Version of 18.5.07

University of Essex Mathematics Department Research Report 92-10

The McShane and Birkhoff integrals of vector-valued functions

D.H.Fremlin

University of Essex, Colchester, England

I discuss the relationship of the McShane and Talagrand integrals to the Birkhoff integral.

Introduction A large number of different methods of integration of Banach-space-valued functions have been introduced, based on the various possible constructions of the Lebesgue integral. They commonly run fairly closely together when the range space is separable (or has w*-separable dual) and diverge more or less sharply for general range spaces. Here I discuss the McShane integral of [Fr95] and compare it with the Birkhoff integral of [Bi35].

The McShane or 'gauge-limit' integral of [McS83] was devised as a construction for the Lebesgue integral on $\mathbb{R}$. In [Go90] the same method is used to define an integral for vector-valued functions defined on $[0,1]$. Various adaptations are described in [McS83] for integrating real-valued functions defined on $\mathbb{R}^n$, $[0,1]^{\mathbb{N}}$ and similar spaces. In [Fr95] I show how, at the cost of significant extra techncal difficulties, the same essential idea can be used to define integrals for vector-valued functions defined on any of a wide variety of topological measure spaces, the 'outer regular σ-finite quasi-Radon' measure spaces.

The Birkhoff integral of [Bi35], on the other hand, does not involve a topology, and may be defined on any σ-finite measure space. The question nevertheless arises whether, in contexts in which the McShane integral is definable, it agrees with the Birkhoff integral. Because the Pettis integral extends both the McShane and Birkhoff integrals, they must agree on any function which is integrable by both methods; so the question is just whether they integrate the same functions. Here I show that if $(S,\mathfrak{T},\Sigma,\mu)$ is an outer regular σ-finite quasi-Radon measure space, and X is a Banach space, then

(i) any Birkhoff integrable function from S to X is McShane integrable (Proposition 4);

(ii) in general, not every McShane integrable function is Birkhoff integrable (Example 8);

(iii) but if the unit ball of the dual X^* of X is w*-separable (equivalently, if X is isometric to a subspace of $\ell^\infty(\mathbb{N})$), then a function from S to X is Birkhoff integrable iff it is McShane integrable (Theorem 10).

In passing I show also that a bounded Birkhoff integrable function on a probability space is Talagrand integrable (Corollary 7).

1. The McShane integral I use this name for a method of integrating vector-valued functions which is adapted from the integration process described in [McS83].

Definitions Let $(S,\mathfrak{T},\Sigma,\mu)$ be a non-empty σ-finite quasi-Radon measure space which is **outer regular**, that is, such that $\mu E=\inf\{\mu G: E\subseteq G\in\mathfrak{T}\}$ for every $E\in\Sigma$. A **generalized McShane partition** of S is a sequence $\langle(E_i,t_i)\rangle_{i\in\mathbb{N}}$ such that $\langle E_i\rangle_{i\in\mathbb{N}}$ is a disjoint family of measurable sets of finite measure, $\mu(S\setminus\bigcup_{i\in\mathbb{N}}E_i)=0$ and $t_i\in S$ for each i. A **gauge** on S is a function $\Delta:S\to\mathfrak{T}$ such that $s\in\Delta(s)$ for every $s\in S$. A generalized McShane partition $\langle(E_i,t_i)\rangle_{i\in\mathbb{N}}$ is **subordinate** to a gauge Δ if $E_i\subseteq\Delta(t_i)$ for every $i\in\mathbb{N}$.

Now let X be a Banach space. I will say that a function $\phi:S\to X$ is **McShane integrable**, with **McShane integral** w, if for every $\epsilon>0$ there is a gauge $\Delta:S\to\mathfrak{T}$ such that

$$\limsup_{n\to\infty}\|w-\sum_{i\le n}\mu E_i\cdot\phi(t_i)\|\le\epsilon$$

for every generalized McShane partition $\langle(E_i,t_i)\rangle_{i\in\mathbb{N}}$ of S subordinate to Δ.

2. Remarks For the elementary theory of quasi-Radon measure spaces see [Fr74], [Fr03], §415, and [Fr84]. For the elementary theory of the McShane integral, as defined here, see [Fr95]. I ought to remark that my version of the McShane integral is well-defined, in the sense that any given function has at most one value of the integral ([Fr95], 1Bd). Readers familiar with [McS83] will observe that my definition of the McShane integral is significantly different from (and more complex than) the most natural generalisations of the work in [McS83]; a much simpler expression is used in [Go90], [FM94] and [Fr94] for the case $S=[0,1]$.

The equivalence of the two definitions in this special case is proved in [Fr95], 1E-1G. In [Fr95], 1K and 3B, I showed that Bochner integrable functions are McShane integrable and McShane integrable functions are Pettis integrable; in particular, if $X = \mathbb{R}$ then the McShane integral agrees with the ordinary integral.

