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Linking filters

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1 Definitions Let $\mathcal{F}$ be a filter on a set I.

(a) $\mathcal{F}$ is **measure-n-linking**, where $n \ge 2$, if whenever $(\mathfrak{A},\bar\mu)$ is a probability algebra and $\langle a_i\rangle_{i\in I}$ is a family in $\mathfrak{A}$ such that $\inf_{i\in I}\bar\mu a_i > 0$, there is a $J \in \mathcal{F}$ such that $\{a_i : i \in J\}$ is n-linked.

$\mathcal{F}$ is **measure-linking** if it is measure-2-linking.

(b) $\mathcal{F}$ is **strongly measure-n-linking**, where $n \ge 2$, if whenever $(\mathfrak{A},\bar\mu)$ is a probability algebra and $\langle a_i\rangle_{i\in I}$ is a family in $\mathfrak{A}$ such that $\inf_{i\in I}\bar\mu a_i > 0$, there is a $J \in \mathcal{F}$ such that $\inf_{K\in[J]^n}\bar\mu(\inf_{i\in K} a_i) > 0$.

$\mathcal{F}$ is **strongly measure-linking** if it is strongly measure-2-linking.

(c) $\mathcal{F}$ is **graph-linking** if whenever $E \subseteq [I]^2$ and $m \in \mathbb{N}$ are such that $[K]^2 \cap E \ne \emptyset$ for every $K \in [I]^m$ (that is, there are no (I,E)-anticliques of size m), there is a $J \in \mathcal{F}$ such that $[J]^2 \subseteq E$ (that is, J is an E-clique).

(d) $\mathcal{F}$ is **Hausdorff**, or has **property C**, if whenever f, g are functions defined on I, then either $\{i : f(i) = g(i)\}$ belongs to $\mathcal{F}$, or there is a $J \in \mathcal{F}$ such that $f[J] \cap g[J] = \emptyset$.

(e) $\mathcal{F}$ is **measure-centering**, or has **property M**, if whenever $(\mathfrak{A},\bar\mu)$ is a probability algebra and $\langle a_i\rangle_{i\in I}$ is a family in $\mathfrak{A}$ such that $\inf_{i\in I}\bar\mu a_i > 0$, there is a $J \in \mathcal{F}$ such that $\{a_i : i \in J\}$ is centered.

(f)(i) $\mathcal{F}$ is **submeasure-n-linking**, where $n \ge 2$, if whenever $\mathfrak{A}$ is a Boolean algebra, $\nu : \mathfrak{A} \to [0,\infty]$ is an exhaustive submeasure, and $\langle a_i\rangle_{i\in I}$ a family in $\mathfrak{A}$ such that $\inf_{i\in I}\nu a_i > 0$, there is a $J \in \mathcal{F}$ such that $\{a_i : i \in J\}$ is n-linked.

$\mathcal{F}$ is **submeasure-linking** if it is submeasure-2-linking.

(ii) $\mathcal{F}$ is **strongly submeasure-n-linking**, where $n \ge 2$, if whenever $\mathfrak{A}$ is a Boolean algebra, $\nu : \mathfrak{A} \to [0,\infty]$ is an exhaustive submeasure, and $\langle a_i\rangle_{i\in I}$ a family in $\mathfrak{A}$ such that $\inf_{i\in I}\nu a_i > 0$, there is a $J \in \mathcal{F}$ such that $\inf_{K\in[J]^n}\nu(\inf_{i\in K} a_i) > 0$.

$\mathcal{F}$ is **strongly submeasure-linking** if it is strongly submeasure-2-linking.

2 Elementary facts (a) A graph-linking filter is strongly measure-linking. **P** Let $\mathcal{F}$ be a graph-linking filter on a set I, $(\mathfrak{A},\bar\mu)$ a probability algebra, and $\langle a_i\rangle_{i\in I}$ a family in $\mathfrak{A}$ such that $\inf_{i\in I}\bar\mu a_i = \epsilon > 0$. If $m \in \mathbb{N}$ and $m\epsilon^2 > 1$, set $\delta = \epsilon^2 - \frac{1}{m}$. For any $K \in [\mathbb{N}]^m$ there are distinct i, $j \in K$ such that $\bar\mu(a_i \cap a_j) \ge \epsilon^2 - \frac{1}{m}$, because $\|\sum_{i\in K}\chi a_i\|_1 \le \|\sum_{i\in K}\chi a_i\|_2$. Set $E = \{\{i,j\} : i \ne j,\ \bar\mu(a_i \cap a_j) \ge \delta\}$; then there is a $J \in \mathcal{F}$ such that $[J]^2 \subseteq E$, that is, $\inf_{i,j\in J}\bar\mu(a_i \cap a_j) \ge \delta$. **Q**

(b) A strongly measure-n-linking filter is measure-n-linking. A submeasure-n-linking filter is measure-n-linking. A strongly submeasure-n-linking filter is submeasure-n-linking and strongly measure-n-linking.

