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from D.H.Fremlin

Note of 5.1.84
formerly Note of 24.5.82

Two new versions of MA

1. The axioms Consider the statements

$P_\downarrow(\kappa)$: If $\lambda \le \kappa$ and $\langle A_\xi\rangle_{\xi<\lambda}$ is a family of infinite subsets of $\mathbb{N}$ such that $A_\xi \setminus A_\eta$ is finite whenever $\eta \le \xi < \lambda$, then there is an infinite $I \subseteq \mathbb{N}$ such that $I \setminus A_\xi$ is finite for every $\xi < \lambda$.

$L(\kappa)$: If P is an upwards-ccc partially ordered set and $\#(P) \le \kappa$ then P is σ-centered upwards.

H^+ : If P is an uncountable upwards-ccc partially ordered set then it has an uncountable upwards-directed subset.

Then we have $MA(\kappa) \Longleftrightarrow L(\kappa) + \kappa<\mathfrak{p}$; $\kappa<\mathfrak{p} \Longrightarrow P_\downarrow(\kappa)$; $P_\downarrow(\omega_1) \Longrightarrow \omega_1 < \mathfrak{p}$; and $MA(\omega_1) \Longrightarrow H^+$ ([1], 14D and 41B).

2. Lemma If $P_\downarrow(\kappa)$ is true, then whenever $F \subseteq \mathbb{N}^{\mathbb{N}}$ and $\#(F) \le \kappa$ there is a $g \in \mathbb{N}^{\mathbb{N}}$ such that $\{ n : f(n) > g(n) \}$ is finite for every $f \in F$.

proof Use the method of [1], 14B.

3. Theorem $MA(\kappa) \Longleftrightarrow L(\kappa) + P_\downarrow(\kappa)$.

proof We have only to show that $L(\kappa) + P_\downarrow(\kappa) \Longrightarrow \kappa<\mathfrak{p}$. ? If this is false, then we ~~will~~ shall have $L(\mathfrak{p}) + P_\downarrow(\mathfrak{p})$. [redacted] Let $\mathcal{A}$ be a family of subsets of $\mathbb{N}$

such that $\#(\mathcal{A}) = \mathfrak{p}$, $\bigcap \mathcal{A}_0$ is infinite for every finite $\mathcal{A}_0 \subseteq \mathcal{A}$, but there is no infinite $I \subseteq \underset{\sim}{N}$ such that $I \setminus A$ is finite for every $A \in \mathcal{A}$. Enumerate $\mathcal{A}$ as $\langle A_\xi \rangle_{\xi<\mathfrak{p}}$. Let $\mathcal{D}$ be $\{ \bigcap \mathcal{A}_0 :$ $\emptyset \neq \mathcal{A}_0 \in [\mathcal{A}]^{<\omega} \}$. Then we can construct inductively a family $\langle B_\xi \rangle_{\xi<\mathfrak{p}}$ of infinite subsets of $\underset{\sim}{N}$ such that

$B_\xi \cap D$ is infinite $\forall\ \xi < \mathfrak{p}$, $D \in \mathcal{D}$;

$B_\xi \setminus B_\eta$ is finite $\forall\ \eta \leq \xi < \mathfrak{p}$;

$B_{\xi+1} \subseteq A_\xi \quad \forall\ \xi < \mathfrak{p}$,

as follows. (i) Start by setting $B_0 = \underset{\sim}{N}$. (ii) For the inductive step to a successor ordinal $\xi + 1$, set $B_{\xi+1} = B_\xi \cap A_\xi$. (iii) For the inductive step to a limit ordinal ξ of countable cofinality, take a sequence $\langle \xi(n) \rangle_{n\in\underset{\sim}{N}}$ in ξ increasing to ξ . Then $B_{\xi(n)} \setminus \bigcap_{i\leq n} B_{\xi(i)}$ is finite for each $n \in \underset{\sim}{N}$, so $D \cap \bigcap_{i\leq n} B_{\xi(i)}$ is infinite for each $n \in \underset{\sim}{N}$, $D \in \mathcal{D}$, by the inductive hypothesis; so for $D \in \mathcal{D}$ we can define $f_D : \underset{\sim}{N} \to \underset{\sim}{N}$ by writing $f_D(n) = \min(D \cap \bigcap_{i\leq n} B_{\xi(i)} \setminus n)$ for each $n \in \underset{\sim}{N}$. By Lemma 2, there is an $f \in \underset{\sim}{N}^{\underset{\sim}{N}}$ such that $\{ n : f(n) \leq f_D(n) \}$ is finite for each $D \in \mathcal{D}$. Set

$$B_\xi = \bigcap_{n\in\underset{\sim}{N}} (B_{\xi(n)} \cup f(n)) \subseteq \underset{\sim}{N} .$$

