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Supplement to 'Sequential convergence in $C_p(X)$'

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I give details of some examples and further results relevant to [Frp92]. For notation see [Frp92]. Some of these results have been previously circulated in [Frn91].

1. I claimed ([Frp92], §14) that if the continuum hypothesis is true, then there is an s_1-space $X \subseteq \mathbb{R}$ such that $X \setminus \mathbb{Q}$ is not an s_1-space. The construction is as follows.

1A Lemma Let $\langle W_r\rangle_{r\in\mathbb{N}}$ be a sequence of perfect subsets of $\mathbb{R}$. Suppose that $W_r \cap \mathbb{Q}$ is dense in W_r and $y_r \in W_r$ for each r. Let $\langle g_{mn}\rangle_{m,n\in\mathbb{N}}$ be a double sequence of continuous real-valued functions on $W = \bigcup_{r\in\mathbb{N}} W_r$ such that $\lim_{n\to\infty} g_{mn}(w) = 0$ for every $m \in \mathbb{N}$, $w \in W$. Then there are $\langle k(m)\rangle_{m\in\mathbb{N}}$, $\langle W'_r\rangle_{r\in\mathbb{N}}$ such that

$k(m) \in \mathbb{N}$ for each m;

$y_r \in W'_r \subseteq W_r$, W'_r is perfect, $W'_r \cap \mathbb{Q}$ is dense in W'_r for each r;

$\lim_{m\to\infty} g_{m,k(m)}(w) = 0$ for all $w \in \bigcup_{r\in\mathbb{N}} W'_r$.

proof Given $m \in \mathbb{N}$, $\langle k(r)\rangle_{r<m}$ and a family $\langle F_{mr}\rangle_{r<m}$ of finite sets such that $F_{mr} \subseteq W_r$ for each $r < m$, $|g_{j,k(j)}(y)| < 2^{-j}$ for $r \le j < m$, $y \in F_{mr}$, take $k(m) \in \mathbb{N}$ such that

$$|g_{m,k(m)}(y)| < 2^{-m} \text{ for all } y \in \{y_m\} \cup \bigcup_{r<m} F_{mr},$$

and for $r \le m$ choose a finite set $F_{m+1,r} \subseteq W_r$ such that

$y_m \in F_{m+1,m}$,

$F_{m+1,r} \subseteq \{y : y \in W_r,\ |g_{j,k(j)}(y)| < 2^{-j} \text{ if } r \le j \le m\}$

if $r \le m$, and if $r < m$ then

$F_{mr} \subseteq F_{m+1,r}$,

$\forall\ q \in F_{mr}\ \exists\ q' \in F_{m+1,r} \cap \mathbb{Q},\ 0 < |q - q'| \le 2^{-m}$.

On completing this construction, set

$$W'_r = \overline{\bigcup_{m>r} F_{mr}};$$

then $|g_{m,k(m)}|(y) \le 2^{-m}$ whenever $m > r$, $y \in W'_r$, while $y_r \in W'_r \subseteq W_r$ for each r.

1B Proposition Assume CH. Then there is a set $X \subseteq \mathbb{R}$ such that X is an s_1-space but $X \setminus \mathbb{Q}$ is not an s_1-space.

proof (a) Start by enumerating as $\langle (E_\xi, \langle g^{(\xi)}_{mn}\rangle_{m,n\in\mathbb{N}})\rangle_{\xi<\mathfrak{c}}$ the family of all pairs $(E, \langle g_{mn}\rangle_{m,n\in\mathbb{N}})$ such that E is a Borel subset of $\mathbb{R}$, $g_{mn} : E \to \mathbb{R}$ is continuous for every m, $n \in \mathbb{N}$ and $\lim_{n\to\infty} g_{mn}(y) = 0$ for every $y \in E$, $m \in \mathbb{N}$. Enumerate as $\langle V_\xi\rangle_{\xi<\mathfrak{c}}$ the family of G_δ subsets of $\mathbb{R}$ including $\mathbb{Q}$. Write $\mathcal{W}$ for the family of perfect non-empty subsets W of $\mathbb{R}$ such that $W \cap \mathbb{Q}$ is dense in W.