(c) A measure-linking filter is an ultrafilter. **P** Let $\mathcal{F}$ be a measure-linking filter on I, and $(\mathfrak{A},\bar\mu)$ any probability algebra with disjoint non-zero elements b, c. Given $A \subseteq I$, define $\langle a_i\rangle_{i\in I}$ by setting $a_i = b$ if $i \in A$, c if $i \in I \setminus A$. There is supposed to be a $J \in \mathcal{F}$ such that $\{a_i : i \in J\}$ is linked; now either $J \subseteq A$ and $A \in \mathcal{F}$, or $J \cap A = \emptyset$ and $I \setminus A \in \mathcal{F}$. **Q**

(d) A Ramsey filter on $\mathbb{N}$ is graph-linking. **P** Let $\mathcal{F}$ be a Ramsey filter on $\mathbb{N}$. If $E \subseteq [\mathbb{N}]^2$ and $m \in \mathbb{N}$ are such that $[K]^2 \cap E \ne \emptyset$ for every $K \in [\mathbb{N}]^m$, then there can be no $J \in \mathcal{F}$ such that $[J]^2 \cap E$ is empty, so there must be a $J \in \mathcal{F}$ such that $[J]^2 \subseteq E$. **Q**

(e) A measure-centering ultrafilter is measure-linking.

(f) A measure-linking ultrafilter is Hausdorff. **P** Let $\mathcal{F}$ be a measure-linking ultrafilter on I, and f, g two functions on I; set $X = f[I] \cup g[I] \cup \{\infty\}$, where ∞ is any set not belonging to $f[I] \cup g[I]$. Let $(\mathfrak{B},\bar\nu)$ be the measure algebra of the usual measure on $\{0,1\}^X$, and $\langle e_x\rangle_{x\in X}$ its standard generating family (Fremlin

08?, 525A). For $i \in I$, set $a_i = e_\infty \cap e_{f(i)} \setminus e_{g(i)}$ if $f(i) \neq g(i)$, $1 \setminus e_\infty$ if $f(i) = g(i)$. Let $J \subseteq I$ be such that $\{a_i : i \in J\}$ is linked. If there is any $i \in J$ such that $f(i) = g(i)$, then $f{\restriction}J = g{\restriction}J$. Otherwise, for all i, $j \in J$, $e_{f(i)} \setminus e_{g(j)} \neq 0$, so $f[J] \cap g[J] = \emptyset$. **Q** (Cf. Benedikt 98.)

(g) Let $\mathcal{F}$ be a free filter on a set I. Then $\mathcal{F} \ltimes \mathcal{F}$ (Fremlin 08?, 538D) is not measure-linking. **P** Let $(\mathfrak{B}, \bar\nu)$ be the measure algebra of the usual measure on $\{0,1\}^I$, and $\langle e_i \rangle_{i \in I}$ its standard generating family. For i, $j \in I$, set $a_{ij} = e_i \setminus e_j$ if $i \neq j$, 1 if $i = j$. Take any $J \in \mathcal{F} \ltimes \mathcal{F}$. Set $A = \{i : J[\{i\}] \in \mathcal{F}\}$, so that $A \in \mathcal{F}$. Take $i \in A$, $j \in (A \cap J[\subset\}]) \setminus \{i\}$, $k \in (A \cap J[\{i\}] \cap J[\{j\}]) \setminus \{i, j\}$. Then (i, j) and $(j, k) \in J$ and $a_{ij} \cap a_{jk} = 0$. So $\{a_x : x \in J\}$ is not linked. **Q**

(h) Let I, J be sets, $f : I \to J$ a function, $n \geq 2$ and $\mathcal{F}$ a filter on I.

(i) If $\mathcal{F}$ is measure-n-linking, $f[[\mathcal{F}]]$ is measure-n-linking.

(ii) If $\mathcal{F}$ is strongly measure-n-linking, $f[[\mathcal{F}]]$ is strongly measure-n-linking.

(iii) If $\mathcal{F}$ is submeasure-n-linking, $f[[\mathcal{F}]]$ is submeasure-n-linking.

(iv) If $\mathcal{F}$ is strongly submeasure-n-linking, $f[[\mathcal{F}]]$ is strongly submeasure-n-linking.