Then $B_\xi \setminus B_{\xi(n)}$ is finite for each $n \in \underset{\sim}{N}$, so $B_\xi \setminus B_\eta$ is finite for each $\eta < \xi$. If $D \in \mathcal{D}$ and $m \in \underset{\sim}{N}$ there is a $k \geq m$ such that $f(n) > f_D(n)$ $\forall\ n \geq k$; now $f_D(k) \in \bigcap_{i\leq k} B_{\xi(i)}$, while for $n \geq k$ we have $f_D(k) \leq f_D(n) < f(n)$. It follows that $f_D(k) \in B_\xi$; but also $f_D(k) \in D \setminus k \subseteq D \setminus m$. As m is arbitrary, $D \cap B_\xi$ is infinite; as D is arbitrary, the inductive hypothesis is satisfied by B_ξ .

(iv) For the inductive step to a limit ordinal ξ of uncountable cofinality, enumerate $\mathcal{D}$ as $\langle D_\alpha \rangle_{\alpha<\mathfrak{p}}$. Construct inductively $\langle E_{\alpha\beta} \rangle_{\alpha\leq\beta<\mathfrak{p}}$ and $\langle F_\alpha \rangle_{\alpha<\mathfrak{p}}$ so that

$E_{\alpha\beta}$ is infinite $\forall\ \alpha \leq \beta < \mathfrak{p}$; $\quad (I_1)$

$E_{\alpha\alpha} \subseteq D_\alpha\ \ \forall\ \alpha < \mathfrak{p}$; $\quad (I_2)$

$E_{\alpha\gamma} \setminus E_{\alpha\beta}$ is finite if $\alpha \leq \beta \leq \gamma < \mathfrak{p}$; $\quad (I_3)$

$F_\beta \supseteq \bigcup_{\alpha\leq\beta} E_{\alpha\beta}$, $F_\beta \setminus B_\eta$ is finite if $\beta < \mathfrak{p}$, $\eta < \xi$; $\quad (I_4)$

as follows. ~~(a) Start~~ Given $\langle E_{\alpha\beta}\rangle_{\alpha\leq\beta<\gamma}$, where $\gamma < \mathfrak{p}$, then by ~~p~~ $P_\downarrow(\mathfrak{p})$ and the inductive hypotheses (I_1), (I_3) there is ~~an infinite~~ ~~$E_{\alpha\gamma}$~~ for each $\alpha < \gamma$ an infinite $E'_{\alpha\gamma}$ such that $E'_{\alpha\gamma} \setminus E_{\alpha\beta}$ is finite for each $\beta < \gamma$. Next, because $\max(\xi,\gamma) < \mathfrak{p}$, we can apply [1], 21A to find a~~n infinite~~ set F_γ such that $F_\gamma \setminus B_\eta$ is finite for $\eta < \xi$, $F_\gamma \cap E'_{\alpha\gamma}$ is infinite for $\alpha < \gamma$, and $F_\gamma \cap D_\gamma$ is infinite. (This works because $E'_{\alpha\gamma} \setminus B_\eta \subseteq (E'_{\alpha\gamma} \setminus E_{\alpha\alpha}) \cup (F_\alpha \setminus B_\eta)$ is finite for $\alpha < \gamma$, $\eta < \xi$.) Now set $E_{\alpha\gamma} = E'_{\alpha\gamma} \cap F_\gamma$ for $\alpha < \gamma$ and $E_{\gamma\gamma} = F_\gamma \cap D_\gamma$, and see that the induction continues.

Having got hold of these, use $P_\downarrow(\mathfrak{p})$ to choose infinite sets G_α such that $G_\alpha \setminus E_{\alpha\beta}$ is finite whenever $\alpha \leq \beta < \mathfrak{p}$. Observe that $G_\alpha \setminus B_\eta$ is finite whenever $\alpha < \mathfrak{p}$, $\eta < \xi$.

