on $\zeta(a)$). For the inductive step to $\zeta(b) = \eta + 1$, there must be a $c \subseteq b$ such that $\phi(c) > \epsilon$, $\zeta(b \setminus c) = \eta$; now, by the inductive hypothesis, $\zeta((a \cup b) \setminus c) = \zeta(a \cup (b \setminus c)) \geq \zeta(a) + \eta$, so $\zeta(a \cup b) \geq (\zeta(a) + \eta) + 1 = \zeta(a) + \zeta(b)$. For the inductive step to a limit ordinal $\eta > 0$, we see that for every $\eta' < \eta$ there is a $b' \subseteq b$ such that $\zeta(b') = \eta'$, so that $\zeta(a \cup b) \geq \zeta(a) + \zeta(b') = \zeta(a) + \eta'$; as η' is arbitrary, $\zeta(a \cup b) \geq \zeta(a) + \zeta(b)$, and the induction proceeds.

(b) Induce on n. If $n = 0$ this is trivial. For the inductive step to $n + 1$, there must be a $c \subseteq a$ such that $\phi(c) > \epsilon$, $\zeta(a \setminus c) = \xi + n$. By the inductive hypothesis, there is a $b \subseteq a \setminus c$ such that $\zeta(b) = \xi$ and $\zeta(a \setminus (b \cup c)) = n$. Now $\zeta(a \setminus b) \geq \zeta(a \setminus (b \cup c)) + 1 \geq n + 1$; but also, by part (a), $\xi + n + 1 = \zeta(a) \geq \zeta(b) + \zeta(a \setminus b) = \xi + \zeta(a \setminus b)$, so $\zeta(a \setminus b) \leq n + 1$. Thus the induction proceeds.

22. Proposition Let $\mathfrak{A}$ be a Boolean algebra, ϕ a submeasure on $\mathfrak{A}$ and $a \in \mathfrak{A}$.

(a) If $0 < \epsilon < (U\phi)(a)$ then $\zeta_{\epsilon\phi}(a) \geq \omega$.

(b) If $0 < \epsilon < \frac{1}{2}(U\phi)(a)$ then $\zeta_{\epsilon\phi}(a) \geq \omega^2$ (ordinal power).

(c) If $0 < \epsilon < \frac{1}{4}(U\phi)(a)$ then $\zeta_{\epsilon\phi}(a) \geq \omega^2 + \omega$.

proof (a) Immediate from 20(i).

(b) It is enough to consider the case $a = \mathbf{1}$. Write ζ for $\zeta_{\epsilon\phi}$.

? Suppose, if possible, that $\zeta(\mathbf{1}) < \omega^2$; then there is an $n < \omega$ such that $\zeta(\mathbf{1}) < \omega.n$. We find that for each $k \leq n$ there is a disjoint family $\langle a_{ki} \rangle_{i<2^k}$ such that $\zeta(\mathbf{1} \setminus a_{ki}) < \omega.k$ for every $i < 2^k$. **P** Use downwards induction from $k = n$ to $k = 0$. Start the induction by setting $a_{ni} = \mathbf{0}$ for every $i < 2^n$. Given $\langle a_{ki} \rangle_{i<2^k}$, where $k \geq 1$, let $m_k < \omega$ be such that $\zeta(\mathbf{1} \setminus a_{ki}) < \omega.(k-1) + m_k$ for every $i < 2^k$. Because $(U\phi)(\mathbf{1}) > 2\epsilon$ we can find a disjoint set $E \subseteq \mathfrak{A}$ such that $\#(E) \geq 2^k m_k$ and $\phi(e) > 2\epsilon$ for every $e \in E$. Set

$$b_0 = \sup_{i<2^{k-1}} a_{ki},$$

$$b_1 = \sup_{2^{k-1} \leq i < 2^k} a_{ki}.$$

Then $b_0 \cap b_1 = \mathbf{0}$ so for each $e \in E$ at least one of $\phi(e \setminus b_0)$, $\phi(e \setminus b_1)$ is greater than ϵ, and there is a $j \in \{0, 1\}$ such that