(b) I am to construct inductively $\langle Y_\xi\rangle_{\xi<\mathfrak{c}}$, $\langle \mathcal{W}_\xi\rangle_{\xi<\mathfrak{c}}$, $\langle\langle k_\xi(m)\rangle_{m\in\mathbb{N}}\rangle_{\xi<\mathfrak{c}}$ such that

(i) $\langle Y_\xi\rangle_{\xi<\mathfrak{c}}$ is an increasing family of countable subsets of $\mathbb{R}$, with $Y_0 = \mathbb{Q}$;

(ii) $\mathcal{W}_\xi$ is a countable subset of $\mathcal{W}$ for each ξ, and $\bigcup \mathcal{W}_\xi \supseteq \bigcup \mathcal{W}_\eta \supseteq Y_\eta$ if $\xi \le \eta < \mathfrak{c}$;

(iii) if $\xi \le \eta < \mathfrak{c}$, $W \in \mathcal{W}_\xi$, $q \in W \cap Y_\xi$ then there is a $W' \in \mathcal{W}_\eta$ such that $q \in W' \subseteq W$;

(iv) if $\xi < \mathfrak{c}$ and $Y_{\xi+1} \subseteq E_\xi$ then $\lim_{m\to\infty} g^{(\xi)}_{m,k_\xi(m)}(y) = 0$ for every $y \in \bigcup \mathcal{W}_{\xi+1}$;

(v) $Y_{\xi+1} \cap V_\xi \setminus \mathbb{Q} \ne \emptyset$ for every ξ.

(c) construction (i) Start with $Y_0 = \mathbb{Q}$, $\mathcal{W}_0 = \{[-r, r] : r \in \mathbb{N} \setminus \{0\}\}$.

(ii) Given Y_ξ, $\mathcal{W}_\xi$ choose $y \in \bigcup \mathcal{W}_\xi \cap V_\xi \setminus \mathbb{Q}$.

case 1 If $E_\xi \not\supseteq Y_\xi \cup \bigcup \mathcal{W}_\xi$ choose $y' \in (Y_\xi \cup \bigcup \mathcal{W}_\xi) \setminus E_\xi$; set $Y_{\xi+1} = Y_\xi \cup \{y, y'\}$, $\mathcal{W}_{\xi+1} = \mathcal{W}_\xi$, $k_\xi(m) = m$ for every $m \in \mathbb{N}$.

case 2 If $E_\xi \supseteq Y_\xi \cup \bigcup \mathcal{W}_\xi$ set $Y_{\xi+1} = Y_\xi \cup \{y\}$ and use 1A to find a countable $\mathcal{W}_{\xi+1} \subseteq \mathcal{W}$ and a sequence $\langle k_\xi(m)\rangle_{m\in\mathbb{N}}$ such that

$$\lim_{m\to\infty} g^{(\xi)}_{m,k_\xi(m)}(y) = 0 \text{ for each } y \in \bigcup \mathcal{W}_{\xi+1};$$

$$\forall\ W \in \mathcal{W}_\xi,\ y \in W \cap Y_{\xi+1}\ \exists\ W' \in \mathcal{W}_{\xi+1},\ y \in W' \subseteq W;$$

every member of $\mathcal{W}_{\xi+1}$ is included in some member of $\mathcal{W}_\xi$.

(iii) Given $\langle Y_\xi\rangle_{\xi<\eta}$, $\langle\mathcal{W}_\xi\rangle_{\xi<\eta}$ where η is a non-zero countable limit ordinal, set $Y_\eta=\bigcup_{\xi<\eta}Y_\xi$. For $\xi<\eta$, $W\in\mathcal{W}_\xi$, $q\in W\cap Y_\xi$ choose $W'_{\xi Wq}$ as follows. Take a strictly increasing sequence $\langle\xi_n\rangle_{n\in\mathbb{N}}$ with limit η and $\xi=\xi_0$. Choose $\langle F_n\rangle_{n\in\mathbb{N}}$, $\langle K_n\rangle_{n\in\mathbb{N}}$ such that

$F_0=\{q\}$, $K_0=W$;

$F_{n+1}\subseteq K_n\cap Y_\xi$ is finite, every member of F_n is within a distance 2^{-n} of some member of $F_{n+1}\cap\mathbb{Q}$ other than itself;

K_{n+1} is a finite union of members of $\mathcal{W}_{\xi_{n+1}}$, $F_n\subseteq F_{n+1}\subseteq K_{n+1}\subseteq K_n$.