Let

$$\mathcal{B} = \{ I \triangle \textstyle\bigcap_{\eta\in J} B_\eta : I \in [\underset{\sim}{N}]^{<\omega},\ \emptyset \neq J \in [\xi]^{<\omega} \} ,$$

$$\mathcal{G} = \{ I \triangle \textstyle\bigcup_{\alpha\in J} G_\alpha : I \in [\underset{\sim}{N}]^{<\omega},\ \emptyset \neq J \in [\mathfrak{p}]^{<\omega} \} .$$

~~so that~~ Let P be the set $\{ (B,G) : B \in \mathcal{B} , G \in \mathcal{G} , B \supseteq G \}$ ordered by saying that $(B,G) \leq (B',H')$ if $B' \subseteq B$ and $G \subseteq G'$. Then P ~~satisfies Knaster's condition upwards if linked upwards~~ satisfies Knaster's condition upwards. **P** Note first that since we have $P_\downarrow(\mathfrak{p})$ and not $P(\mathfrak{p})$, $\mathfrak{p} > \omega_1$; as $\mathfrak{p}$ is regular ([1], 21O), $\mathrm{cf}(\mathfrak{p}) > \omega_1$. If therefore $\langle (C_\zeta, H_\zeta)\rangle_{\zeta<\omega_1}$ is any family in P , there is a $\gamma < \mathfrak{p}$ such that every H_ζ belongs to $\{ I \triangle \bigcup_{\alpha\in J} G_\alpha : I \in [\underset{\sim}{N}]^{<\omega} ,\ \emptyset \neq J \in [\gamma]^{<\omega}\}$.

In this case, since $G_\alpha \setminus F_\gamma \subseteq G_\alpha \setminus E_{\alpha\gamma}$ is finite for every $\alpha < \gamma$, $H_\zeta \setminus F_\gamma$ is finite for every $\zeta < \omega_1$; while at the same time $F_\gamma \setminus B_\eta$ is finite for every $\eta < \xi$, so $F_\gamma \setminus C_\zeta$ is finite for every $\zeta < \omega_1$. We can therefore find an uncountable $Z \subseteq \omega_1$ such that

$$F_\gamma \setminus C_\zeta = F_\gamma \setminus C_\theta \ , \quad H_\zeta \setminus F_\gamma = H_\theta \setminus F_\gamma \quad \forall\, \theta, \zeta \in Z .$$

~~Now Z is upwards-centered in P~~ Now $\{ (C_\zeta, H_\zeta) : \zeta \in Z \}$ is upwards-centered in P. Q

At the same time, $\#(P) \leq \mathfrak{p}$. So by $L(\mathfrak{p})$ P is expressible as $\bigcup_{n \in \mathbb{N}} P_n$ where each P_n is upwards-centered in P. For each $n \in \mathbb{N}$ set

$$C_n = \bigcap \{ C : (C,H) \in P_n \} \ , \quad H_n = \bigcup \{ H : (C,H) \in P_n \} \ ,$$

so that $H \subseteq H_n \subseteq C_n \subseteq C$ whenever $(C,H) \in P_n$. Set

$$I = \{ n : n \in \mathbb{N} \, , \exists\, \eta < \xi \, , \ C_n \setminus B_\eta \text{ is infinite} \} \ ,$$

$$\zeta_n = \min\{ \eta : \eta < \xi \, , \ C_n \setminus B_\eta \text{ is infinite} \} \ \forall\, n \in I .$$

As $\mathrm{cf}(\xi) > \omega$, $\zeta = \sup_{n \in \mathbb{N}} \zeta_n < \xi$. As $\xi < \mathfrak{p}$, there is a B_ξ such that $B_\xi \setminus B_\eta$ is finite for each $\eta < \xi$ and $H_n \setminus B_\xi$ is finite for each $n \in \mathbb{N} \setminus I$ ([1], 21A again, or otherwise).

Let $\alpha < \mathfrak{p}$. Then $(B_{\zeta+1}, G_\alpha \cap B_{\zeta+1}) \in P$, so $(B_{\zeta+1}, G_\alpha \cap B_{\zeta+1}) \in P_n$ for some $n \in \mathbb{N}$. As $C_n \subseteq B_{\zeta+1}$, $C_n \setminus B_\eta$ is finite for every $\eta \leq \zeta$; so $n \in \mathbb{N} \setminus I$. Accordingly

$$G_\alpha \setminus B_\xi \subseteq (G_\alpha \setminus B_{\zeta+1}) \cup (H_n \setminus B_\xi)$$

is finite, and $D_\alpha \setminus B_\xi$ is infinite, because $G_\alpha \setminus D_\alpha \subseteq G_\alpha \setminus E_{\alpha\alpha}$ is finite. Thus the induction continues.

