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Banach limits on ordinals

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1 Proposition There is a family $\langle F_\alpha\rangle_{\alpha\in\mathrm{On}\setminus\{0\}}$ such that

for each non-zero $\alpha\in\mathrm{On}$, F_α is a positive linear functional from $\ell^\infty(\alpha)$ to $\mathbb{R}$ and $F_\alpha(\chi\alpha)=1$,

whenever α is the ordinal sum $\beta+\gamma$ where $\gamma>0$, $F_\alpha(\langle z_\xi\rangle_{\xi<\alpha})=F_\gamma(\langle z_{\beta+\eta}\rangle_{\eta<\gamma})$ for every $\langle z_\xi\rangle_{\xi<\alpha}\in\ell^\infty(\alpha)$.

proof Induce on α.

When $\beta\le\alpha\in\mathrm{On}$ write $\alpha-\beta$ for that ordinal such that $\beta+(\alpha-\beta)=\alpha$, and set $T_{\alpha\beta}(\langle z_\xi\rangle_{\xi<\alpha})=\langle z_{\beta+\eta}\rangle_{\eta<\alpha-\beta}$ when $\langle z_\xi\rangle_{\xi<\alpha}\in\ell^\infty(\alpha)$. Observe that if $\gamma\le\beta\le\alpha$ then $\gamma+(\beta-\gamma)+(\alpha-\beta)=\alpha$, $\alpha-\beta=(\alpha-\gamma)-(\beta-\gamma)$ and $T_{\alpha-\gamma,\beta-\gamma}T_{\alpha\gamma}=T_{\alpha\beta}$. Our target is to arrange that $F_\alpha=F_{\alpha-\beta}T_{\alpha\beta}$ whenever $\beta<\alpha$.

Start If $\alpha=1$ set $F(z)=z_0$.

Inductive step to an ordinal which is not indecomposable Suppose that $\gamma\le\beta<\alpha<\gamma+\alpha$. Then

$$\begin{aligned}F_{\alpha-\beta}T_{\alpha\beta}&=F_{\alpha-\beta}T_{\alpha-\gamma,\beta-\gamma}T_{\alpha\gamma}\\&=F_{(\alpha-\gamma)-(\beta-\gamma)}T_{\alpha-\gamma,\beta-\gamma}T_{\alpha\gamma}=F_{\alpha-\gamma}T_{\alpha\gamma}\end{aligned}$$

by the inductive hypothesis. We therefore can (and must) take $F_\alpha=F_{\alpha-\beta}T_{\alpha\beta}$ whenever $\beta<\alpha<\beta+\alpha$. We now find that if $\gamma<\alpha=\gamma+\alpha$, there is a δ such that $\delta<\alpha<\delta+\alpha$, and if we set $\beta=\gamma+\delta$ we have $\gamma<\beta<\alpha$ and

$$\begin{aligned}F_\alpha&=F_{\alpha-\beta}T_{\alpha\beta}=F_{(\alpha-\gamma)-(\beta-\gamma)}T_{\alpha-\gamma,\beta-\gamma}T_{\alpha\gamma}\\&=F_{\alpha-\delta}T_{\alpha\delta}T_{\alpha\gamma}=F_\alpha T_{\alpha\gamma}=F_{\alpha-\gamma}T_{\alpha\gamma}.\end{aligned}$$

So we have $F_\alpha=F_{\alpha-\gamma}T_{\alpha\gamma}$ for every $\gamma<\alpha$, as demanded by the inductive hypothesis.

Inductive step to an indecomposable ordinal Suppose that $\alpha\ge\omega$ is an indecomposable ordinal. Recall that every ordinal is expressible as a finite sum

$$\beta_0+\beta_1+\ldots+\beta_m$$

of non-zero indecomposable ordinals (allowing the empty string with sum 0), and that this expression is unique if we insist that $\beta_{i+1}\le\beta_i$ for $i<n$. Moreover, if we say that for two expressions of this type we write

$$(\beta_0+\beta_1+\ldots+\beta_m)\oplus(\gamma_0+\ldots+\gamma_n)=\delta_0+\ldots+\delta_{m+n+1}$$

where the δ_k enumerate the β_i and γ_j, with repetitions, in descending order (so that

$$\#(\{k:\delta_k=\alpha\})=\#(\{i:\beta_i=\alpha\})+\#(\{j:\gamma_j=\alpha\})$$

for every α), then On, with $\oplus$, is an abelian semigroup. Note that we always have $\beta+\gamma\le\beta\oplus\gamma$.

Because α is indecomposable, it is a subgroup of On. As an abelian semigroup, it is amenable, and there is a positive linear functional $F_\alpha:\ell^\infty(\alpha)\to\mathbb{R}$ such that $F_\alpha(\chi\alpha)=1$ and $F_\alpha S_\gamma=F_\alpha$ for every $\gamma<\alpha$, where

$$S_\gamma(\langle z_\xi\rangle_{\xi<\alpha})=\langle z_{\gamma\oplus\xi}\rangle_{\xi<\alpha}$$

whenever $\langle z_\xi\rangle_{\xi<\alpha}\in\ell^\infty(\alpha)$. I need to show that $F_\alpha T_{\alpha\gamma}=F_\alpha$ for every $\gamma<\alpha$.

case 1 Suppose that α is a successor indecomposable, that is, is the ordinal product $\beta\omega$ for an indecomposable β. Then $\beta n+\xi=\beta n\oplus\xi$ for every $n\in\mathbb{N}$ and $\xi<\alpha$, so $S_{\beta n}=T_{\alpha,\beta n}$ for every n. If $\gamma<\alpha$ there is an $m\in\mathbb{N}$ such that $\beta m\le\gamma<\beta(m+1)$ and $\gamma+\xi=\beta m+\xi$ whenever $\beta\le\xi<\alpha$. But this means that

$$S_\beta T_{\alpha\gamma} = S_\beta T_{\alpha,\beta m} = S_\beta S_{\beta m} = S_{\beta(m+1)}$$

and

$$F_\alpha T_{\alpha\gamma} = F_\alpha S_\beta T_{\alpha\gamma} = F_\alpha S_{\beta(m+1)} = F_\alpha.$$

case 2 Suppose that α is a limit of indecomposables. Take any $\gamma < \alpha$. Let β be an indecomposable ordinal such that $\gamma < \beta < \alpha$. Then $\gamma + \beta = \beta$ so $\gamma + \xi = \xi$ for every $\xi \geq \beta$ and $S_\beta T_{\alpha\gamma} = S_\beta$, so that

$$F_\alpha T_{\alpha\gamma} = F_\alpha S_\beta T_{\alpha\gamma} = F_\alpha S_\beta = F_\alpha.$$

Thus in either case we have the required property for F_α and the induction can proceed.

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