3. The Birkhoff integral Now let (S, Σ, μ) be a σ-finite measure space and X a Banach space. A function $\phi : S \to X$ is **Birkhoff integrable**, with **Birkhoff integral** w, if for every $\epsilon > 0$ there is a partition $\langle E_i \rangle_{i\in\mathbb{N}}$ of S into measurable sets of finite measure such that $\sum_{i\in\mathbb{N}} \mu E_i \cdot \phi(t_i)$ is unconditionally summable and $\|w - \sum_{i\in\mathbb{N}} \mu E_i \cdot \phi(t_i)\| \le \epsilon$ whenever $\langle t_i \rangle_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} E_i$.

(This wording leads to formal difficulties if there is no partition of S into infinitely many non-empty measurable sets. But that case is of course quite trivial, and I will pass it over throughout.)

For the elementary properties of this integral see [Bi35]. In particular, a Birkhoff integrable function is Pettis integrable (see [Bi35], Theorem 14); consequently the McShane and Birkhoff integrals agree where they are both defined.

4. I come now to the first new result of this note.

Proposition Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space, and X a Banach space. If $\phi : S \to X$ is Birkhoff integrable it is McShane integrable.

proof Let $\phi : S \to X$ be a Birkhoff integrable function, with Birkhoff integral w; let $\epsilon > 0$. Let $\langle E_i \rangle_{i\in\mathbb{N}}$ be a partition of S into measurable sets of finite measure such that $\|w - \sum_{i\in\mathbb{N}} \mu E_i \cdot \phi(t_i)\| \le \epsilon$ whenever $\langle t_i \rangle_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} E_i$. For each $i \in \mathbb{N}$ let C_i be the closed convex hull of $\phi[E_i]$ in X; then $\|w - \sum_{i\in\mathbb{N}} \mu E_i . x_i\| \le \epsilon$ whenever $\langle x_i \rangle_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} C_i$ (see the arguments of [Bi35], in particular his Theorem 3). Set $\gamma_i = 1 + \sup_{x\in C_i} \|x\|$; then $\gamma_i < \infty$ whenever $\mu E_i > 0$. Because $\sum_{i\in\mathbb{N}} \mu E_i \cdot \phi(t_i)$ is unconditionally summable whenever $t_i \in E_i$ for every i, there must be an $m \in \mathbb{N}$ such that $\|\sum_{i\in I} \mu E_i \cdot \phi(t_i)\| \le \epsilon$ whenever $I \subseteq \mathbb{N} \setminus m$ is finite and $t_i \in E_i$ for every i; now $\|\sum_{i\in I} \mu E_i . x_i\| \le \epsilon$ whenever $I \subseteq \mathbb{N} \setminus m$ is finite and $x_i \in C_i$ for every $i \in I$.

Set $K = \{i : i \in \mathbb{N}, \mu E_i > 0\}$, and for $i \in K$ choose a closed set F_i and an open set G_i such that $F_i \subseteq E_i \subseteq G_i$ and $\mu(G_i \setminus F_i) < \min(\mu E_i, 2^{-i}\epsilon/\gamma_i)$. Write $E = \bigcup_{i\in\mathbb{N}\setminus K} E_i$, so that $\mu E = 0$, and for each $i \in \mathbb{N}$ choose an open set $G_i' \supseteq E$ such that $2^i(i + 1)\mu G_i' \le \epsilon$. For $t \in S$ define $\Delta(t) \in \mathfrak{T}$ by saying

if $i \in K$, $t \in E_i$ then $\Delta(t) = G_i \setminus \bigcup\{F_j : j \in K \cap m, j \ne i\}$,

if $t \in E$, $i \le \|\phi(t)\| < i + 1$ then $\Delta(t) = G_i' \setminus \bigcup\{F_j : j \in K \cap m\}$.

Suppose that $\langle (H_j, t_j) \rangle_{j\in\mathbb{N}}$ is a generalised McShane partition subordinate to Δ. Let $M \in \mathbb{N}$ be such that

$$\mu(F_i \cap \bigcup_{j<M} H_j) \ge \mu E_i - 2^{-i}\epsilon/\gamma_i$$

for every $i \in K \cap m$; take any $k \ge M$. For $i \in K$, set $J_i = \{j : j < k, t_j \in E_i\}$, $\tilde{E}_i = \bigcup\{H_j : j \in J_i\} \subseteq G_i$; for $i \in \mathbb{N}$ set $J_i' = \{j : j < k, t_j \in E, i \le \|\phi(t_j)\| < i + 1\}$, $\tilde{E}_i' = \bigcup\{H_j : j \in J_i'\} \subseteq G_i'$. Note that if $i \in K \cap m$, $j < k$ then $F_i \cap H_j = \emptyset$ unless $j \in J_i$, so that

$$\mu\tilde{E}_i \ge \mu(F_i \cap \tilde{E}_i) = \mu(F_i \cap \bigcup_{j<k} H_j) \ge \mu E_i - 2^{-i}\epsilon/\gamma_i,$$

while of course $\mu\tilde{E}_i \le \mu G_i \le \mu E_i + 2^{-i}\epsilon/\gamma_i$, so $\gamma_i|\mu\tilde{E}_i - \mu E_i| \le 2^{-i}\epsilon$.