Having got the $\langle B_\xi \rangle_{\xi < \mathfrak{p}}$, use $P_\downarrow(\mathfrak{p})$ to construct an infinite set B such that $B \setminus B_\xi$ is finite for each $\xi < \mathfrak{p}$, and observe that $B \setminus A$ is finite for every $A \in \mathcal{A}$; contradicting the choice of $\mathcal{A}$. X

This contradiction proves the theorem.

4. <u>Theorem</u> $H^+ \Leftrightarrow MA(\omega_1)$.

<u>proof</u> (<u>a</u>) $H^+ \Rightarrow L(\omega_1)$. **P** Assume H^+, and let P be an upwards-ccc partially ordered set of cardinal ω_1 . Let $f : P \rightarrow \omega_1$ be a injection, and for $p \in P$ choose a sequence $\langle s(p,n) \rangle_{n \in \mathbb{N}}$ running over $\{ q : f(q) \leq f(p) \}$, with $s(p,0) = p$, ~~and the $s(p,n)$ all distinct if $f(p) \geq$~~ .

Let $\tilde{P}$ be $\bigcup_{n \in \mathbb{N}} P^{n+1}$, ordered by saying

$$(p_0,\ldots,p_m) < (q_0,\ldots,q_n) \text{ iff } m < n , \quad p_i \leq q_i \quad \forall \; i \leq m ,$$
$$\text{and } \forall \; i \leq m, \; j \leq m \;\; \exists \;\; k \leq n , \; s(p_i,j) = q_k .$$

(This is a partial order because $s(p,0) = p$ for every p .) Now $\tilde{P}$ is upwards-ccc. **P** Note first that H^+ implies that every ccc partially ordered set satisfies Knaster's condition, so that P^{n+1} is upwards-ccc for every $n \in \mathbb{N}$; and secondly that $(p_0,\ldots,p_m)$ and $(q_0,\ldots,q_m)$ have a common upper bound in $\tilde{P}$ iff every pair $\{p_i,q_i\}$ is bounded above in P . **Q** Also, $\#(\tilde{P}) = \omega_1$. So by H^+ there is an uncountable upwards-directed $\tilde{R} \subseteq \tilde{P}$. Observe that, for any $\xi < \omega_1$,

$$\{ (p_0,\ldots,p_n) : \max_{i \leq n} f(p_i) \leq \xi \}$$

is countable; so

$$\sup\{ f(p_i) : (p_0,\ldots,p_n) \in R , \; i \leq n \} = \omega_1 .$$

Also, for any $\tilde{q} \in \tilde{P}$, $\{ \tilde{p} : \tilde{p} \leq \tilde{q} \}$ is finite, so $\tilde{R}$ can have no greatest member.

Set $P_k = \{ p_k : (p_0,\ldots,p_m) \in \tilde{R} , \; m \geq k \}$. Because $\tilde{R}$ is upwards-directed in $\tilde{P}$, each P_k is upwards-directed in P . Now $\bigcup_{k \in \mathbb{N}} P_k = P$.
P Let $p \in P$. Let $(p_0,\ldots,p_m) \in R$ be such that $\max_{i \leq m} f(p_i) \geq f(p)$. Let $i \leq m$ be such that $f(p_i) \geq f(p)$, and $j \in \mathbb{N}$ such that $p = s(p_i,j)$. As $\tilde{R}$ has no greatest member we can find $\tilde{r}_0 , \ldots , \tilde{r}_{j+1} \in \tilde{R}$ such that $(p_0,\ldots,p_m) < \tilde{r}_0 < \ldots < \tilde{r}_{j+1}$. Expressing each $\tilde{r}_l$ as $(q_{l0},\ldots,q_{ln_l})$, we see that $m < n_0 < \ldots < n_j$ so that $j \leq n_j$; also there is an $s \leq n_j$ such that $p_i = s(p_i,0) = q_{js}$; and now $p = s(q_{js},j) = q_{j+1,k} \in P_k$ for some $k \leq n_{j+1}$. **Q**

~~$<\tilde{r}_{i+1}$. Express $\tilde{r}_j$ as $(J_j, q_{jo}, \ldots, q_{jn_j})$ and observe that $p' \in J_o \subseteq J_i$ and that $i \leq n_i$, so that $p = s(p', i) \leq q_{i+1,k}$ for some $k \leq n_{i+1}$, and $p \in P_k$. Q~~