Now setting $W'_{\xi Wq}=\overline{\bigcup_{n\in\mathbb{N}}F_n}\subseteq\bigcap_{n\in\mathbb{N}}K_n$, we see that $W'_{\xi Wq}\in\mathcal{W}$, $q\in W'_{\xi Wq}\subseteq W\cap\bigcup\mathcal{W}_{\xi_n}$ for every n. So taking

$$\mathcal{W}_\eta=\{W'_{\xi Wq}:\xi<\eta,\ W\in\mathcal{W}_\xi,\ q\in W\cap Y_\xi\}$$

the induction will proceed.

(d) On completing the construction, set $X=\bigcup_{\xi<\mathfrak{c}}Y_\xi$.

(i) To see that X is an s_1-space, let $\langle f_{mn}\rangle_{m,n\in\mathbb{N}}$ be a double sequence in $C(X)$ such that $\lim_{n\to\infty}f_{mn}(x)=0$ for every $m\in\mathbb{N}$, $x\in X$. For each m, $n\in\mathbb{N}$ there is a G_δ set $E_{mn}\supseteq X$ such that f_{mn} has a continuous extension to a function $f'_{mn}:E_{mn}\to\mathbb{R}$. Set

$$E=\{x:x\in\bigcap_{m,n\in\mathbb{N}}E_{mn},\ \lim_{n\to\infty}f'_{mn}(x)=0\ \forall\ m\in\mathbb{N}\}.$$

Then E is a Borel set, so there is a $\xi<\mathfrak{c}$ such that

$$E_\xi=E,\ g^{(\xi)}_{mn}=f'_{mn}\restriction E\ \forall\ m,\ n\in\mathbb{N}.$$

Now we see that $Y_{\xi+1}\subseteq X\subseteq E$, so

$$\lim_{m\to\infty}g^{(\xi)}_{m,k_\xi(m)}(y)=0\ \forall\ y\in\bigcup\mathcal{W}_{\xi+1},$$

and

$$\lim_{m\to\infty}f_{m,k_\xi(m)}(x)=0\ \forall\ x\in X.$$

(ii) To see that $X\setminus\mathbb{Q}$ is not an s_1-space, let $H\subseteq\mathbb{R}\setminus\mathbb{Q}$ be any K_σ set. Then there is a $\xi<\mathfrak{c}$ such that $V_\xi=\mathbb{R}\setminus H$, so $X\setminus\mathbb{Q}\not\subseteq H$. This means that if $h:\mathbb{R}\setminus\mathbb{Q}\to\mathbb{N}^\mathbb{N}$ is any homeomorphism, $h[X\setminus\mathbb{Q}]$ is not included in any K_σ set in $\mathbb{N}^\mathbb{N}$, that is, is essentially unbounded. So $X\setminus\mathbb{Q}$ is not an s_1-space.

1C Remark The space X of 1B has the following property: there is a double sequence $\langle f_{mn}\rangle_{m,n\in\mathbb{N}}$ in $C(X)$ such that $\lim_{n\to\infty}f_{mn}=0$ in $C(X)$ for every m, but whenever $J\subseteq\mathbb{N}$ is infinite and $\langle k(m)\rangle_{m\in\mathbb{N}}$ is a sequence in $\mathbb{N}$, there are $x\in X$, $\langle n(m)\rangle_{m\in\mathbb{N}}$ such that $n(m)\ge k(m)$ for every m and $\limsup_{m\to J}f_{m,n(m)}(x)\ne 0$. To see this, take an enumeration $\langle q_m\rangle_{m\in\mathbb{N}}$ of $\mathbb{Q}$ and for each $m\in\mathbb{N}$ choose sequences $\langle\gamma_{mn}\rangle_{n\in\mathbb{N}}$, $\langle\delta_{mn}\rangle_{n\in\mathbb{N}}$ in $\mathbb{R}\setminus X$ such that $\langle\gamma_{mn}\rangle_{n\in\mathbb{N}}$ is strictly increasing, $\langle\delta_{mn}\rangle_{n\in\mathbb{N}}$ is strictly decreasing and both converge to q_m. For m, $n\in\mathbb{N}$ write

$$G_{mn}=\bigcup_{i\le m}]\gamma_{in},\delta_{in}[,$$

and set

$$f_{mn}(x)=1 \text{ if } x\in X\cap G_{mn}\setminus G_{m,n+1},$$
$$=0 \text{ for other } x\in X.$$

Of course every f_{mn} is continuous (because $X\cap\overline{G_{mn}}\setminus G_{mn}=\emptyset$) and $\lim_{n\to\infty}f_{mn}=0$.