(iv) If $\mathcal{F}$ is graph-linking, $f[[\mathcal{F}]]$ is graph-linking. **P** If $E \subseteq [J]^2$ and $m \in \mathbb{N}$ are such that $[K]^2 \cap E \neq \emptyset$ for every $K \in [J]^m$, set

$$E' = \{L : L \in [I]^2, \text{ either } f \text{ is constant on } L \text{ or } f[L] \in E\}.$$

If $K \in [I]^m$, then either f is injective on K and $f[K] \in [J]^m$ and there is an $L \in [K]^2$ such that $f[L] \in E$, or there is an $L \in [K]^2$ such that f is constant on L and $L \in E'$. In either case, $[K]^2 \cap E'$ is non-empty. So there is an $A \in \mathcal{F}$ such that $[A]^2 \subseteq E'$. Now $f[A] \in f[[\mathcal{F}]]$ and $[f[A]]^2 \subseteq E$. **Q**

(i) A Ramsey ultrafilter on $\mathbb{N}$ is strongly submeasure-n-linking for every n. **P** Induce on n. For $n \leq 1$ the result is trivial. For the inductive step to $n+1 \geq 2$, let $\mathfrak{A}$ be a Boolean algebra, $\nu : \mathfrak{A} \to [0, \infty]$ an exhaustive submeasure and $\langle a_i \rangle_{i \in \mathbb{N}}$ a sequence in $\mathfrak{A}$ such that $\inf_{i \in \mathbb{N}} \nu a_i > 0$. Set $\nu' a = \min(1, \nu a)$ for $a \in \mathfrak{A}$; then $\nu' : \mathfrak{A} \to [0,1]$ is an exhaustive submeasure and $\epsilon = \inf_{i \in \mathbb{N}} \nu' a_i$ is greater than 0. By the inductive hypothesis, there are an $I^* \in \mathcal{F}$ and a $\delta > 0$ such that $\nu'(\inf_{i \in K} a_i) \geq \delta$ for every $K \in [I]^n$. For each $k \in \mathbb{N}$, set

$$\delta_k = \sup_{I \in \mathcal{F}} \inf\{\nu'(a_k \cap \inf_{i \in K} a_i) : K \in [I]^n\}.$$

Because $\mathcal{F}$ is a Ramsey ultrafilter, there is an $I_k \in \mathcal{F}$ such that

either $\delta_k - 2^{-k} \leq \nu'(a_k \cap \inf_{i \in K} a_i) \leq \delta_k + 2^{-k}$ for every $K \in [I_k]^n$

or $\nu'(a_k \cap \inf_{i \in K} a_i) \geq \delta_k + 2^{-k}$ for every $K \in [I_k]^n$

or $\nu'(a_k \cap \inf_{i \in K} a_i) \leq \delta_k - 2^{-k}$ for every $K \in [I_k]^n$.

The second and third are impossible, by the choice of δ_k, so we have the first. Again because $\mathcal{F}$ is a Ramsey ultrafilter, there is an $I \in \mathcal{F}$ such that $i \in I_k$ whenever i, $k \in I$ and $i > k$. **?** If $\lim_{k \to \mathcal{F}} \delta_k = 0$, then (once more because $\mathcal{F}$ is a Ramsey ultrafilter, therefore rapid) there is a $J \in \mathcal{F}$ such that $J \subseteq I \cap I^*$ and $\sum_{k \in J} \delta_k + 2^{-k} \leq \frac{1}{2}\delta$. Let $\langle K_l \rangle_{l \in \mathbb{N}}$ be a disjoint sequence in $[J]^n$ such that $\max K_l < \min K_m$ whenever $l < m$; set $k_l = \min K_l$ and $b_l = \inf_{i \in K_l} a_i$ for each l. In this case, whenever $l < m$,

$$\nu' b_l \geq \delta, \quad \nu'(b_l \cap b_m) \leq \nu'(a_{k_l} \cap b_m) \leq \delta_{k_l} + 2^{-k_l}$$

because $K_m \in [I_{k_l}]^n$. So, setting $c_m = b_m \setminus \sup_{l<m} b_l$ for each m,

$$\nu' c_m \geq \delta - \sum_{l=0}^{m-1} (\delta_{k_l} + 2^{-k_l}) \geq \frac{1}{2}\delta;$$

but $\langle c_m \rangle_{m \in \mathbb{N}}$ is disjoint and ν' is supposed to be exhaustive. **X**

We conclude that $\lim_{k \to \mathcal{F}} \delta_k > 0$. So we have a $J \in \mathcal{F}$ such that $J \subseteq I$ and $\delta' = \inf_{k \in J} \delta_k - 2^{-k} > 0$. Now if $K \in [J]^{n+1}$ and $\min K = k$, $K \setminus \{k\} \in [I_k]^n$ so

$$\nu(\inf_{i \in K} a_i) \geq \nu'(a_k \cap \inf_{i \in K \setminus \{k\}} a_i) \geq \delta_k - 2^{-k} \geq \delta'.$$

Thus the induction proceeds. **Q**

3 Lemma Let $E \subseteq [\mathbb{N} \times \mathbb{N}]^2$ and $m \in \mathbb{N}$ be such that there is no $(\mathbb{N} \times \mathbb{N}, E)$-anticlique of size m, and I a subset of $\mathbb{N} \times \mathbb{N}$ such that infinitely many vertical sections of I are infinite. Then there is an E-clique $J \subseteq I$ such that infinitely many vertical sections of J are infinite.

proof (a) Induce on m. If $m \le 2$, then we can take $J = I$. So for the rest of the proof I will look at the inductive step to $m+1 > 2$. Write $\mathcal{I}$ for the ideal $[\mathbb{N}]^{<\omega} \ltimes [\mathbb{N}]^{<\omega}$ of sets $J \subseteq \mathbb{N}\times\mathbb{N}$ such that $\{i : J[\{i\}]$ is infinite$\}$ is finite; then $I \notin \mathcal{I}$ and we are seeking an E-clique $J \subseteq I$ such that $J \notin \mathcal{I}$.