This proves $L(\omega_1)$. Q

(b) $H^+ \Rightarrow P_\downarrow(\omega_1)$. P Assume H^+ , and let $\langle A_\xi \rangle_{\xi < \omega_1}$ be a family of infinite subsets of $\underset{\sim}{N}$ such that $A_\xi \setminus A_\eta$ is finite for $\eta \leq \xi < \omega_1$. For $A \in [\underset{\sim}{N}]^\omega$, $n \in \underset{\sim}{N}$ set $W(A,n) = \{ i : i \in A , \#(A \cap i) < n \}$. Let P be $[\omega_1]^{<\omega}$ ordered by saying that $I \leq J$ if $I \subseteq J$ and $W(\bigcap_{\xi \in I} A_\xi, \#(I)) \subseteq \bigcap_{\xi \in J} A_\xi$. (Conventionally take $\emptyset \leq J \ \forall \ J \in P$.) Then P is σ-centered upwards. P For $C \in [\underset{\sim}{N}]^{<\omega}$ set

$$Q_C = \{ I : W(\textstyle\bigcap_{\xi \in I} A_\xi, \#(I)) = C \}$$

and observe that Q_C is upwards-~~directed~~centered in P . Q

By H^+, P has an uncountable upwards-directed subset R . Consider $K = \bigcup R$, $A = \bigcap_{\xi \in K} A_\xi$. K is uncountable so $A \setminus A_\xi$ is finite for every $\xi < \omega_1$. Now let $m \in \underset{\sim}{N}$. As R is upwards-directed, there is an $I \in R$ with $\#(I) \geq m$; let $C = W(\bigcap_{\xi \in I} A_\xi, \#(I))$ so that $\#(C) \geq m$. If $\xi \in K$ then there is a $J \in R$ such that $I \leq J$ and $\xi \in J$; in which case $C \subseteq \bigcap_{\eta \in J} A_\eta \subseteq A_\xi$. Thus $C \subseteq A$, and $\#(A) \geq m$. As m is arbitrary, A is infinite, as required by $P_\downarrow(\omega_1)$. Q

(c) Now the theorem follows from the remarks in §1 above.

Reference 1. D.H.Fremlin, Consequences of Martin's Axiom, to be published by Cambridge U.P.

Problems (a) Does $P_\downarrow(\kappa)$ imply $\kappa < \mathfrak{p}$ for $\kappa > \omega_1$?

(b) Does $L(\omega_1)$ imply $MA(\omega_1)$?

<u>Note added 8.6.82</u>

5. <u>Lemma</u> If $L(\kappa)$ is true, and P is an upwards-ccc partially ordered set, and $A \subseteq P$ has cardinal $\leq \kappa$, then there is a sequence $\langle P_n \rangle_{n \in \mathbb{N}}$ of upwards-centered subsets of P such that $A \subseteq \bigcup_{n \in \mathbb{N}} P_n$.

<u>proof</u> Set $A_0 = A$. Given $A_n \subseteq P$ with $\#(A_n) \leq \kappa$, let A_{n+1} be xa subset of P such that $A_n \subseteq A_{n+1}$, $\#(A_{n+1}) \leq \kappa$, and every finite subset of A_n which has an upper bound in P has an upper bound in A_{n+1}. Set $P' = \bigcup_{n \in \mathbb{N}} A_n$. Then P' is upwards-ccc and has cardinal $\leq \kappa$, so by $L(\kappa)$ is expressible as $\bigcup_{n \in \mathbb{N}} P_n$ where each P_n is upwards-centered in P' i.e. is upwards-centered in P. Now $A \subseteq \bigcup_{n \in \mathbb{N}} P_n$.

6. <u>Theorem</u> $L(\kappa) + \mathfrak{p} > \omega_1 \Rightarrow MA(\kappa)$.

<u>proof</u> (<u>a</u>) Assume $L(\kappa)$. If $A \subseteq \ell^1(\mathbb{N})$ and $\#(A) \leq \kappa$ there is a $z \in \ell^1$ such that $\{ i : |x(i)| > z(i) \}$ is finite for each $x \in A$.