Now suppose that $J\subseteq\mathbb{N}$ is infinite and that $\langle k(m)\rangle_{m\in\mathbb{N}}$ is a sequence in $\mathbb{N}$. Then $V=\bigcap_{p\in\mathbb{N}}\bigcup_{m\in J\setminus p}G_{m,k(m)}$ is a G_δ set including $\mathbb{Q}$, so is equal to V_ξ for some $\xi<\mathfrak{c}$, and there is an $x\in X\cap V\setminus\mathbb{Q}$. For each $m\in\mathbb{N}$ set

$$n(m)=\min\{n:n\ge k(m),\ f_{mn}(x)=1\}$$

if this is defined, $k(m)$ otherwise. For every $p\in\mathbb{N}$ there is an $m\in J\setminus p$ such that $x\in G_{m,k(m)}$; but $x\in X\setminus\mathbb{Q}$, so there is an $n\ge k(m)$ such that $x\in G_{m,n}\setminus G_{m,n+1}$ and $f_{mn}(x)=1$; accordingly $f_{m,n(m)}(x)=1$. Thus $\limsup_{m\to J}f_{m,n(m)}(x)=1$.

2. I claim in [Frp92] that for metrizable X, $\Sigma(B_1(C(X)))=\Sigma(C(X))$. My argument proceeds by dealing (i) with discrete spaces of cardinal $\mathfrak{b}$ (2D) (ii) with spaces with dense subsets of cardinal less than $\mathfrak{b}$ (2E).

2A Definition Let α be an ordinal. Say that a quadruple $(Z,\mathfrak{T},A,z)$ is **α-acceptable** if

(i) $(Z,\mathfrak{T})$ is a zero-dimensional, compact, sequentially compact Hausdorff space of weight at most $\mathfrak{b}$;

(ii) A is a countable subset of Z;

(iii) $z \in s_\alpha(A,Z) \setminus \bigcup_{\beta<\alpha} s_\beta(A,Z)$;

(iv) there is a family $\langle U_\xi\rangle_{\xi<\mathfrak{b}}$ of open-and-closed neighbourhoods of z such that $\bigcap_{\xi<\mathfrak{b}}\bigcup_{\eta\ge\xi} U_\eta$ is countable.

2B Lemma For every $\alpha<\omega_1$ there is an α-acceptable $(Z,\mathfrak{T},A,z)$.

proof Induce on α.

(a) The induction starts with $\alpha=0$, $Z=A=\{z\}$. For the inductive step to $\alpha>0$, let $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ be a sequence of ordinals such that $\alpha=\lim_{n\to\infty}(\alpha_n+1)=\sup_{n\in\mathbb{N}}(\alpha_n+1)$. By the inductive hypothesis, there is for each n an α_n-acceptable $(Z_n,\mathfrak{T}_n,A_n,z_n)$; we may choose the Z_n in such a way that they are disjoint from each other and from $\mathfrak{b}+1=\mathfrak{b}\cup\{\mathfrak{b}\}$, the ordinal successor of $\mathfrak{b}$. For each $n\in\mathbb{N}$ choose a family $\langle U_{n\xi}\rangle_{\xi<\mathfrak{b}}$ of open-and-closed neighbourhoods of z_n in Z_n such that $B_n=\bigcap_{\xi<\mathfrak{b}}\bigcup_{\eta\ge\xi}U_{n\eta}$ is countable; now let $\langle G_{ni}\rangle_{i\in\mathbb{N}}$ be a disjoint cover of $B_n\setminus\{z_n\}$ by open-and-closed sets in Z_n.

(b) As observed in [vD84], Theorem 3.3, there is a family $\langle g_\xi\rangle_{\xi<\mathfrak{b}}$ of strictly increasing functions in $\mathbb{N}^\mathbb{N}$ such that $\{n: g_\xi(n)\le g_\eta(n)\}$ is finite whenever $\eta<\xi<\mathfrak{b}$, while for every $g\in\mathbb{N}^\mathbb{N}$ there is a $\xi<\mathfrak{b}$ such that $\{n: g(n)<g_\xi(n)\}$ is infinite.