(b) For $x \in \mathbb{N}\times\mathbb{N}$, set $E'_x = \{y : y \in I,\ \{x,y\} \notin E\}$. If there is any x such that $E'_x \notin \mathcal{I}$, then there is an E-clique $J \subseteq E'_x$ such that $J \notin \mathcal{I}$. **P** Set $F = E \cup \{\{y,z\} : y \in (\mathbb{N}\times\mathbb{N}) \setminus E'_x,\ z \in (\mathbb{N}\times\mathbb{N}) \setminus \{y\}\}$. Then there is no $(\mathbb{N}\times\mathbb{N}, F)$-anticlique of size m, so by the inductive hypothesis there is an F-clique $J \subseteq E'_x$ not belonging to $\mathcal{I}$, and J is now an E-clique. **Q**

So henceforth I will suppose that $E'_x \in \mathcal{I}$ for every $x \in \mathbb{N}\times\mathbb{N}$.

(c) Set $L = \{i : i \in \mathbb{N},\ I[\{i\}]$ is infinite$\}$, so that L is infinite. For $i \in L$, let $\mathcal{F}_i$ be a non-principal ultrafilter on $I[\{i\}]$.

Set

$$F_1 = \{\{i,k\} : i,\ k \in L,\ i < k,\ \{j : \{l : \{(i,j),(k,l)\} \in E\} \in \mathcal{F}_k\} \in \mathcal{F}_i\},$$

$$F_2 = \{\{i,k\} : i,\ k \in L,\ i < k,\ \{l : \{j : \{(i,j),(k,l)\} \in E\} \in \mathcal{F}_i\} \in \mathcal{F}_k\},$$

so that $F_1,\ F_2 \subseteq [L]^2$. Then there is no (L, F_1)-anticlique in L of size $m+1$. **P?** If $i_0 < \ldots < i_m$ and $\{i_0,\ldots,i_m\}$ is an (L,F_1)-anticlique, then for $n \le m$ let $M_n \in \mathcal{F}_{i_n}$ be such that $\{l : l \in \mathbb{N},\ \{\{(i_n,j),(i_r,l)\} \notin E\} \in \mathcal{F}_{i_r}$ whenever $j \in M_n$ and $n < r \le m$. Now choose $j_n \in M_n$, for $n \le m$, such that $\{(i_r,j_r),(i_n,j_n)\} \notin E$ whenever $r < n$; this gives us an $(\mathbb{N}\times\mathbb{N}, E)$-anticlique of size $m+1$. **XQ**

Similarly, there is no (L,F_2)-anticlique of size $m+1$. There is therefore no $(L, F_1 \cap F_2)$-anticlique of size the Ramsey number $R(m+1,m+1)$, and there must be an infinite $(F_1 \cap F_2)$-clique $L_0 \subseteq L$.

(d) Note next that for each $i \in L$ the set

$$\tilde{M}_i = \{j : j \in I[\{i\}],\ \{j' : \{(i,j),(i,j')\} \in E\} \in \mathcal{F}_i\}$$

belongs to $\mathcal{F}_i$. **P?** Otherwise, we can choose $j_0 < \ldots < j_m$ in $I[\{i\}] \setminus \tilde{M}_i$ such that $\{(i,j_r),(i,j_s)\} \notin E$ whenever $r < s \le m$. **XQ**

(e) Let S be the set of finite sequences $((i_0,j_0),\ldots,(i_n,j_n))$ of distinct points in I such that

$i_r \in L_0$, $j_r \in \tilde{M}_{i_r}$ for every $r \le n$,

$\{(i_r,j_r) : r \le n\}$ is an E-clique,

if $r,\ s \le n$ and $i_r < i_s$, then $\{j : \{(i_r,j),(i_s,j_s)\} \in E\}$ belongs to $\mathcal{F}_{i_r}$.