$$E_1 = \{e : e \in E,\ \phi(e \setminus b_j) > \epsilon\}$$

has cardinal at least $2^{k-1} m_k$. Re-labelling the a_{ki} if necessary, we may suppose that $j = 0$. Let $\langle J_i \rangle_{i<2^{k-1}}$ be a disjoint family of subsets of E_1 with $\#(J_i) = m_k$ for each $i < 2^{k-1}$. Set $a_{k-1,i} = a_{ki} \cup (\sup J_i \setminus b_0)$ for each $i < 2^{k-1}$. Then $\langle a_{k-1,i} \rangle_{i<2^{k-1}}$ is disjoint because $\langle a_{ki} \rangle_{i<2^{k-1}}$ is disjoint, $\langle \sup J_i \rangle_{i<2^{k-1}}$ is disjoint, and $a_{ki} \subseteq b_0$ for every $i < 2^{k-1}$. Next, given $i < 2^{k-1}$,

$$\zeta(\mathbf{1}\setminus a_{k-1,i})+m_k\le\zeta(\mathbf{1}\setminus a_{k-1,i})+\sum_{e\in J_i}\zeta(e\setminus b_0)\le\zeta(\mathbf{1}\setminus a_{ki})<\omega.(k-1)+m_k,$$

so $\zeta(\mathbf{1}\setminus a_{k-1,i})<\omega.(k-1)$, and the induction continues. **Q**

But now when we reach $k=0$ we have $\zeta(\mathbf{1}\setminus a_{00})<\omega.0=0$, which is absurd. **X**

So $\zeta(\mathbf{1})\ge\omega^2$, as claimed.

(c) Again, it is enough to consider the case $a=\mathbf{1}$; again, write ζ for $\zeta_{\epsilon\phi}$.

(i) Suppose that there is no a such that $\epsilon<(U\phi)(a)\le 2\epsilon$. In this case,

$$\mathcal{I}=\{a:(U\phi)(a)\le\epsilon\}$$

is an ideal in $\mathfrak{A}$ (compare Proposition 2), so there is a maximal ideal $\mathcal{J}\supseteq\mathcal{I}$. Now we may choose inductively a decreasing sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}\setminus\mathcal{J}$, as follows. $a_0=\mathbf{1}$. Given a_n, then $(U\phi)(a_n)>2\epsilon$, so there is an $a'\subseteq a_n$ such that both $\phi(a')$ and $\phi(a\setminus a')$ are greater than 2ϵ; take a_{n+1} to be either a' or $a_n\setminus a'$, whichever does not belong to $\mathcal{J}$. Continue. Now observe that $\langle a_n\setminus a_{n+1}\rangle_{n\in\mathbb{N}}$ is disjoint and witnesses that $\zeta_{2\epsilon,\phi}(\mathbf{1})=\infty$, so surely $\zeta(\mathbf{1})\ge\omega^2+\omega$.

(ii) Now suppose that there is an a such that $\epsilon<(U\phi)(a)\le 2\epsilon$. In this case $(U\phi)(\mathbf{1}\setminus a)>2\epsilon$. By (a), $\zeta(a)\ge\omega$; by (b), $\zeta(\mathbf{1}\setminus a)\ge\omega^2$; by Lemma 21a, $\zeta(\mathbf{1})\ge\omega^2+\omega$.

23. Proposition Let $\mathfrak{A}$ be a Boolean algebra and ϕ a submeasure on $\mathfrak{A}$. Then $\zeta_{\epsilon\phi}(a)\ge\omega.\zeta_{\epsilon,U\phi}(a)$ for every $a\in\mathfrak{A}$, $\epsilon>0$.

proof Induce on $\xi=\zeta_{\epsilon,U\phi}(a)$. For $\xi=0$, trivial. For the inductive step to $\xi+1$, take $b\subseteq a$ such that $\zeta_{\epsilon,U\phi}(b)=\xi$, $(U\phi)(a\setminus b)>\epsilon$. By Proposition 20, $\zeta_{\epsilon,\phi}(a\setminus b)\ge\omega$; by the inductive hypothesis, $\zeta_{\epsilon\phi}(b)\ge\omega.\xi$; by Lemma 21a, $\zeta_{\epsilon\phi}(a)\ge\omega.\xi+\omega=\omega.(\xi+1)$, as required. The inductive step to limit $\xi>0$ is now elementary.