Set $Z=\bigcup_{n\in\mathbb{N}}Z_n\cup(\mathfrak{b}+1)$, $z=\mathfrak{b}\in Z$, $A=\bigcup_{n\in\mathbb{N}}A_n\subseteq Z$. For $\xi<\mathfrak{b}$ set

$$V_\xi=\bigcup_{n\in\mathbb{N}}(Z_n\setminus U_{n\xi})\cup\bigcup\{G_{ni}: n\in\mathbb{N},\ i\le g_\xi(n)\}\cup(\xi+1).$$

Let $\mathfrak{T}$ be the topology on Z generated by

$$\bigcup_{n\in\mathbb{N}}\mathfrak{T}_n\cup\{V_\xi:\xi<\mathfrak{b}\}\cup\{Z\setminus V_\xi:\xi<\mathfrak{b}\}\cup\{Z\setminus Z_n: n\in\mathbb{N}\}.$$

Each Z_n is an open-and-closed subset of Z and the topology on Z_n induced by $\mathfrak{T}$ is precisely $\mathfrak{T}_n$ (because the Z_n are disjoint and $V_\xi\cap Z_n$ is always open-and-closed in Z_n). The topology on $\mathfrak{b}+1$ induced by $\mathfrak{T}$ is the usual order topology of $\mathfrak{b}+1$. Observe also that $\langle z_n\rangle_{n\in\mathbb{N}}$ converges to z.

(c) Now for the conditions (i)-(iv) of 2A.

(i) $(Z,\mathfrak{T})$ is Hausdorff because (α) each Z_n is open-and-closed in Z (β) the topology induced on each Z_n is Hausdorff, by the inductive hypothesis (γ) if $\eta<\xi\le\mathfrak{b}$ then V_η is an open-and-closed subset of Z containing η and not containing ξ.

$(Z,\mathfrak{T})$ is zero-dimensional because all the V_ξ, Z_n are open-and-closed for $\mathfrak{T}$.

Let $\mathcal{F}$ be any ultrafilter on Z. If some Z_n belongs to $\mathcal{F}$ then $\mathcal{F}$ converges to some point of Z_n, because Z_n is compact for $\mathfrak{T}_n$. If $\mathfrak{b}+1\in\mathcal{F}$ then $\mathcal{F}$ converges to some point of $\mathfrak{b}+1$. Otherwise $\bigcup_{n\le i\in\mathbb{N}}Z_i\in\mathcal{F}$ for every $n\in\mathbb{N}$. If $Z\setminus V_\xi\in\mathcal{F}$ for every $\xi<\mathfrak{b}$, then $\mathcal{F}\to z$. Otherwise, let ξ be the least ordinal such that $V_\xi\in\mathcal{F}$; then $\mathcal{F}\to\xi$.

Thus every ultrafilter on Z has a limit in Z and $(Z,\mathfrak{T})$ is compact.

Let $\langle y_i\rangle_{i\in\mathbb{N}}$ be any sequence in Z. If $\{i: y_i\in Z_n\}$ is infinite for some $n\in\mathbb{N}$, then $\langle y_i\rangle_{i\in\mathbb{N}}$ has a subsequence converging to some point of Z_n. If $\{i: y_i\in\mathfrak{b}+1\}$ is infinite, then $\langle y_i\rangle_{i\in\mathbb{N}}$ has a subsequence converging to some point of $\mathfrak{b}+1$. If $\{i:\exists\ n,\ y_i=z_n\}$ is infinite, then $\langle y_i\rangle_{i\in\mathbb{N}}$ has a subsequence converging to some point of $\{z\}\cup\{z_n: n\in\mathbb{N}\}$. Otherwise, $\langle y_i\rangle_{i\in\mathbb{N}}$ has a subsequence of the form $\langle y'_i\rangle_{i\in\mathbb{N}}$ where $y'_i\in Z_{n(i)}\setminus\{z_{n(i)}\}$ for every i, $\langle n(i)\rangle_{i\in\mathbb{N}}$ being a strictly increasing sequence in $\mathbb{N}$. For each i, take $f(i)\in\mathbb{N}$ such that

$$y'_i\in(Z_{n(i)}\setminus B_{n(i)})\cup\bigcup_{j\le f(i)}G_{n(i),j};$$

let $\zeta(i)<\mathfrak{b}$ be such that

$$y'_i\in(Z_{n(i)}\setminus\bigcup_{\eta\ge\zeta(i)}U_{n(i),\eta})\cup\bigcup_{j\le f(i)}G_{n(i),j}.$$