(i) If $((i_0,j_0),\ldots,(i_n,j_n)) \in S$, there is a point $(k,l) \in I$ such that $k > i_r$ for every $r \le n$ and $((i_0,j_0),\ldots,(i_n,j_n),(k,l)) \in S$. **P** By (b), there is a $k \in L_0$ such that $k > \max_{r\le n} i_r$ and $E'_{(i_r,j_r)}[\{k\}]$ is finite for every $r \le n$. Now, because $\{i_r,k\} \in F_2$ for every $r \le n$, the set $\{l : \{j : \{(i_r,j),(k,l)\} \in E\} \in \mathcal{F}_{i_r}\}$ belongs to $\mathcal{F}_k$ for each r, and we can easily find an $l \in \tilde{M}_k$ such that $\{(i_0,j_0),\ldots,(i_n,j_n),(k,l)\}$ is an E-clique and $\{j : \{(i_r,j),(k,l)\} \in E\} \in \mathcal{F}_{i_r}$ for every $r \le n$. **Q**

(ii) If $((i_0,j_0),\ldots,(i_n,j_n)) \in S$ and $r \le n$, there is an $l \in I$ such that $l > j_r$ for every $r \le n$ and $((i_0,j_0),\ldots,(i_n,j_n),(i_r,l)) \in S$. **P** For each $s \le n$ such that $i_s = i_r$, $j_s \in \tilde{M}_{i_r}$, so $\{l : \{(i_s,j_s),(i_r,l)\} \in E\}$ belongs to $\mathcal{F}_{i_r}$. If $s \le n$ and $i_s < i_r$, then $\{l : \{(i_s,l),(i_r,j_r)\} \in E\}$ belongs to $\mathcal{F}_{i_s}$; while if $i_r < i_s$, then, as in (i), $\{l : \{j : \{(i_r,j),(i_s,l)\} \in E\} \in \mathcal{F}_{i_r}\} \in \mathcal{F}_{i_s}$. So again we have a set in $\mathcal{F}_{i_s}$ to choose from. **Q**

(f) We can therefore choose inductively a sequence $\langle(i_n,j_n)\rangle_{n\in\mathbb{N}}$ such that $(i_0,j_0),\ldots,(i_n,j_n)) \in S$ for every n, $\{i_n : n \in \mathbb{N}\}$ is infinite, and $\{j_n : n \in \mathbb{N},\ i_n = i_r\}$ is infinite for every $r \in \mathbb{N}$. So we have an E-clique $\{(i_n,j_n) : n \in \mathbb{N}\}$ included in I and not belonging to $\mathcal{I}$.

4 Proposition [CH] There is a graph-linking ultrafilter on $\mathbb{N}\times\mathbb{N}$ which is not a p-point ultrafilter.

proof Enumerate $\mathcal{P}([\mathbb{N}\times\mathbb{N}]^2)$ as $\langle E_\xi\rangle_{\xi<\omega_1}$. As in the proof of Lemma 3, set $\mathcal{I} = [\mathbb{N}]^{<\omega} \ltimes [\mathbb{N}]^{<\omega}$. Choose $\langle I_\xi\rangle_{\xi<\omega_1}$ as follows. The inductive hypothesis will be that $I_\zeta \setminus I_\eta \in \mathcal{I}$ whenever $\eta < \zeta < \xi$, and that $I_\eta \notin \mathcal{I}$ for every $\eta < \xi$. Start with $I_0 = \mathbb{N}\times\mathbb{N}$. For the inductive step to $\xi+1$, if there are $(\mathbb{N}\times\mathbb{N}, E_\xi)$-anticliques of all finite sizes, set $I_{\xi+1} = I_\xi$. Otherwise, by Lemma 3, we can find an E_ξ-clique $I_{\xi+1} \subseteq I_\xi$ such that $I_{\xi+1} \notin \mathcal{I}$. For non-zero limit ordinals ξ, let I_ξ be such that $I_\xi \notin \mathcal{I}$ but $I_\xi \setminus I_\eta \in \mathcal{I}$ for every $\eta < \xi$.

Let $\mathcal{F}$ be the filter generated by $\{I_\xi \setminus J : \xi < \omega_1,\ J \in \mathcal{I}\}$; then $\mathcal{F}$ has the properties required.

5 Lemma (M.Talagrand) For I, $J \subseteq \mathbb{N}$, say that $I \preccurlyeq J$ if there is a bijective function $f : I \to J$ such that $f(i) \geq i$ for every $i \in J$.

(a) If $I \preccurlyeq J$ and $f : I \to J$ is a bijective function such that $\#(\{i : i \in I,\ f(i) > i\}) = r$, there are $I_0, \dots, I_r$ such that $I = I_0 \preccurlyeq I_1 \preccurlyeq \dots \preccurlyeq I_r = J$ and $\#(I_k \triangle I_{k+1}) = 2$ for every $k < r$.