Now observe that, for $i \in K$, there is an $x_i \in C_i$ such that $\sum_{j\in J_i} \mu H_j \cdot \phi(t_j) = \mu\tilde{E}_i . x_i$. If $i \in K \cap m$, we have $\|\mu\tilde{E}_i . x_i - \mu E_i . x_i\| \le 2^{-i}\epsilon$. But also, because $|\mu\tilde{E}_i - \mu E_i| \le \mu E_i$ for every $i \in K$, we have, for any finite $J \subseteq K \setminus m$,

$$\begin{aligned}\|\sum_{i\in J}(\mu\tilde{E}_i - \mu E_i)x_i\| &\le 2\sup\{\|\sum_{i\in I} \mu E_i . y_i\| : I \subseteq J,\ y_i \in C_i\ \ \forall\ i \in I\}\\ &\le 2\epsilon,\end{aligned}$$

by the choice of m. Consequently, for any $l \in \mathbb{N}$,

$$\|\textstyle\sum_{i\in l\cap K} \mu\tilde{E}_i . x_i - \mu E_i . x_i\| \le 2\epsilon + \sum_{i<m} 2^{-i}\epsilon \le 4\epsilon.$$

On the other hand,

$$\textstyle\sum_{j<k, t_j\in E} \|\mu H_j \cdot \phi(t_j)\| \le \sum_{i\in\mathbb{N}} \mu E_i'(i + 1) \le \sum_{i\in\mathbb{N}} \mu G_i'(i + 1) \le 2\epsilon.$$

So if we take x_i to be an arbitrary member of C_i for $i \in \mathbb{N} \setminus K$, we see that for any finite $J \subseteq \mathbb{N} \setminus K$ we have

$$\|\textstyle\sum_{j<k, t_j\in E} \mu H_j \cdot \phi(t_j) - \sum_{i\in J} \mu E_i . x_i\| \le 2\epsilon.$$

Let $l \in \mathbb{N}$ be so large that $t_j \in \bigcup_{i<l} E_i$ for every $j < k$. Then

$$\textstyle\sum_{i\in l\cap K} \mu\tilde{E}_i.x_i = \sum_{j<k, t_j\notin E} \mu H_j \cdot \phi(t_j),$$

so

$$\textstyle\|\sum_{j<k} \mu H_j \cdot \phi(t_j) - \sum_{i<l} \mu E_i.x_i\| \le 6\epsilon.$$

This is true for all l large enough. But

$$\textstyle\|w - \sum_{i\in\mathbb{N}} \mu E_i.x_i\| \le \epsilon,$$

by the original choice of $\langle E_i\rangle_{i\in\mathbb{N}}$, so

$$\textstyle\|w - \sum_{j<k} \mu H_j \cdot \phi(t_j)\| \le 7\epsilon.$$

And this is true for all $k \ge M$.

Referring to the definitions in §1, we see that ϕ is McShane integrable, with integral w.

5. Properly measurable functions I come now to a result connecting the Birkhoff and Talagrand integrals. Recall that if (S, Σ, μ) is a probability space, a set A of real-valued functions on S is **stable** (in Talagrand's terminology) if for every $E \in \Sigma$ with $\mu E > 0$, and all real numbers $\alpha < \beta$, there are m, $n \ge 1$ such that $\mu^*_{m+n} Z(A, E, m, n, \alpha, \beta) < (\mu E)^{m+n}$, where throughout the rest of this paper I write $Z(A, E, m, n, \alpha, \beta)$ for

$$\{(t, u) : t \in E^m,\ u \in E^n,\ \exists\ f \in A,\ f(t(i)) \le \alpha\ \forall\ i < m,\ f(u(j)) \ge \beta\ \forall\ j < n\},$$

and μ^*_{m+n} is the ordinary product outer measure on $S^m \times S^n$. Now if X is a Banach space, a function $\phi : S \to X$ is **properly measurable** if $\{h\phi : h \in X^*, \|h\| \le 1\}$ is stable. For the elementary properties of stable sets, see [Ta84]. In particular, if $A \subseteq \mathbb{R}^S$ is stable, then every function in A is measurable for the completion of the measure μ.