Of course $\mathfrak{b}$ has uncountable cofinality, so $\zeta=\sup_{i\in\mathbb{N}}\zeta(i)<\mathfrak{b}$; now let $\zeta'\ge\zeta$ be such that $J=\{i: f(i)\le g_{\zeta'}(i)\}$ is infinite. Then (because $g_{\zeta'}$ is non-decreasing) $f(i)\le g_{\zeta'}(n(i))$ for each $i\in J$, so that $y'_i\in V_{\zeta'}$ for every $i\in J$. We can therefore find a least $\xi<\mathfrak{b}$ such that $I=\{i: y'_i\in V_\xi\}$ is infinite, and see that $\xi=\lim_{i\to I}y'_i$ is the limit of a subsequence of $\langle y_i\rangle_{i\in\mathbb{N}}$.

As $\langle y_i\rangle_{i\in\mathbb{N}}$ is arbitrary, $(Z,\mathfrak{T})$ is sequentially compact.

(ii) Of course A is countable.

(iii) The discussion of convergent sequences in (i) just above makes it plain that a sequence in Z can be convergent to z only if it lies eventually in $\{z\}\cup\{z_n: n\in\mathbb{N}\}$. Now an induction on β shows that if $\beta<\alpha$ then

$$\bigcup_{n\in\mathbb{N}}s_\beta(A_n,Z_n)\subseteq s_\beta(A,Z)\subseteq\bigcup_{n\in\mathbb{N}}s_\beta(A_n,Z_n)\cup\mathfrak{b},$$

because $\{n: z_n\in s_\beta(A_n,Z_n)\}=\{n:\alpha_n\le\beta\}$ is finite. Consequently $z\in s_\alpha(A,Z)\setminus\bigcup_{\beta<\alpha}s_\beta(A,Z)$.

(iv) Finally, observe that

$$\bigcap_{\xi<\mathfrak{b}}\bigcup_{\eta\ge\xi}(X\setminus V_\eta)\subseteq\{z\}\cup\bigcup_{n\in\mathbb{N}}B_n$$

is countable.

(d) Thus the induction proceeds and the lemma is proved.

2C Theorem There is a compact, sequentially compact, zero-dimensional Hausdorff space Z, of weight $\mathfrak{b}$, with a countable set $A\subseteq Z$ such that $\sigma(A,Z)=\omega_1$.

proof For each $\alpha<\omega_1$ take a quadruple $(Z_\alpha,\mathfrak{T}_\alpha,A_\alpha,z_\alpha)$ which is α-acceptable in the sense of 2A-2B. Let W be the topological product $\prod_{\alpha<\omega_1}Z_\alpha$ and for each α let $\langle a_{\alpha n}\rangle_{n\in\mathbb{N}}$ be a sequence running over A_α. Let $f:\omega_1\to\mathbb{N}^{\mathbb{N}}$ be an injective function. For $n\in\mathbb{N}$, $s\in\mathrm{Seq}$ set

$$\begin{aligned}z_{sn}(\alpha)&=a_{\alpha n}\text{ if }s\subseteq f(\alpha),\\&=a_{\alpha 0}\text{ otherwise.}\end{aligned}$$

Set $A=\{z_{sn}:s\in\mathrm{Seq},\ n\le\#(s)\}\subseteq W$. If we set

$$\begin{aligned}z^*_{\alpha n}(\beta)&=a_{\alpha n}\text{ if }\beta=\alpha,\\&=a_{\beta 0}\text{ otherwise,}\end{aligned}$$

then $z^*_{\alpha n}=\lim_{m\to\infty}z_{f(\alpha)\restriction m,n}\in s_1(A,W)$. So $w_\alpha\in s_{\omega_1}(A,W)$, where $w_\alpha(\beta)=z_\alpha$ if $\beta=\alpha$, $a_{\beta 0}$ otherwise. On the other hand, $w_\alpha\notin s_\beta(A,W)$ for $\beta<\alpha$, because, writing π_α for the canonical map from W to Z_α, we have $\pi_\alpha[A]\subseteq A_\alpha$, so $\pi_\alpha[s_\beta(A,W)]\subseteq s_\beta(A_\alpha,Z_\alpha)$ does not contain $z_\alpha=\pi_\alpha(w_\alpha)$.