24. Proposition There is a Boolean algebra $\mathfrak{A}$ with a submeasure ϕ on $\mathfrak{A}$, taking only the values 0, $\frac{1}{3}$, $\frac{2}{3}$ and 1, such that $(U\phi)(\mathbf{1})=\phi(\mathbf{1})=1$, $(E\phi)(\mathbf{1})=\frac{1}{3}$.

proof Take $\mathfrak{A}$, $\langle e_{ni}\rangle_{i\le n+1}$, $\langle\mathfrak{A}_n\rangle_{n\in\mathbb{N}}$ as in the proof of Proposition 12. For each $n\in\mathbb{N}$ set $E_n=\{e_{ni}:1\le i\le n+1\}$, $F_n=\bigcup_{m<n}E_m$; write $E=\bigcup_{n\in\mathbb{N}}E_n$.

(a) Our aim is to build a family $\langle Z(e)\rangle_{e\in E}$ of subsets of $\mathfrak{A}$ and a function $\zeta:\mathfrak{A}\to\omega^2+1$ such that

(i) for each $e\in E$, $a\in\mathfrak{A}$,

$$a\in Z(e)\iff a\cap e\in Z(e)\iff \mathbf{1}\setminus a\notin Z(e)\Longrightarrow a'\in Z(e)\ \forall\ a'\supseteq a;$$

(ii) if $e\in E$, $a\in Z(e)$, $b\cap a=\mathbf{0}$ then $\zeta(b)<\zeta(a\cup b)$.

In addition, the inductive procedure will depend on ensuring that

(iii) if a, $a'\in\mathfrak{A}$, $a\cup a'=\mathbf{1}$, $\zeta(a)<\omega k+p$, $\zeta(a')<\omega k+p'$ (where k, p, $p'\in\mathbb{N}$), then $k\ge 1$ and $\zeta(a\cap a')<\omega(k-1)+p+p'$.

(iv) $\zeta(\mathbf{1})=\omega^2$, $\zeta(a)<\omega^2$ for $a\in\mathfrak{A}\setminus\{\mathbf{1}\}$.

We will explain in (f) below how to obtain ϕ from $\langle Z(e)\rangle_{e\in E}$.

(b) We proceed by induction, defining $\zeta_n=\zeta\restriction\mathfrak{A}_n$ and $Z_n(e)=Z(e)\cap\mathfrak{A}_n$, for $e\in F_n$, for each n in turn. The inductive hypothesis will be

$(\mathrm{i})_n$ as (i), but only for $e\in F_n$, a, $a'\in\mathfrak{A}_n$;

$(\mathrm{ii})_n$ a strengthening of (ii): for $a\in\mathfrak{A}_n$, write $r_n(a)=\#(\{e:e\in F_n,\ a\in Z_n(e)\})$; then we require $\zeta_n(a)+r_n(b)\le\zeta_n(a\cup b)$ whenever a, $b\in\mathfrak{A}_n$ and $a\cap b=\mathbf{0}$;

(iii)$_n$ as (iii), but only for a, $a' \in \mathfrak{A}_n$;
(iv)$_n$ as (iv), but only for $a \in \mathfrak{A}_n$.
Note that (ii)$_n$ will ensure that ζ_n is monotonic non-decreasing.

(c) The induction starts with $\mathfrak{A}_0 = \{\mathbf{0}, \mathbf{1}\}$, $F_0 = \emptyset$, $\zeta_0(\mathbf{0}) = 0$, $\zeta_0(\mathbf{1}) = \omega^2$. For the inductive step to $n+1$, we must begin by defining the $Z_{n+1}(e)$, for $e \in F_{n+1}$. We do this as follows. Let W be the set

$$\{b \setminus b' : b,\, b' \in \mathfrak{A}_n,\, b' \subseteq b,\, \zeta_n(b) < \zeta_n(b') + \omega\}.$$