For the purposes of this note, we do not need the ordinary definition of the Talagrand integral ([FM94], 1Ab); all we need to know is that a function $\phi : S \to X$ is Talagrand integrable iff it is properly measurable and the upper integral $\overline{\int}\|\phi\|$ is finite (see [Ta87], Theorem 8), and that in this case its Talagrand integral is equal to its Pettis integral.

6. Proposition If (S, Σ, μ) is a probability space, X is a Banach space, and $\phi : S \to X$ is a Birkhoff integrable function, then ϕ is properly measurable.

proof Take $E \in \Sigma$, with $\mu E > 0$, and $\alpha < \beta$ in $\mathbb{R}$. Set $\epsilon = \frac{1}{3}(\beta - \alpha)\mu E > 0$. Let $\langle E_i\rangle_{i\in\mathbb{N}}$ be a partition of S into measurable sets such that

$$\textstyle\|w - \sum_{i\in\mathbb{N}} \mu E_i \cdot \phi(t_i)\| \le \epsilon$$

whenever $t_i \in E_i$ for every i, where w is the Birkhoff integral of ϕ. Because any rearrangement of $\langle E_i\rangle_{i\in\mathbb{N}}$ will have the same property, I may suppose that there is an $n \in \mathbb{N}$ such that

$$\textstyle\mu(E \cap \bigcup_{i<n} E_i) > \frac{2\epsilon}{\beta-\alpha}$$

and $\mu(E_i \cap E) > 0$ for every $i < n$. Now we must have

$$\textstyle\|\sum_{i<n} \mu E_i(\phi(t(i)) - \phi(u(i)))\| \le 2\epsilon$$

whenever $t(i)$, $u(i) \in E_i$ for $i < n$.

Set

$$\textstyle W = (\prod_{i<n} E_i \cap E) \times (\prod_{i<n} E_i \cap E) \subseteq E^n \times E^n.$$

Then $\mu_{2n}(W) > 0$. Suppose, if possible, that we have a point $(t, u) \in W \cap Z(A, E, n, n, \alpha, \beta)$, where $A = \{h\phi : h \in X^*, \|h\| \le 1\}$. Then there is an $h \in X^*$ such that $\|h\| \le 1$ and $h(\phi(t(i)i)) \le \alpha$, $h(\phi(u(i))) \ge \beta$ for every $i < n$. Now

$$\begin{aligned}2\epsilon &\ge \|\sum_{i<n} \mu E_i(\phi(u(i)) - \phi(t(i)))\| \ge h(\sum_{i<n} \mu E_i(\phi(u(i)) - \phi(t(i))))\\ &\ge (\beta - \alpha)\sum_{i<n} \mu E_i \ge (\beta - \alpha)\mu(E \cap \bigcup_{i<n} E_i) > 2\epsilon,\end{aligned}$$

which is absurd.

So $W \cap Z(A, E, n, n, \alpha, \beta) = \emptyset$ and $\mu_{2n}^* Z(A, E, n, n, \alpha, \beta) < \mu_{2n}(E^n \times E^n)$. Because α, β and E are arbitrary, ϕ is properly measurable.

7. Corollary A bounded Birkhoff integrable function on a probability space is Talagrand integrable.

8. From §§5 and 7 we see that a bounded Birkhoff integrable function on a quasi-Radon probability space is McShane integrable and Talagrand integrable. The converse is not true, as the following example shows.

Example There is a bounded function $\phi : [0,1] \to \ell^\infty(\mathfrak{c})$ which is McShane integrable and Talagrand integrable but not Birkhoff integrable.

proof Enumerate as $\langle\langle E_{\xi n}\rangle_{n\in\mathbb{N}}\rangle_{\xi<\mathfrak{c}}$ the family of all sequences of Borel sets in $[0,1]$ of positive Lebesgue measure. Construct inductively a family $\langle s_\xi\rangle_{\xi<\mathfrak{c}}$ in $[0,1]$ such that (i) $s_\xi - s_\eta$ is irrational whenever $\eta < \xi < \mathfrak{c}$ (ii) $E_{\xi n} \cap (s_\xi + \mathbb{Q}) \neq \emptyset$ for every $n \in \mathbb{N}$, $\xi < \mathfrak{c}$. (This is possible because if $\mu E > 0$ then $\mu([0,1] \cap (E - \mathbb{Q})) = 1$, where μ is Lebesgue measure.)