P Set $P = \{ y : y \in (\ell^1)^+ , \sum_{i \in \mathbb{N}} y(i) < 1 \}$. Then P satisfies Knaster's condition upwards ([1], 33C). For each $x \in A$ choose a $y_x \in P$ such that $\{ i : |x(i)| > y_x(i) \}$ is finite. By Lemma 5, there is a sequence $\langle P_n \rangle_{n \in \mathbb{N}}$ of upwards-centered sets in P covering $\{ y_x : x \in A \}$. We can suppose that each P_n is non-empty. Set $z_n(i) = \sup\{ y(i) : y \in P_n \}$ for each $n \in \mathbb{N}$. Then $\sum_{i \in \mathbb{N}} z_n(i) \leq 1$ for each $n \in \mathbb{N}$. Now there is a $z \in (\ell^1)^+$ such that $\{ i : z_n(i) > z(i) \}$ is finite for each $n \in \mathbb{N}$. This z works. **Q**

(<u>b</u>) Assume $L(\kappa)$. If $A \subseteq \mathbb{N}^{\mathbb{N}}$ and $\#(A) \leq \kappa$ there is a $g \in \mathbb{N}^{\mathbb{N}}$ such that $\{ i : f(i) > g(i) \}$ is finite for each $f \in A$. **P** For $f \in A$, define $f^*(n) = \max_{i \leq n} f(i) + n$ for each $n \in \mathbb{N}$, and $w_f \in c_0(\mathbb{N})$ by

$$w_f(i) = 1 \quad \text{if } i < f^*(0)$$
$$= 2^{-n} \quad \text{if } f^*(n) \leq i < f^*(n+1) .$$

Now set $x_f(n) = w_f(n) - w_f(n+1)$ for each $n \in \mathbb{N}$ so that $x_f \in \ell^1$. By (a), there is a $z \in (\ell^1)^+$ such that $\{ i : x_f(i) > z(i) \}$ is finite for each $f \in A$. Set $v(n) = \sum_{i \geq n} z(i) + 2^{-n}$ for each $n \in \mathbb{N}$, so that $v \in c_0$ and $\{ n : w_f(n) > v(n) \}$ is finite for each $f \in A$. Define $g \in \mathbb{N}^{\mathbb{N}}$ by saying

$$g(n) = \max\{ i : v(i) \geq 2^{-n} \} .$$

Now if $f \in A$ there is an $m \in \mathbb{N}$ such that $w_f(i) \leq v(i)$ for every $i \geq m$. In this case, if $n \geq m$, $f^*(n) \geq n \geq m$, so

$$2^{-n} = w_f(f^*(n)) \leq v(f^*(n))$$

and $f^*(n) \leq g(n)$. So this g serves.

(c) ? Suppose that $L(\kappa) + \omega_1 < \mathfrak{p}$ is true but $MA(\kappa)$ is false. In this case [redacted] $\kappa \geq \mathfrak{p}$, and $L(\mathfrak{p})$ is true. By Theorem 3, $P_{\downarrow}(\mathfrak{p})$ must be false. Let $\langle A_\xi \rangle_{\xi < \mathfrak{p}}$ be a family of infinite subsets of $\mathbb{N}$ such that $A_\xi \setminus A_\eta$ is finite whenever $\eta < \xi < \mathfrak{p}$, and there is no infinite $A \subseteq \mathbb{N}$ such that $A \setminus A_\xi$ is finite for every $\xi < \mathfrak{p}$. For each $\xi < \mathfrak{p}$ let f_ξ be the increasing enumeration of A_ξ. By (b), there is a $g \in \mathbb{N}^{\mathbb{N}}$ such that $\{ i : f_\xi(2i) + 1 > g(i) \}$ is finite for each $\xi < \mathfrak{p}$. We can take g to be increasing. Set

$$P = \{ B : B \subseteq \mathbb{N} , \#(B \cap g(n)) \geq n \ \forall n \in \mathbb{N} ,$$
$$\exists \xi < \mathfrak{p} , A_\xi \setminus B \text{ is finite} \} .$$

Then P is downwards-ccc. Let $R \subseteq P$, $\#(R) = \omega_1$. Then as we know that $\mathfrak{p}$ is regular ([1], 21K) and we are hypothesizing that $\mathfrak{p} > \omega_1$, we have $\mathrm{cf}(\mathfrak{p}) > \omega_1$. Consequently there is a $\zeta < \mathfrak{p}$ such that $A_\zeta \setminus B$ is finite for every $B \in R$. Now there is an $n \in \mathbb{N}$ such that $\{ B : B \in R , A_\zeta \setminus B \subseteq n \}$ is uncountable. Let $m \in \mathbb{N}$ be such that $f_\zeta(2i) + 1 \leq g(i)$ for every $i \geq m$ i.e. $\#(A \cap g(i)) \geq 2i$ for every $i \geq m$ and $\#(A \cap g(i) \setminus n) \geq i$ for every $i \geq \max(m,n)$.