Then $v \cup w \neq \mathbf{1}$ for all v, $w \in W$. **P?** If we have b, b', c, $c' \in \mathfrak{A}_n$ such that $b' \subseteq b$, $c' \subseteq c$, $\zeta_n(b) < \zeta_n(b') + \omega$, $\zeta_n(c) < \zeta_n(c') + \omega$, $(b \setminus b') \cup (c \setminus c') = \mathbf{1}$ then $b \cup c = \mathbf{1}$ and $b' \cup c' \subseteq b \cap c$. Take k, $l \in \mathbb{N}$ such that

$$\omega k \le \zeta_n(b') \le \zeta_n(b) < \omega(k+1),\ \omega l \le \zeta_n(c') \le \zeta_n(c) < \omega(l+1).$$

Then

$$\omega k \le \zeta_n(b' \cup c') \le \zeta_n(b \cap c) < \omega(l+1),$$

so $k \le l$; similarly, $l \le k$, and $k = l$. By the hypothesis (iii)$_n$, $\zeta_n(b \cap c) < \omega k$; but of course this is impossible. **XQ**

We can therefore find a set $W' \supseteq W$ which is maximal subject to the requirements that $W' \subseteq \mathfrak{A}_n$ and $w \cup w' \neq \mathbf{1}$ for all w, $w' \in W'$. It is easy to check that for each $b \in \mathfrak{A}_n$ exactly one of b, $\mathbf{1} \setminus b$ belongs to W'. Let Z^* be $\mathfrak{A}_n \setminus W'$; then Z^* is a maximal linked set in $\mathfrak{A}_n$.

For $e \in E_n$ take

$$Z_{n+1}(e) = \{a : a \in \mathfrak{A}_{n+1},\ \exists\ b \in Z^* \text{ such that } a \supseteq b \cap e\}.$$

Next, for $e \in F_n$, take

$$Z_{n+1}(e) = \{a : a \in \mathfrak{A}_{n+1},\ \exists\ b \in Z_n(e) \text{ such that } a \supseteq b \cap e_{n0}\}.$$

It is easy to check that this assignment satisfies the inductive hypothesis (i)$_{n+1}$ and that $Z_n(e) = Z_{n+1}(e) \cap \mathfrak{A}_n$ for $e \in F_n$.

We are now in a position to examine $r_{n+1} : \mathfrak{A}_{n+1} \to \mathbb{N}$; observe that of course $r_{n+1}(b) \ge r_n(b)$ for every $b \in \mathfrak{A}_n$. In fact we find that if b_1, $b_2 \in \mathfrak{A}_n$ and $b_1 \cap b_2 = \mathbf{0}$, then $\zeta_n(b_1) + r_{n+1}(b_2) \le \zeta_n(b_1 \cup b_2)$. **P** By the inductive hypothesis (ii)$_n$, we know that this will be true if $r_{n+1}(b_2) = r_n(b_2)$. But the alternative is that $b_2 \in Z^*$, so that $b_2 \notin W'$, and we shall have $\zeta_n(b_1 \cup b_2) \ge \zeta_n(b_1) + \omega$; so that we shall be perfectly safe. **Q**

We shall need also to know that $r_{n+1}(a \cup a') \ge r_{n+1}(a) + r_{n+1}(a')$ whenever $a \cap a' = \mathbf{0}$; this is because every $Z_{n+1}(e)$ is linked, so that at most one of a, a' can belong to any $Z_{n+1}(e)$.

(d) Now we can proceed with the definition of ζ_{n+1}. Let $\langle c_i \rangle_{i \le M}$ be an enumeration of $\mathfrak{A}_{n+1}$ such that $i \le j$ whenever $c_i \subseteq c_j$. We seek to define each $\zeta_{n+1}(c_i)$ in turn, starting with $\zeta_{n+1}(c_0) = \zeta_{n+1}(\mathbf{0}) = 0$ and ending with $\zeta_{n+1}(c_M) = \zeta_{n+1}(\mathbf{1}) = \omega^2$. In order to do this we shall need to ensure, when we come to define c_j, that
(α) if $c_j \subseteq b \in \mathfrak{A}_n$ then $\zeta_{n+1}(c_j) + r_{n+1}(b \setminus c_j) \le \zeta_n(b)$;
(β) if $c_j \supseteq b \in \mathfrak{A}_n$ then $\zeta_n(b) + r_{n+1}(c_j \setminus b) \le \zeta_{n+1}(c_j)$;
(γ) if $i < j$ and $c_i \subseteq c_j$ then $\zeta_{n+1}(c_i) + r_{n+1}(c_j \setminus c_i) \le \zeta_{n+1}(c_j)$;