Define $\phi : [0,1] \to \ell^\infty(\mathfrak{c})$ by saying that $\phi(s) = e_\xi$ if $s \in s_\xi + \mathbb{Q}$, while $\phi(s) = 0$ if $s \notin \bigcup_{\xi<\mathfrak{c}}(s_\xi + \mathbb{Q})$; here $e_\xi \in \ell^\infty(\mathfrak{c})$ is the usual unit vector, with $e_\xi(\eta) = 1$ if $\eta = \xi$, 0 otherwise. Because $\mu(s_\xi + \mathbb{Q}) = 0$ for every ξ, ϕ is McShane integrable, with McShane integral 0, just as in [FM94], 3A. To see that ϕ is properly measurable, take $E \subseteq [0,1]$ with $\mu E > 0$, and $\alpha < \beta$ in $\mathbb{R}$. Then either $\alpha < 0$ or $\beta > 0$; suppose that $\beta > 0$. Take $n > 1/\beta$. Let $(t, u) \in Z(A, E, n, n, \alpha, \beta)$, where $A = \{h\phi : h \in (\ell^\infty(\mathfrak{c}))^*, \|h\| \le 1\}$. Let $h \in (\ell^\infty(\mathfrak{c}))^*$ be such that $\|h\| \le 1$ and $h(\phi(t(i))) \le \alpha$, $h(\phi(u(i))) \ge \beta$ for $i < n$. Then $\#(\{\xi : h(e_\xi) \ge \beta\}) < n$; so there must be distinct i, $j < n$ such that $\phi(u(i)) = \phi(u(j))$ and $u(i) - u(j) \in \mathbb{Q}$. Accordingly

$$Z(A, E, n, n, \alpha, \beta) \subseteq \{(t, u) : \exists\ i < j \text{ such that } u(i) - u(j) \in \mathbb{Q}\},$$

and $\mu_{2n}(Z(A, E, n, n, \alpha, \beta)) = 0$. The same argument applies if $\alpha < 0$.

Accordingly ϕ is McShane integrable and Talagrand integrable. But it is not Birkhoff integrable, for if $\langle E_i\rangle_{i\in\mathbb{N}}$ is any partition of $[0,1]$ into non-empty measurable sets, there must be a $\xi < \mathfrak{c}$ such that $s_\xi + \mathbb{Q}$ meets E_i for every i such that $\mu E_i > 0$. Taking $\langle t_i\rangle_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} E_i$ such that $t_i - s_\xi \in \mathbb{Q}$ whenever $\mu E_i > 0$, we see that

$$\|\textstyle\sum_{i\in\mathbb{N}} \mu E_i \cdot \phi(t_i)\| \ge \sum_{i\in\mathbb{N}} \mu E_i \cdot \phi(t_i)(\xi) = 1.$$

Accordingly 0, at least, cannot be the Birkhoff integral of ϕ; but as 0 is certainly the Pettis integral of ϕ, ϕ is not Birkhoff integrable.

9. The work of [FM94] makes it plain that arbitrary Banach spaces give rise to many difficulties which do not arise in separable spaces, or even in spaces with w^*-separable dual unit balls (note that these are precisely the spaces which are isometric to closed subspaces of $\ell^\infty(\mathbb{N})$). If we accept limitations of this kind, the situation changes.

It will be helpful to begin with a lemma on the Birkhoff integral.

Lemma Let (S, Σ, μ) be a σ-finite measure space, X a Banach space, $\phi : S \to X$ a function, and $\langle S_i\rangle_{i\in\mathbb{N}}$ a cover of S by measurable sets. Then ϕ is Birkhoff integrable iff it is Pettis integrable and $\phi\restriction S_i : S_i \to X$ is Birkhoff integrable for every $i \in \mathbb{N}$.

proof (a) If ϕ is Birkhoff integrable, then $\phi\restriction E$ is Birkhoff integrable for every $E \in \Sigma$ ([Bi35], Theorem 14); consequently, ϕ is Pettis integrable.

(b) Now suppose that ϕ is Pettis integrable and that $\phi\restriction S_i$ is Birkhoff integrable for every i. Let $\epsilon > 0$. Set $S_i' = S_i \setminus \bigcup_{j<i} S_j$; then $\phi\restriction S_i'$ is Birkhoff integrable, by (a), for each $i \in \mathbb{N}$. For $E \in \Sigma$ write νE for the Pettis integral of ϕ over E; then $\nu : \Sigma \to X$ is countably additive ([Ta84], 4-2-2). For each i, let $\langle E_{ij}\rangle_{j\in\mathbb{N}}$ be a partition of S_i' into sets of finite measure such that $\sum_{j\in\mathbb{N}} \mu E_{ij} \cdot \phi(t_j)$ is unconditionally summable and $\|\nu(S_i') - \sum_{j\in\mathbb{N}} \mu E_{ij} \cdot \phi(t_j)\| \le 2^{-i}\epsilon$ whenever $\langle t_j\rangle_{j\in\mathbb{N}} \in \prod_{j\in\mathbb{N}} E_{ij}$. (I leave it to you to find appropriate phrasing to deal with the case in which S_i' has no partition into a sequence of non-empty measurable sets.) Take $t_{ij} \in E_{ij}$ for all i, $j \in \mathbb{N}$.