(b) Suppose that $m \geq 1$. For $n \in \mathbb{N}$, set

$$\mathcal{I}_n = \{I : I \subseteq n,\ \#(I \cap J) \geq m \text{ whenever } J \subseteq n \text{ and } J \succcurlyeq I\}.$$

Then $\#(\mathcal{I}_n) \leq \frac{2^n(1+\sqrt{n})}{m}$.

proof (a) Induce on r. If $r = 0$ then $I = J$ and we can stop. For the inductive step to $r + 1$, take the least i_0 such that $f(i_0) > i_0$, and express f as gh, where

$$\begin{aligned} h(i) &= f(i) \text{ if } i \in I \setminus \{i_0\}, \\ &= i_0 \text{ if } i = i_0, \\ g(i) &= i \text{ if } i \in h[I] \setminus \{i_0\}, \\ &= f(i_0) \text{ if } i = i_0. \end{aligned}$$

Now $\#(\{i : i \in I,\ h(i) > i\}) = r$, so there are $I_0, \dots, I_r$ such that $I = I_0 \preccurlyeq I_1 \preccurlyeq \dots \preccurlyeq I_r = h[I]$ and $\#(I_k \triangle I_{k+1}) = 2$ for every $k < r$; while g witnesses that $h[I] \preccurlyeq J$, and $h[I] \triangle J = \{i_0, f(i_0)\}$ has just two members, so we can set $I_{r+1} = J$ to show that the induction proceeds.

(b) For $I \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, set $h_n(I) = \min\{\#(I \cap J) : I \cap n \preccurlyeq J \subseteq n\}$, $g_n(I) = n - 2\#(I \cap n)$. Then we find that

$$\begin{aligned} h_{n+1}(I) &= h_n(I) + 1 \text{ if } n \in I, \\ &= \max(0, h_n(I) - 1) \text{ otherwise}, \\ g_{n+1}(I) &= g_n(I) + 1 \text{ if } n \in I, \\ &= g_n(I) - 1 \text{ otherwise}. \end{aligned}$$

Inducing on n, we see that $0 \leq h_n(I) \leq 1 + |g_n(I)|$ for all n and I. Giving $\mathcal{P}\mathbb{N}$ its usual probability measure, we have

$$\mathbb{E}(h_n) \leq 1 + \mathbb{E}(|g_n|) \leq 1 + \sqrt{\mathbb{E}(g_n^2)} = 1 + \sqrt{n}$$

for every n. So

$$\#(\mathcal{I}_n) = 2^n \Pr(h_n \geq m) \leq \frac{2^n(1+\sqrt{n})}{m}.$$

6 Lemma (M.Talagrand) If $1 \leq m \leq n \in \mathbb{N}$, $\mathcal{I} \subseteq \mathcal{P}n$ and $\#(I \cap J) \geq m$ for all I, $J \in \mathcal{I}$, then $\#(\mathcal{I}) \leq \frac{2^n(1+\sqrt{n})}{m}$.

proof Induce on $w(\mathcal{I}) = \sum_{I \in \mathcal{I}} \sum_{i \in I}(n - i)$. If $w(\mathcal{I}) = 0$ the result is trivial. For the inductive step to $w(\mathcal{I}) = k + 1$, then if $\mathcal{I} \subseteq \mathcal{I}_n$, as defined in Lemma 5, we can stop. Otherwise, there must be an $I_0 \in \mathcal{I} \setminus \mathcal{I}_n$ and $J_0 \succcurlyeq I_0$ such that $\#(I_0 \cap J_0) < m$, so that $J_0 \notin \mathcal{I}$.

By 5a, there must the $I_1 \in \mathcal{I}$, $J_1 \notin \mathcal{I}$ such that $I_1 \preccurlyeq J_1$ and $I_1 \triangle J_1$ has just two members; let i_1 be the member of $I_1 \setminus J_1$ and j_1 the member of $J_1 \setminus I_1$; of course $i_1 < j_1$. Define $\phi(\mathcal{I}) \to \mathcal{P}n$ by saying that

$$\begin{aligned} \phi(I) &= I \triangle \{i_1, j_1\} \text{ if } i_1 \in I,\ j_1 \notin I \text{ and } I \triangle \{i_1, j_1\} \notin \mathcal{I}, \\ &= I \text{ otherwise}. \end{aligned}$$

Then ϕ is injective. Set $\mathcal{J} = \phi[\mathcal{I}]$, so that $\#(\mathcal{I}) = \#(\mathcal{J})$; while $w(\mathcal{J}) < w(\mathcal{I})$. Now $\#(J \cap J') \geq m$ for all J, $J' \in \mathcal{J}$. **P** Set $I = \phi^{-1}(J)$, $I' = \phi^{-1}(J')$. Then

— if $I = J$ and $I' = J'$, then $\#(J \cap J') = \#(I \cap I') \geq m$;

— if $I \neq J$ and $I' \neq J'$, then

$$\#(J\cap J')=\#((I\triangle\{i_1,j_1\})\cap(I'\triangle\{i_1,j_1\})=\#(I\cap I')\ge m$$

because $i_1\in I\cap I'$ and $j_1\notin I\cup I'$;