Next, let $I \subseteq g(\max(m,n))$ be such that $R'' = \{ B : B \in R ,$ $B \cap g(\max(m,n)) = I \}$ is uncountable. Consider $C = I \cup (A_\zeta \setminus n)$. We see that $A_\zeta \setminus C$ is finite. Also, for $i \leq \max(m,n)$, $\#(C \cap g(i)) \geq \#(I \cap g(i)) = \#(B_0 \cap g(i)) \geq i$, where B_0 is any member of R'' ; while for $i > \max(m,n)$, $\#(C \cap g(i)) \geq \#(A_\zeta \cap g(i) \setminus n) \geq i$. So $C \in P$. And $C \subseteq B$ for every $B \in R''$, so R'' is surely dwonwards-linked. **Q**

For each $\xi < \mathfrak{p}$, there is a $B_\xi \in P$ such that $B_\xi \setminus A_\xi$ is finite. **P** Take $B_\xi = A_\xi \cup g(m)$ where m is such that $f_\xi(2i) + 1 \leq g(i)$ for every $i \geq m$. **Q** By Lemma 5, there is a sequence $\langle P_n \rangle_{n \in \mathbb{N}}$ of downwards-centered subsets of P covering $\{ B_\xi : \xi < \mathfrak{p} \}$. As $\mathrm{cf}(\mathfrak{p}) > \omega$, there is an $n \in \mathbb{N}$ such that $D = \{ \xi : B_\xi \in P_n \}$ is cofinal in $\mathfrak{p}$. Examine $C = \bigcap P_n$. If $Q \subseteq P_n$ is finite then $\#(\bigcap Q \cap g(m)) \geq m$ for every $m \in \mathbb{N}$, because Q has a lower bound in P . So $\#(C \cap g(m)) \geq m$ for every $m \in \mathbb{N}$, and C is infinite. On the other hand, $C \subseteq B_\xi$ for every $\xi \in D$, so $C \setminus A_\xi$ is finite for every $\xi \in D$, and $C \setminus A_\xi$ is finite for every $\xi < \mathfrak{p}$ because D is cofinal in $\mathfrak{p}$. But such a C is not supposed to exist. **X**

This proves the result.

7. <u>The principle H</u> Consider the statement

H : If P is an uncountable upwards-ccc partially ordered set then P has an uncountable upwards-centered subset.

Clearly $H^+ \Rightarrow H$ and $L(\omega_1) \Rightarrow H$.

8. <u>Lemma</u> Assume H . Then if P is an uncountable upwards-ccc partially ordered set and $A \subseteq P$ is uncountable, there is an uncountable upwards-centered $R \subseteq A$.

proof Apply H to $\{ I : I \in [A]^{<\omega}$, I has an upper bound in $P \}$.

9. Theorem $H \Leftrightarrow L(\omega_1)$.

proof I have only to show that $H \Rightarrow L(\omega_1)$. Assume H , and let P be an upwards-ccc partially ordered set of cardinal ω_1 . Enumerate P as $\langle p_\xi \rangle_{\xi<\omega_1}$.

Let X be the set of finite up-antichains in P . Define $\leq$ on X by saying that $I \leq J$ if for every $p \in I$ there is a $q \in J$ such that $p \leq q$. Then $\leq$ is a partial order on X . Let $Z \subseteq X^{\underset{\sim}{N}}$ be the set of sequences $\langle I_n \rangle_{n \in \underset{\sim}{N}}$ such that $\#(I_n) \leq n$ for every $n \in \underset{\sim}{N}$ and $\sup_{n \in \underset{\sim}{N}} \#(I_n) < \infty$. Order Z by saying that $\langle I_n \rangle_{n \in \underset{\sim}{N}} \leq \langle J_n \rangle_{n \in \underset{\sim}{N}}$ if $I_n \leq J_n$ for every $n \in \underset{\sim}{N}$. Then Z is upwards-ccc. P If $R \subseteq Z$ is uncountable, then there is an $m \in \underset{\sim}{N}$ such that

$$R_1 = \{ \langle I_n \rangle_{n \in \underset{\sim}{N}} : \langle I_n \rangle_{n \in \underset{\sim}{N}} \in R , \ \#(I_n) \leq m \ \forall \ n \in \underset{\sim}{N} \}$$