Let $\delta > 0$. Then there is an $n \ge 1$ such that $2^{-n}\epsilon \le \delta$ and $\|\nu E\| \le \delta$ whenever $E \in \Sigma$ and $E \cap \bigcup_{i<n} S_i' = \emptyset$. There is an $m \in \mathbb{N}$ such that $\|\sum_{j\in J} \mu E_{ij} \cdot \phi(t_{ij})\| \le \delta/n$ whenever $i < n$ and $J \subseteq \mathbb{N} \setminus m$ is finite. Now if $I \subseteq (\mathbb{N} \times \mathbb{N}) \setminus (n \times m)$ is finite, set $J_i = \{j : (i, j) \in I\}$, $F_i = \bigcup_{j\in J_i} E_{ij}$ for $i \in \mathbb{N}$, and $F = \bigcup_{i\ge n} F_i$. We have

$$\begin{aligned}\|\sum_{(i,j)\in I}\mu E_{ij}\cdot\phi(t_{ij})\| &\le \sum_{i<n}\|\sum_{j\in J_i}\mu E_{ij}\cdot\phi(t_{ij})\| + \|\sum_{i\ge n}\sum_{j\in J_i}\mu E_{ij}\cdot\phi(t_{ij})\|\\ &\le n\delta/n + \sum_{i\ge n}\|\nu F_i - \sum_{j\in J_i}\mu E_{ij}\cdot\phi(t_{ij})\| + \|\nu F\|\\ &\le \delta + \sum_{i\ge n} 2^{-i}\epsilon + \delta \le 4\delta.\end{aligned}$$

As δ is arbitrary, $w=\sum_{(i,j)\in\mathbb{N}}\mu E_{ij}\cdot\phi(t_{ij})$ exists as an unconditional sum. Now

$$\|\nu S - w\| \le \sum_{i\in\mathbb{N}}\|\nu S'_i - \sum_{j\in\mathbb{N}}\mu E_{ij}\cdot\phi(t_{ij})\| \le 2\epsilon;$$

as ϵ is arbitrary, ϕ is Birkhoff integrable.

10. Theorem Let $(S,\mathfrak{T},\Sigma,\mu)$ be an outer regular σ-finite quasi-Radon measure space, and X a Banach space such that the unit ball of X^* is w*-separable. Then a function $\phi : S\to X$ is Birkhoff integrable iff it is McShane integrable.

proof (a) We already know (Prop. 4) that if ϕ is Birkhoff integrable it is McShane integrable; so for the rest of this proof I assume that ϕ is McShane integrable and seek to prove that it is Birkhoff integrable. Now the sets

$$\{t : \|\phi(t)\|\le m\} = \bigcap_{n\in\mathbb{N}}\{t : |f_n(\phi(t))|\le m\},$$

where $\langle f_n\rangle_{n\in\mathbb{N}}$ runs over a w*-dense subset of the unit ball of X^*, are all measurable. So we can cover S by a sequence of sets of finite measure on each of which ϕ is bounded, and moreover is either an atom for μ or atomless. By Lemma 9 it will be enough to prove that ϕ is Birkhoff integrable over each of these. Now the atoms are trivial (because on an atom for μ, ϕ is constant except on a negligible set), so we are reduced to the case in which ϕ is bounded and S is atomless and totally finite. Evidently we may suppose further that $\mu S = 1$ and that $\|\phi(t)\|\le 1$ for every t. Set $h_n = f_n\phi : S\to\mathbb{R}$ for each n, and let w be the McShane integral of ϕ. Suppose that $0<\epsilon\le\frac{1}{6}$.

(b) By 4-1-5 and 4-1-6 of [Ta84], $\{h_n : n\in\mathbb{N}\}$ is totally bounded for the seminorm $\|\;\|_1$. So there is an $r\in\mathbb{N}$ such that

$$\forall\ i\in\mathbb{N}\ \exists\ l_i\le r,\ \int|h_i - h_{l_i}|\le\epsilon^4.$$

Set

$$g_i = h_i - h_{l_i},\qquad F_i = \{t : |g_i(t)|\ge\epsilon^2\}$$

so that $\mu F_i\le\epsilon^2$ for $i\in\mathbb{N}$.

Let $\Delta : S\to\mathfrak{T}$ be a gauge such that

$$\limsup_{n\to\infty}\|w - \sum_{i\le n}\mu E_i\cdot\phi(t_i)\|\le\epsilon^4$$

for every generalized McShane partition $\langle(E_i,t_i)\rangle_{i\in\mathbb{N}}$ subordinate to Δ. Set

$$G_i = \bigcup\{\Delta(t) : t\in F_i\}$$

for each $i\in\mathbb{N}$.