(δ) if $b \in \mathfrak{A}_n$, $b \cup c_j = \mathbf{1}$, $k \geq 1$, p, $p' \in \mathbb{N}$, $\zeta_n(b) < \omega k + p$, $\zeta_{n+1}(b \cap c_j) \geq \omega(k-1)+p+p'$ then $\zeta_{n+1}(c_j) \geq \omega k + p'$;

(ϵ) if $b \in \mathfrak{A}_n$, $b \cup c_j = \mathbf{1}$, $\zeta_n(b) < \omega$ then $\zeta_{n+1}(c_j) \geq \omega$;

(η) if $i < j$, $c_i \cup c_j = \mathbf{1}$, $k \geq 1$, p, $p' \in \mathbb{N}$, $\zeta_{n+1}(c_i) < \omega k + p$, $\zeta_{n+1}(c_i \cap c_j) \geq \omega(k-1) + p + p'$ then $\zeta_{n+1}(c_j) \geq \omega k + p'$;

(ζ) if $i < j$, $c_i \cup c_j = \mathbf{1}$, $\zeta_{n+1}(c_i) < \omega$ then $\zeta_{n+1}(c_j) \geq \omega$.

The point is that of these conditions only (α) gives an upper bound for $\zeta_{n+1}(c_j)$; all the others give lower bounds. Also, of course, there are, in effect, only finitely many conditions to be satisfied. So take $\zeta_{n+1}(c_j) = \xi$ to be the least ordinal compatible with all the conditions (β)-(ζ); now take a $b^* \in \mathfrak{A}_n$ such that $c_j \subseteq b^*$; we need to show that $\xi + r^* \leq \zeta_n(b^*)$, where $r^* = r_{n+1}(b^* \setminus c_j)$. We do this by checking through each of the possible things which ξ could be.

Case (β) It might be that $\xi = \zeta_n(b) + r_{n+1}(c_j \setminus b)$ where $b \in \mathfrak{A}_n$ and $b \subseteq c_j$. In this case we have $b \subseteq b^*$, so

$$\xi + r^* = \zeta_n(b) + r_{n+1}(c_j \setminus b) + r^* \leq \zeta_n(b) + r_{n+1}(b^* \setminus b) \leq \zeta_n(b^*),$$

by the last remark in (c) above.

Case (γ) It might be that $\xi = \zeta_{n+1}(c_i) + r_{n+1}(c_j \setminus c_i)$ where $i < j$ and $c_i \subseteq c_j$. In this case $\zeta_{n+1}(c_i)$ will have been chosen so that $\zeta_{n+1}(c_i) + r_{n+1}(b^* \setminus c_i) \leq \zeta_n(b^*)$, so we can use the same argument as in case (β), with c_i in place of b.

Case (δ) It might be that $\xi = \omega k + p'$ where we have $b \in \mathfrak{A}_n$, $k \geq 1$, p, $p' \in \mathbb{N}$, $b \cup c_j = \mathbf{1}$, $\zeta_n(b) < \omega k + p$, $\zeta_{n+1}(b \cap c_j) \geq \omega(k-1) + p + p'$. In this case $b^* \setminus c_j \subseteq b \cap b^*$, so

$$\zeta_n(b \cap b^*) \geq \zeta_{n+1}(b \cap b^* \cap c_j) + r_{n+1}(b \cap b^* \setminus c_j)$$

(by the requirement (α) when applied to the choice of $\zeta_{n+1}(b \cap b^* \cap c_j)$)

$$= \zeta_{n+1}(b \cap c_j) + r^*$$

$$\geq \omega(k-1) + p + p' + r^*;$$

but as we also have $b \cup b^* = \mathbf{1}$ and $\zeta_n(b) < \omega k + p$, hypothesis $(\mathrm{iii})_n$ tells us that $\zeta_n(b^*) \geq \omega k + p' + r^* = \xi + r^*$, as required.