— if $I=J$ and $j_1\in I$ and $I'\ne J'$, then $i_1\in I'$, $J'=I'\triangle\{i_1,j_1\}$ so

$$\#(J\cap J')=\#(I\cap J')\ge\#(I\cap I')\ge m;$$

— if $I=J$ and $i_1\notin I$ and $I'\ne J'$, then $i_1\in I'$, $J'=I'\triangle\{i_1,j_1\}$ so

$$\#(J\cap J')=\#(I\cap J')\ge\#(I\cap I')\ge m;$$

— if $I=J$ and $i_1\in I$ and $j_1\notin I$ and $I'\ne J'$, then $I''=I\triangle\{i_1,j_1\}\in\mathcal{I}$ so

$$\#(J\cap J')=\#((I''\triangle\{i_1,j_1\})\cap(I'\triangle\{i_1,j_1\}))=\#(I''\cap I')\ge m$$

because $i_1\in I'\setminus I''$ and $j_1\in I''\setminus I'$.

Thus in all cases we have $\#(J\cap J')\ge m$. **Q**

By the inductive hypothesis,

$$\frac{2^n(1+\sqrt{n})}{m}\ge\#(\mathcal{J})=\#(\mathcal{I})$$

and the induction continues.

7 Proposition (V.Bergelson-M.Talagrand) There are a probability algebra $(\mathfrak{A},\bar\mu)$ and a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\bar\mu a_n\ge\frac12$ for every n and I has zero asymptotic density whenever $I\subseteq\mathbb{N}$ and $\inf_{m,n\in I}\bar\mu(a_m\cap a_n)>0$.

proof For $r\in\mathbb{N}$, set $K_r=\{n:2^r-1\le n\le 2^{r+1}-2\}$ and

$$\mathcal{K}_r=\{I:I\subseteq r+1,\,\#(I)>\tfrac{r+1}{2}\}\cup\{I:0\in I\subseteq r+1,\,\#(I)=\tfrac{r+1}{2}\},$$

so that $\#(K_r)=\#(\mathcal{K}_r)=2^r$. For each $r\in\mathbb{N}$ let $h_r:K_r\to\mathcal{K}_r$ be a bijection. Set $X=\prod_{r\in\mathbb{N}}(r+1)$, with the product μ of the uniform probabilities on the factors; let $(\mathfrak{A},\bar\mu)$ be the measure algebra of μ. Define $A_n\subseteq X$, for $n\in\mathbb{N}$, by saying that $A_n=\{x:x\in X,\,x(r)\in h_r(n)\}$ where $n\in K_r$; as $\#(h_r(n))\ge\frac{r+1}{2}$, $\mu A_n\ge\frac12$. Set $a_n=A_n^\bullet\in\mathfrak{A}$, so that $\bar\mu a_n\ge\frac12$ for every n.

Let $I\subseteq\mathbb{N}$ and $\epsilon>0$ be such that $\bar\mu(a_m\cap a_n)\ge\epsilon$ for all m, $n\in I$. For $r\in\mathbb{N}$, set $\mathcal{J}_r=h_r[I\cap K_r]$. If J, $J'\in\mathcal{J}_r$, then $\#(J\cap J')\ge\epsilon(r+1)$. So, setting $m_r=\lceil\epsilon(r+1)\rceil$,

$$\#(I\cap K_r)=\#(\mathcal{J}_r)\le\frac{2^{r+1}(1+\sqrt{r+1})}{m_r}$$

for each r, by Lemma 6, and

$$2^{-r}\#(I\cap K_r)\le\frac{2(1+\sqrt{r+1})}{\epsilon(r+1)}\to 0$$

as $r\to\infty$. So I has zero asymptotic density, as claimed.

8 Proposition (a) If $\mathcal{F}$ is a strongly measure-linking ultrafilter on $\mathbb{N}$, it contains a set of zero asymptotic density.

(b) [$\operatorname{cov}\mathcal{N}=\mathfrak{c}$] There is a measure-centering ultrafilter on $\mathbb{N}$ which contains no set of zero asymptotic density.

(c) [$\operatorname{cov}\mathcal{N}=\mathfrak{c}$] There is a measure-centering ultrafilter on $\mathbb{N}$ which is not strongly measure-linking.

proof For $I\subseteq\mathbb{N}$ write $d^*(I)$ for its upper asymptotic density; write $\mathcal{Z}$ for the asymptotic density ideal $\{I:d^*(I)=0\}$.

(a) Let $(\mathfrak{A},\bar\mu)$ and $\langle a_n\rangle_{n\in\mathbb{N}}$ be as in Proposition 7. If $\mathcal{F}$ is a strongly measure-linking ultrafilter on $\mathbb{N}$, there is an $I\in\mathcal{F}$ such that $\inf_{m,n\in I}\bar\mu(a_m\cap a_n)>0$, so $I\in\mathcal{Z}$.

(b)(i) Let (X,Σ,μ) be a probability space, $I\in\mathcal{P}\mathbb{N}\setminus\mathcal{Z}$, and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence in Σ with $\inf_{n\in\mathbb{N}}\mu E_n=\gamma>0$. For $x\in X$ set $J_x=\{n:n\in I,\,x\in E_n\}$. Then $\mu\{x:d^*(J_x)>0\}\ge\gamma$.