(c) We find that $\mu G_i\le 6\epsilon^2$ for each i. To see this, start by setting $F'_i = \{t : g_i(t)\ge\epsilon^2\}$, $G'_i = \bigcup\{\Delta(t) : t\in F'_i\}$. Then we can find a generalized McShane partition $\langle(E_j,t_j)\rangle_{j\in\mathbb{N}}$ of S, subordinate to Δ, such that $\mu(G'_i\setminus E) = 0$, where

$$E = \bigcup\{E_j : j\in\mathbb{N},\ t_j\in F'_i\}.$$

The arguments of [Fr95], Theorem 1N, make it plain that if w_E is the integral of ϕ over E, then

$$\limsup_{n\to\infty}\|w_E - \sum_{j\in J, j\le n}\mu E_j\cdot\phi(t_j)\|\le\epsilon^4,$$

where $J = \{j : t_j\in F'_i\}$. Consequently

$$\limsup_{n\to\infty}|\int_E h_i - \sum_{j\in J, j\le n}\mu E_j\cdot h_i(t_j)|\le\epsilon^4$$

for every m. Now $g_i = h_i - h_{l_i}$, so that

$$\limsup_{n\to\infty} |\int_E g_i - \sum_{j\in J, j\le n} \mu E_j \cdot g_i(t_j)| \le 2\epsilon^4,$$

and

$$\epsilon^2 \mu G_i' \le 2\epsilon^4 + |\int_E g_i| \le 3\epsilon^4.$$

Accordingly $\mu G_i' \le 3\epsilon^2$. Similarly,

$$\mu(\bigcup\{\Delta(t) : g_i(t) \le -\epsilon^2\}) \le 3\epsilon^2,$$

and $\mu G_i \le 6\epsilon^2$.

(d) For $t \in S$ set

$$\tilde{\Delta}(t) = \bigcap\{G_i : i \in \mathbb{N},\, t \in F_i\} \supseteq \Delta(t).$$

Then whenever $\mu V > 0$ there is a $W \subseteq V$ such that $\mu W > 0$ and

$$\mu\{t : t \in W,\, \mu(W \cap \tilde{\Delta}(t)) \le \epsilon \mu W\} \le 7\epsilon \mu W.$$

For note first that if $E \in \Sigma$ then the function

$$t \mapsto \mu(E \cap \tilde{\Delta}(t)) = \lim_{n\to\infty} \mu(E \cap \bigcap\{G_i : i < n,\, t \in F_i\})$$

is measurable. Now set

$$\Delta_n(t) = V \cap \bigcap\{G_i : i < n,\, t \in F_i\}$$

for $t \in V$, $n \in \mathbb{N}$. Note that $\mu(V \cap \Delta(t)) > 0$ for almost all $t \in V$ (because $\Delta(t)$ is open and contains t), so we can find an n such that $\mu V_1 > 0$, where

$$V_1 = \{t : t \in V,\, \mu(\Delta_n(t)) \le 2\mu(V \cap \tilde{\Delta}(t))\}.$$

For $I \subseteq n$ set

$$H_I = V \cap \bigcap_{i\in I} G_i, \quad H_I' = \{t : t \in V_1,\, I = \{i : i < n,\, t \in F_i\}\}.$$

Then $H_I' \subseteq H_I$ and $V_1 = \bigcup_{I\subseteq n} H_I'$; fix $I \subseteq n$ such that $\mu H_I' > 0$. Note that $\Delta_n(t) = H_I$ for every $t \in H_I'$.

Because S is atomless, we can choose a sequence $\langle W_k\rangle_{k\in\mathbb{N}}$ of subsets of S, independent of each other and of H_I, and all of measure $6\epsilon\mu H_I'/\mu H_I$. Then for any $t \in H_I'$,

$$\begin{aligned}\lim_{k\to\infty} \mu(W_k \cap H_I \cap \tilde{\Delta}(t)) &= 6\epsilon\mu(H_I \cap \tilde{\Delta}(t))\mu H_I'/\mu H_I \\ &= 6\epsilon\mu(V \cap \tilde{\Delta}(t))\mu H_I'/\mu(\Delta_n(t)) \ge 3\epsilon\mu H_I'.\end{aligned}$$

So there is a $k \in \mathbb{N}$ such that

$$\mu\{t : t \in H_I',\, \mu(W_k \cap H_I \cap \tilde{\Delta}(t)) \le 2\epsilon\mu H_I'\} \le \epsilon\mu H_I'.$$