Case (ϵ) It might be that $\xi = \omega$ and that there is a $b \in \mathfrak{A}_n$ such that $b \cup c_j = \mathbf{1}$, $\zeta_n(b) < \omega$. Again, $b^* \setminus c_j \subseteq b \cap b^*$ so $r^* \leq r_{n+1}(b \cap b^*) \leq \zeta_n(b \cap b^*)$, while $\zeta_n(b) < \omega + 0$, $b \cup b^* = \mathbf{1}$; so applying $(\mathrm{iii})_n$ again we get $\zeta_n(b^*) \geq \omega + r_{n+1}(b \cap b^*) \geq \xi + r^*$, as required.

Cases (η)-(ζ) We need also to deal with the cases corresponding to (δ) and (ϵ) when the rôle of b is taken by some c_i, where $i < j$. The argument is essentially the same (writing $\zeta_{n+1}(c_i)$ in place of $\zeta_n(b)$ at each opportunity) but the individual steps require slightly different justifications. To be sure that $\zeta_{n+1}(c_i \cap b^*) \geq \zeta_{n+1}(c_i \cap b^* \cap c_j) + r_{n+1}(c_i \cap b^* \setminus c_j)$, for instance, given that $c_i \cup c_j = \mathbf{1}$, observe that this was a consequence of the requirement (γ) applied when we came to fix $\zeta_{n+1}(c_i \cap b^*)$, because $c_i \cap b^* \cap c_j = c_i \cap c_j$ must have been dealt with previously. The conclusion $\zeta_n(b^*) \geq \omega k + p' + r^*$ in this argument now relies on (δ) as applied to the choice of $\zeta_{n+1}(c_i)$. In the same way, when tracing through the argument of case (ϵ) with a c_i replacing b, we shall need to apply (δ) with b^* and c_i to conclude that $\zeta_n(b^*) \geq \omega + r^*$.

(e) Thus the induction does proceed; since the conditions (α) and (β) clearly ensure that every ζ_{n+1} is an extension of ζ_n, we obtain a function $\zeta = \bigcup_{n\in\mathbb{N}} \zeta_n$ satisfying (iii) and (iv) of (a) above; moreover, setting $Z(e) = \bigcup_{n\ge m} Z_n(e)$ for $e \in E_m$, we get a family $\langle Z(e)\rangle_{e\in E}$ satisfying (i) and (ii) of (a).

(f) Now at last we are ready to define ϕ. For $a \in \mathfrak{A}$, set

$$\begin{aligned} \phi(a) &= 1 \text{ if there is an } e \in E \text{ such that } a \supseteq e, \\ &= \frac{2}{3} \text{ if there is no such } e, \text{ but } a \in \bigcup_{e\in E} Z(e), \\ &= \frac{1}{3} \text{ if } a \notin \bigcup_{e\in E} Z(e) \cup \{\mathbf{0}\}, \\ &= 0 \text{ if } a = \mathbf{0}. \end{aligned}$$

ϕ is a submeasure because every $Z(e)$ is a maximal linked subset based on e, so that if $a \cup a' \supseteq e$ then at least one of a, a' belongs to $Z(e)$. Of course $(U\phi)(\mathbf{1}) = 1$ because $\phi(e_{ni}) = 1$ whenever $1 \le i \le n+1$. Finally, $(E\phi)(\mathbf{1}) \le \frac{1}{3}$ by the criterion of Proposition 19(iii), taking any ϵ, δ with $\frac{1}{3} < \epsilon < \delta$. (Of course $(E\phi)(a) \ge \frac{1}{3}$ for every non-zero $a \in \mathfrak{A}$, just because $\phi(a) \ge \frac{1}{3}$ for every non-zero a.)

So we have the claimed example.

Remark We believe that this was known some years previously by J.W.Roberts.

Note that we have $\zeta_{\frac{1}{3},\phi}(\mathbf{1}) \le \zeta(\mathbf{1}) = \omega^2$, which is best possible, by 22(b).