Thus we have a zero-dimensional compact Hausdorff space W, of weight at most $\mathfrak{b}$, and a countable set $A\subseteq W$ with $\sigma(A,W)=\omega_1$. Any sequence in A either converges to $z^*=\langle a_{\alpha 0}\rangle_{\alpha<\omega_1}$ or has a subsequence of the form $\langle z_{s_i,n_i}\rangle_{i\in\mathbb{N}}$ where $\bigcup_{i\in\mathbb{N}}s_i\subseteq f(\alpha)$ for some α. In the latter case, taking a further subsequence if necessary, we may suppose that either the sequence is constant or that $s_i\subset s_{i+1}$ for each i, so that $\lim_{i\to\infty}z_{s_i,n_i}(\beta)=a_{\beta 0}$ for every $\beta\ne\alpha$; now $\langle a_{\alpha n_i}\rangle_{i\in\mathbb{N}}$ has a convergent subsequence, so $\langle z_{s_i,n_i}\rangle_{i\in\mathbb{N}}$ has a convergent subsequence.

Now if we set $Z=\overline{A}\subseteq W$, we see that

$$Z\subseteq A\cup\{w:w\in W,\ \#(\{\alpha:w(\alpha)\ne a_{\alpha 0}\})\le 1\};$$

but this is sequentially compact, because each Z_α is sequentially compact, and (as we have just seen) every sequence in A has a subsequence convergent to a point of Z. So Z is sequentially compact; but of course it is still compact, Hausdorff, zero-dimensional and of weight at most $\mathfrak{b}$; while $\sigma(A,Z)=\sigma(A,W)=\omega_1$, so that the weight of Z is precisely $\mathfrak{b}$, by Proposition 4 of [Frp92].

2D Corollary There is a countable $A\subseteq[0,1]^{\mathfrak{b}}$ such that $\sigma(A,[0,1]^{\mathfrak{b}})=\omega_1$.

proof The space Z of Theorem 2C, being a compact Hausdorff space of weight $\mathfrak{b}$, can be embedded into $[0,1]^{\mathfrak{b}}$; so that the set $A\subseteq Z$ of 2C is carried onto a countable set $A'\subseteq[0,1]^{\mathfrak{b}}$ with $\sigma(A',[0,1]^{\mathfrak{b}})=\omega_1$.

2E Proposition Suppose that X is a topological space which is not an s_1-space and has a dense subset of cardinal less than $\mathfrak{b}$. Then $\Sigma(B_1(C(X)))=\omega_1$.

proof (a) By Proposition 12 of [Frp92] there is a uniformly bounded double sequence $\langle f_{mn}\rangle_{m,n\in\mathbb{N}}$ in $C(X)$ such that $\lim_{n\to\infty}f_{mn}=0$ for every $m\in\mathbb{N}$ but $\langle f_{m(i),n(i)}\rangle_{i\to\infty}\not\to 0$ for any sequences $\langle m(i)\rangle_{i\in\mathbb{N}}$, $\langle n(i)\rangle_{i\in\mathbb{N}}$ of which the first is strictly increasing. We may take it that $|f_{mn}(x)|\le 1$ for all m, n and x. Let $Q\subseteq X$ be a dense subset of cardinal less than $\mathfrak{b}$. For each $q\in Q$, $m\in\mathbb{N}$ let $k_q(m)\in\mathbb{N}$ be such that $|f_{mn}(q)|\le 2^{-m}$ for every $n\ge k_q(m)$. Because $\#(Q)<\mathfrak{b}$, there is a sequence $\langle k(m)\rangle_{m\in\mathbb{N}}$ such that $\{m:k(m)<k_q(m)\}$ is finite for every $q\in Q$. Set $f'_{mn}=f_{m,n+k(m)}$ for m, $n\in\mathbb{N}$; then

(i) $\lim_{n\to\infty}f'_{mn}(x)=0$ for every $m\in\mathbb{N}$, $x\in X$,

(ii) $\lim_{m\to\infty}\sup_{n\in\mathbb{N}}|f'_{mn}(q)|=0$ for every $q\in Q$,

(iii) for any strictly increasing sequence $\langle m(i)\rangle_{i\in\mathbb{N}}$, any sequence $\langle n(i)\rangle_{i\in\mathbb{N}}$ there is an $x\in X$ such that $\langle f'_{m(i),n(i)}(x)\rangle_{i\in\mathbb{N}}$ does not converge to 0.