Try

$$W = H_I' \cup (W_k \cap H_I).$$

Then

$$\mu(W \setminus H_I') \le \mu(W_k \cap H_I) = \mu W_k \cdot \mu H_I = 6\epsilon\mu H_I' \le \mu H_I', \quad \mu W \le 2\mu H_I'$$

and

$$\begin{aligned}&\mu\{t : t \in W,\, \mu(W \cap \tilde{\Delta}(t)) \le \epsilon\mu W\} \\ &\qquad \le \mu\{t : t \in W,\, \mu(W_k \cap H_I \cap \tilde{\Delta}(t)) \le 2\epsilon\mu H_I'\} \\ &\qquad \le \mu(W \setminus H_I') + \mu\{t : t \in H_I',\, \mu(W_k \cap H_I \cap \tilde{\Delta}(t)) \le 2\epsilon\mu H_I'\} \\ &\qquad \le 7\epsilon\mu H_I' \le 7\epsilon\mu W.\end{aligned}$$

(e) Consequently there is a partition $\langle W_j\rangle_{j\in\mathbb{N}}$ of S into measurable sets such that

$$\mu\{t : t \in W_j,\, \mu(W_j \cap \tilde{\Delta}(t)) \le \epsilon\mu W_j\} \le 7\epsilon\mu W_j$$

for every $j \in \mathbb{N}$.

Set

$$Z = \bigcup_{j\in\mathbb{N}}\{t : t \in W_j,\ \mu(W_j \cap \tilde{\Delta}(t)) \le \epsilon\mu W_j\},$$

so that $\mu Z \le 7\epsilon$. For $j \in \mathbb{N}$ set $W'_j = W_j \setminus Z$.

Now suppose that $t \in Z$, $t_j \in W'_j$ for $j \in \mathbb{N}$. Take $i \in \mathbb{N}$ and examine

$$|\mu Z \cdot g_i(t) + \sum_{j\in\mathbb{N}} \mu W'_j \cdot g_i(t_j)| \le 2\mu Z + \epsilon^2\mu S + 2\sum_{j\in J} \mu W'_j,$$

where $J = \{j : t_j \in F_i\}$. But if $j \in J$ then $t_j \in W_j \setminus Z$ so $\mu(W_j \cap \tilde{\Delta}(t_j)) \ge \epsilon\mu W_j$. Also $t_j \in F_i$ so $\tilde{\Delta}(t_j) \subseteq G_i$ and $\mu(W_j \cap G_i) \ge \epsilon\mu W_j$. Accordingly $\sum_{j\in J} \mu W_j \le \frac{1}{\epsilon}\mu G_i \le 6\epsilon$, and

$$|\mu Z \cdot g_i(t) + \sum_{j\in\mathbb{N}} \mu W'_j \cdot g_i(t_j)| \le 27\epsilon.$$

(f) It follows that

$$|\sum_{j\in\mathbb{N}} \mu E_j \cdot g_i(t_j)| \le 27\epsilon$$

whenever $\langle E_j\rangle_{j\in\mathbb{N}}$ is a partition of S into measurable sets refining $\{Z\} \cup \{W'_j : j \in \mathbb{N}\}$, $i \in \mathbb{N}$ and $t_j \in E_j$ for every j. But of course we can find such a partition such that

$$|\int h_l - \sum_{j\in\mathbb{N}} \mu E_j \cdot h_l(t_j)| \le \epsilon$$

whenever $l \le r$ and $t_j \in E_j$ for every j. If now we take any $i \in \mathbb{N}$, we shall have $h_i = h_{l_i} + g_i$, so if $t_j \in E_j$ for every j then

$$\begin{aligned}\Bigl|\int h_i - \sum_{j\in\mathbb{N}} \mu E_j \cdot h_i(t_j)\Bigr| &\le \Bigl|\int h_{l_i} - \sum_{j\in\mathbb{N}} \mu E_j \cdot h_{l_i}(t_j)\Bigr| + \Bigl|\int g_i\Bigr| + \Bigl|\sum_{j\in\mathbb{N}} \mu E_j \cdot g_i(t_j)\Bigr| \\ &\le \epsilon + \epsilon^4 + 27\epsilon \le 29\epsilon.\end{aligned}$$

Accordingly, if $t_j \in E_j$ for every j,

$$\|w - \sum_{j\in\mathbb{N}} \mu E_j \cdot \phi(t_j)\| \le 29\epsilon,$$

and the sum is unconditional because $\sum_{j\in\mathbb{N}} \mu E_j\|\phi(t_j)\| \le 1$. As ϵ is arbitrary, w is the Birkhoff integral of ϕ.

Remark Compare [So05].

11. Corollary Let (S, Σ, μ) be a σ-finite measure space, and $\mathfrak{T}_1$, $\mathfrak{T}_2$ two topologies on S rendering it an outer regular quasi-Radon measure space. Then for any Banach space X with w*-separable dual unit ball, the McShane integrable functions from S to X are the same for both topologies.

Remark This is a partial answer to Problem 3Hb of [Fr95].

Acknowledgements Conversations and correspondence with J.Rodriguez.

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