25. The following is of course trivialized by Proposition 24, but was a useful crutch on the way.

Proposition Suppose there is a Boolean algebra $\mathfrak{B}$ with a submeasure ϕ such that $(E\phi)(\mathbf{1}) < \frac{1}{2}(U\phi)(\mathbf{1})$. Then there is a submeasure ψ defined on the algebra $\mathfrak{A}$ of Proposition 12 such that $\psi(\mathbf{1}) = (U\psi)(\mathbf{1}) = 1$, $(E\psi)(\mathbf{1}) \le \frac{1}{3}$, ψ takes only the values 0, $\frac{1}{3}$, $\frac{2}{3}$ and 1, and $\psi(e_{ni}) = 1$, $\zeta_{\frac{1}{3},\psi}(e_{ni}) = 1$ for $i < n$.

proof (a) We may suppose that $(U\phi)(\mathbf{1}) = 1$. Take $\alpha > 2$ such that $\alpha(E\phi)(\mathbf{1}) < 1$, and set $\phi'(b) = \min(1, \frac{1}{3}\lceil\alpha\phi(b)\rceil)$ for each $b \in \mathfrak{B}$, writing $\lceil\gamma\rceil$ for the least integer greater than or equal to γ. Then ϕ' is a submeasure on $\mathfrak{B}$ which takes at most the values 0, $\frac{1}{3}$, $\frac{2}{3}$ and 1, and $(U\phi')(\mathbf{1}) = 1$, $(E\phi')(\mathbf{1}) \le \frac{1}{3}$.

(b) Set

$$R = \{(b, b') : b,\ b' \in \mathfrak{B},\ \phi'(b \setminus b') \ge \tfrac{2}{3},\ \phi'(b' \setminus b) \ge \tfrac{2}{3}\}.$$

Then there is a family $\langle b_{ni}\rangle_{n\in\mathbb{N}, i\le n+1}$ in $\mathfrak{B}$ such that

$\phi'(b_{ni}) = 1$ whenever $i \le n+1$,

$(b_{ni}, b_{mj}) \in R$ whenever $(m, j) \ne (n, i)$,

$b_{ni} \cap b_{nj} = \mathbf{0}$ whenever $i < j \le n+1$.

P Choose $\langle b_{ni}\rangle_{i\le n+1}$ for each n in turn in such a way as to satisfy the three conditions above and moreover, writing $E_n = \{b : \phi'(b) = 1,\ (b, b_{ni}) \in R\ \ \forall\ \ m < n,\ i \le m+1\}$,

for every $k \in \mathbb{N}$ there is a disjoint sequence $\langle c_j\rangle_{j\le k}$ in E_n.

The induction starts because $E_0 = \{b : \phi'(b) = 1\}$ and $(U\phi')(\mathbf{1}) > \frac{2}{3}$. For the inductive step to $n \geq 0$, settle on a disjoint sequence $\langle d_i \rangle_{i \leq n+2}$ in E_n, and for each $k \in \mathbb{N}$ choose a disjoint sequence $\langle c_{ki} \rangle_{i<(k+1)(n+3)}$ in E_n. For each $i \leq n+2$, $\phi'(d_i \setminus c_{kj}) \leq \frac{1}{3}$ for at most one $j < (k+1)(n+3)$, because $d_i \subseteq (d_i \setminus c_{kj}) \cup (d_i \setminus c_{kj'})$ for any distinct j, j', and $\phi'(d_i) > \frac{2}{3}$. So

$$J_k = \{j : \phi'(d_i \setminus c_{kj}) \geq \tfrac{2}{3} \ \forall \ i \leq n+1\}$$

has cardinal at least $k(n+3)$. Next, for each $j \in J_k$, $\phi'(c_{kj} \setminus d_i) \leq \frac{1}{3}$ for at most one i, so there is an $i_k \leq n+2$ such that

$$J'_k = \{j : j \in J_k,\ \phi'(c_{kj} \setminus d_i) \geq \tfrac{2}{3} \ \forall \ i \neq i_k\}$$

has at least k members. Now there is an $i^* \leq n+2$ such that $\{k : i_k = i*\}$ is infinite. Take $\langle b_{ni} \rangle_{i \leq n+1}$ to be a re-enumeration of $\langle d_i \rangle_{i \leq n+2, i \neq i^*}$, and continue.