(b) For $t=\langle r_i\rangle_{i<m}\in\mathrm{Seq}$ write $|t|=m$, $\|t\|=\sum_{i<m}r_i$; then $\|t\|<\|s\|$ whenever $t<s$ in Seq. Define $\langle g_t\rangle_{t\in\mathrm{Seq}}$ inductively by saying

$$g_t=0\text{ if }|t|\le 1,$$

$$g_{t^\frown i^\frown j} = g_{t^\frown i} + 2^{-\|t\|} f'_{ij}$$

for $t \in \mathrm{Seq}$, $i \in \mathbb{N}$, $j \in \mathbb{N}$. Then we see that $|g_s(x) - g_{t^\frown i}(x)| \le 2.2^{-\|t\|}$ whenever $t^\frown i \le s$ in Seq and $x \in X$. Consequently we have

(i) $\lim_{i\to\infty} g_{t^\frown i} = g_t$ for every $t \in \mathrm{Seq}$;

(ii) if $t \in \mathrm{Seq}$, $\langle m(i)\rangle_{i\in\mathbb{N}} \to \infty$, $\langle n(i)\rangle_{i\in\mathbb{N}}$ is any sequence in $\mathbb{N}$, then $\lim_{i\to\infty} g_{t^\frown m(i)^\frown n(i)}(q) = g_t(q)$ for every $q \in Q$, but there is an $x \in X$ such that $\langle g_{t^\frown m(i)^\frown n(i)}(x)\rangle_{i\in\mathbb{N}}$ does not converge to $g_t(x)$;

(iii) if $t \in \mathrm{Seq}$, $\langle m(i)\rangle_{i\in\mathbb{N}} \to \infty$, $t^\frown m(i) < t_i$ in Seq for every i, then $\lim_{i\to\infty} g_{t_i}(q) = g_t(q)$ for every $q \in Q$, but there is an $x \in X$ such that $\langle g_{t_i}(x)\rangle_{i\in\mathbb{N}}$ does not converge to $g_t(x)$;

(iv) consequently, under the conditions of (iv), $\langle g_{t_i}\rangle_{i\in\mathbb{N}}$ has no limit in $C(X)$, because Q is dense in X.

(c) Finally, as in Theorem 9 of [Frp92], we can find a family $\langle \delta_t\rangle_{t\in\mathrm{Seq}}$ in $[0,1]$ such that $t \mapsto e_t : \mathrm{Seq} \to B_1(C(X))$ is a sequentially regular embedding, where $e_t = \frac{1}{3}(g_t + \delta_t \chi X)$.

So $\Sigma(B_1(C(X))) = \omega_1$, by Lemma 8 of [Frp92].

2F Corollary If X is metrizable then $\Sigma(B_1(C(X))) = \Sigma(C(X))$.

proof (a) If $X = \emptyset$ then $\Sigma(B_1(C(X))) = \Sigma(C(X)) = 0$.

(b) If X is a non-empty s_1-space then $\Sigma(B_1(C(X))) = \Sigma(C(X)) = 1$.

(c) If X is not an s_1-space and has a dense subset of cardinal less than $\mathfrak{b}$, then $\Sigma(B_1(C(X))) = \Sigma(C(X)) = \omega_1$, by 2E.

(d) Let ρ be a metric on X defining its topology. If X is not an s_1-space and has no dense subset of cardinal less than $\mathfrak{b}$, then it must have a metrically discrete subset of cardinal $\mathfrak{b}$, that is, there are $\delta > 0$ and a family $\langle x_\xi\rangle_{\xi<\mathfrak{b}}$ such that $\rho(x_\xi, x_\eta) \ge 3\delta$ whenever $\xi < \eta < \mathfrak{b}$. (This is because $\mathrm{cf}(\mathfrak{b}) = \mathfrak{b} > \omega$; see [En89], 4.1.15.) Define $e_\xi \in C(X)$ by setting $e_\xi(x) = \max(0, 1 - \delta^{-1}\rho(x, x_\xi))$ for $x \in X$, $\xi < \mathfrak{b}$. Then we have a map $\psi : [0,1]^{\mathfrak{b}} \to B_1(C(X))$ defined by setting $\psi(g) = \sum_{\xi<\mathfrak{b}} g(\xi)e_\xi$, and this embeds $[0,1]^{\mathfrak{b}}$ as a closed subspace of $B_1(C(X))$; so that $\Sigma(B_1(C(X))) \ge \Sigma([0,1]^{\mathfrak{b}}) \ge \omega_1$, by 2D, and we must have $\Sigma(B_1(C(X))) = \Sigma(C(X)) = \omega_1$.

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