is uncountable. Now observe that P satisfies Knaster's condition upwards, by Lemma 8. It follows easily that $X \cap [P]^{\leq n}$ satisfies Knaster's condition upwards for each $n \in \underset{\sim}{N}$. Consequently there is an uncountable $R_2 \subseteq R$ such that if $\langle I_n \rangle_{n \in \underset{\sim}{N}}$ and $\langle J_n \rangle_{n \in \underset{\sim}{N}}$ belong to R_2 , then for each $n < 2m$ there is a $K_n \in X \cap [P]^{\leq n}$ which is a common upp~~w~~er bound of I_n and J_n . However, if we now take, for $n \geq 2m$, a ~~minimal~~ set $K_n \subseteq P$ of minimal size such that for every $p \in I_n \cup J_n$ there is a $q \in K_n$ with $q \geq p$, we see that $K_n \in X \cap [P]^{\leq 2m}$. Hence $\langle K_n \rangle_{n \in \underset{\sim}{N}} \in Z$ and is a common upper bound for $\langle I_n \rangle_{n \in \underset{\sim}{N}}$ and $\langle J_n \rangle_{n \in \underset{\sim}{N}}$. Q

For each $\xi < \omega_1$ there is an $\langle I^\xi_n \rangle_{n \in \underset{\sim}{N}} \in Z$ such that $\bigcup_{n \in \underset{\sim}{N}} I^\xi_n \supseteq \{ p_\eta : \eta \leq \xi \}$. By Lemma 8, there is an uncountable $C \subseteq \omega_1$ such that $\{ \langle I^\xi_n \rangle_{n \in \underset{\sim}{N}} : \xi \in C \}$ is upwards-centered in Z .

For each $L \in [C]^{<\omega}$ and $n \in \underset{\sim}{N}$, let $J_{Ln} \in X \cap [P]^{\leq n}$ be an upper bound of $\{ I_n^{\xi} : \xi \in C \}$ in X. Enumerate J_{Ln} as $\langle q_{Lni} \rangle_{i<m(L,n)}$. ~~Define~~ Choose $f_{Ln\xi} : I_n^{\xi} \to m(L,n)$ so that $p \leq q_{Lnf_{Ln\xi}(p)}$ whenever $\xi \in L \in [C]^{<\omega}$ and $p \in I_n^{\xi}$. Let $\mathcal{F}$ be an ultrafilter on $[C]^{<\omega}$ such that $\{ L : \xi \in L \in [C]^{<\omega} \} \in \mathcal{F}$ whenever $\xi \in C$. Set $f_{n\xi}(p) = \lim_{L \to \mathcal{F}} f_{Ln\xi}(p) < n$ for every $\xi \in C$ and $p \in I_n^{\xi}$. Now set $P_{nl} = \bigcup_{\xi \in C} \{ p : p \in I_n^{\xi} , f_{n\xi}(p) = l \}$, for $l < n \in \underset{\sim}{N}$. Then each P_{nl} is upwards-centered. **P** If $L \in [C]^{<\omega}$ and $A = \bigcup_{\xi \in L} \{ p : p \in I_n^{\xi} , f_{n\xi}(p) = l \}$, then there is an $M \in [C]^{<\omega}$ such that $M \supseteq L$ and $f_{n\xi}(p) = f_{Mn\xi}(p)$ whenever $\xi \in L$ and $p \in I_n$; in which case q_{Mnl} is an upper bound of A. **Q** Also

$$\bigcup_{l<n \in N} P_{nl} = \bigcup_{\xi \in C, n \in \underset{\sim}{N}} I_n \supseteq \bigcup_{\xi \in C} \{ p_{\eta} : \eta \leq \xi \} = P .$$

So P is σ-centered upwards, as required.

10. Remark added 12.12.84 P.Nyikos has pointed out that Theorem 9 implies that $\mathfrak{m} > \omega_1$ is equivalent to the principle Δ of K.Kunen & F.D.Tall, "Between Martin's Axiom and Souslin's hypothesis", Fund. Math. 102 (1979) 173-181.

11. Remark added 25.11.85 S.Todorčević & B.Veličković ("MA and precalibers of ccc posets", note of August 1985) have shown that there is a ccc poset of cardinal $\mathfrak{t}$ (the least cardinal for which $P_{\downarrow}(\kappa)$ is false) with no centred subset of cardinal $\mathfrak{t}$. Consequently $\mathfrak{l} > \omega_1 \Rightarrow \mathfrak{t} > \omega_1 \Rightarrow \mathfrak{m} > \omega_1$. It follows ~~ak~~ that $\mathfrak{l} = \mathfrak{m}$ and that $H \Leftrightarrow \mathfrak{m} > \omega_1$. (Here $\mathfrak{l}$ is the least cardinal for which $L(\kappa)$ is false.)