(c) Choose a Boolean homomorphism $\theta : \mathfrak{A} \to \mathfrak{B}$ such that $\theta(e_{ni}) = b_{ni}$ whenever $i \leq n+1$; this is possible because $\langle b_{ni} \rangle_{i \leq n+1}$ is disjoint for each n. Set $\psi_0(a) = \phi'(\theta a)$ for each $a \in \mathfrak{A}$. We see at once that

ψ_0 takes at most the values 0, $\frac{1}{3}$, $\frac{2}{3}$ and 1;
$\psi_0(e_{ni}) = 1$ whenever $n \geq 1$, $i \leq n+1$ so $(U\psi_0)(\mathbf{1}) = 1$;
$(E\psi_0)(\mathbf{1}) \leq (E\phi')(\mathbf{1}) \leq \frac{1}{3}$;
$\psi_0(e_{ni} \setminus e_{mj}) \geq \frac{2}{3}$ if $i \leq n+1$, $j \leq m+1$ and $(m, j) \neq (n, i)$.

For $n \in \mathbb{N}$, $i \leq n+1$ consider

$$W_{ni} = \{e_{ni} \cap e_{mj} : j \leq m+1,\ (m, j) \neq (n, i)\} \cup \{a : a \in \mathfrak{A},\ \psi_0(e_{ni} \cap a) \leq \frac{1}{3}\}.$$

We see that $w \cup w' \not\supseteq e_{ni}$ whenever w, $w' \in W_{ni}$. So we may take a maximal set $W'_{ni} \supseteq W_{ni}$ such that $w \cup w' \not\supseteq e_{ni}$ whenever w, $w' \in W'_{ni}$. Set $Z_{ni} = \mathfrak{A} \setminus W'_{ni}$. If $a \in Z_{ni}$ there is a $w \in W'_{ni}$ such that $a \cup w \supseteq e_{ni}$; so $e_{ni} \setminus a \in W'_{ni}$. If a, $a' \in Z_{ni}$ then $e_{ni} \cap a \cap a' \neq \mathbf{0}$. Also if $j \leq m+1$ and $(m, j) \neq (n, i)$ then $e_{mj} \cap e_{ni} \in W_{ni}$ so $e_{ni} \setminus e_{mj} \in Z_{ni}$ and $e_{mj} \notin Z_{ni}$.

(d) Set

$$\begin{aligned}\psi(a) =& 1 \text{ if there are } i \leq n+1 \text{ such that } e_{ni} \subseteq a;\\ & \frac{2}{3} \text{ if there are } i \leq n+1 \text{ such that } a \in Z_{ni},\ a \not\supseteq e_{ni};\\ & \frac{1}{3} \text{ if } a \neq \mathbf{0} \text{ and } a \notin Z_{ni} \text{ whenever } i \leq n+1;\\ & 0 \text{ if } a = \mathbf{0}.\end{aligned}$$

Then ψ is a submeasure because if $a \cup a' \supseteq e_{ni}$ then at least one of a, a' belongs to Z_{ni}. Now $\psi_0(a) \geq \frac{2}{3}$ whenever $\psi(a) \geq \frac{2}{3}$, so $(E\psi)(\mathbf{1}) \leq \frac{1}{3}$. Finally, consider $\zeta_{\frac{1}{3},\psi}(e_{ni})$. If a, $a' \subseteq e_{ni}$ and $a \cap a' = \mathbf{0}$ then at most one of a, a' can belong to Z_{ni} and neither belongs to any other Z_{mj}, so $\min(\psi(a), \psi(a')) \leq \frac{1}{3}$; accordingly $\zeta_{\frac{1}{3},\psi}(e_{ni}) = 1$.

Problems

(a) In Theorem 8, is $\frac{1}{3}$ best possible? (See §§9-10.)

(b) If ϕ is a submeasure such that $\lim_{n\to\infty}(U^n\phi)(\mathbf{1}) > 0$, can ϕ be exhaustive? (Of course, if it is, $U\phi$ will be a pathological exhaustive non-zero submeasure.)

(c) Let ϕ be a pathological submeasure. By Theorem 8, $U^2\phi \geq \frac{1}{9}\phi$. Is $\frac{1}{9}$ best possible?

(d) Given $\epsilon > 0$, is there always a (countable) Boolean algebra $\mathfrak{A}$ with a submeasure ϕ such that $(U\phi)(\mathbf{1}) = \phi(\mathbf{1}) = 1$, $(E\phi)(\mathbf{1}) \leq \epsilon$? If there is, what is the lowest possible value of $\zeta_{\epsilon\phi}(\mathbf{1})$? See Propositions 22, 24.

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