P Set $\delta=d^*(I)$; then there is a disjoint sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ of non-empty finite subsets of $\mathbb{N}$ such that $\#(K_n)=\min K_n$ and $\#(I\cap K_n)\ge\frac13\delta\#(K_n)$ for every n. If $n\in\mathbb{N}$ then

$$\int \#(J_x \cap K_n)\mu(dx) \geq \gamma\#(I \cap K_n),$$

so if $\eta > 0$

$$\mu\{x : \#(J_x \cap K_n) \geq \eta\#(I \cap K_n)\} \geq \gamma - \eta.$$

Consequently

$$\begin{aligned}\mu\{x : d^*(J_x) \geq \frac{1}{6}\delta\eta\} &\geq \mu\{x : \text{for infinitely many } n,\ \#(J_x \cap K_n) \geq \eta\#(I \cap K_n)\}\\ &\geq \gamma - \eta.\end{aligned}$$

As η is arbitrary, $\mu\{x : d^*(J_x) > 0\} \geq \gamma$. **Q**

(ii) Now, given that $\operatorname{cov}\mathcal{N} = \mathfrak{c}$, let ν be the usual measure on $\{0,1\}^{\mathbb{N}}$ and enumerate the family of all sequences $\langle E_n\rangle_{n\in\mathbb{N}}$ of Borel subsets of $\{0,1\}^{\mathbb{N}}$ such that $\inf_{n\in\mathbb{N}} \nu E_n > 0$ as $\langle\langle E_{\xi n}\rangle_{n\in\mathbb{N}}\rangle_{\xi<\mathfrak{c}}$. Build filter bases $\mathcal{E}_\xi \subseteq \mathcal{P}\mathbb{N}$, for $\xi \leq \mathfrak{c}$, as follows. Start with $\mathcal{E}_0 = \{\mathbb{N}\}$. The inductive hypothesis will be that $\#(\mathcal{E}_\xi) \leq \max(\omega, \xi)$ and $\mathcal{E}_\xi \cap \mathcal{Z} = \emptyset$. For the inductive step to $\xi + 1$, set $\epsilon = \inf_{n\in\mathbb{N}} \nu E_{\xi n}$. For each $I \in \mathcal{E}_\xi$, set

$$F_I = \{x : d^*(J_x \cap I) > 0\},$$

where $J_x = \{n : x \in E_{\xi n}\}$. By (i), $\nu F_I \geq \epsilon$; because $\langle F_I\rangle_{I\in\mathcal{E}_\xi}$ is downwards-directed, and $\#(\mathcal{E}_\xi) < \operatorname{cov}\mathcal{N}$, there is a point $x \in \bigcap_{I\in\mathcal{E}_\xi} F_I$. Set $I_\xi = J_x$,

$$\mathcal{E}_{\xi+1} = \mathcal{E}_\xi \cup \{I \cap I_\xi : I \in \mathcal{E}_\xi\}.$$

Then $\mathcal{E}_{\xi+1}$ is a filter base, including $\mathcal{E}_\xi$, disjoint from $\mathcal{Z}$, and containing a set I_ξ such that $\bigcap_{n\in I_\xi} E_{\xi n}$ is non-empty. At non-zero limit ordinals $\xi \leq \mathfrak{c}$, set $\mathcal{E}_\xi = \bigcup_{\eta<\xi} \mathcal{E}_\eta$.

Let $\mathcal{F}$ be the filter on $\mathbb{N}$ generated by $\mathcal{E}_\mathfrak{c}$. Then $\mathcal{F} \cap \mathcal{Z} = \emptyset$. If $\langle F_n\rangle_{n\in\mathbb{N}}$ is any sequence of measurable subsets of $\{0,1\}^{\mathbb{N}}$ such that $\inf_{n\in\mathbb{N}} \nu F_n > 0$, there is a $\xi < \mathfrak{c}$ such that $E_{\xi n} \subseteq F_n$ and $\nu(F_n \setminus E_{\xi n}) = 0$ for every n; now $I_\xi \in \mathcal{F}$ and $\bigcap_{n\in I_\xi} F_n \neq \emptyset$. By Fremlin 08?, 538G, $\mathcal{F}$ is measure-centering.

(c) Immediate from (a) and (b).

9 Proposition $[\operatorname{cov}\mathcal{N} = \omega_1]$ No uniform filter on ω_1 is measure-centering.

proof Let $\langle E_\xi\rangle_{\xi<\omega_1}$ be a family of negligible subsets of $[0,1]$ covering ω_1. For $\xi < \omega_1$, let $K_\xi \subseteq [0,1]\setminus\bigcup_{\eta<\xi} E_\eta$ be a compact set of measure at least $\frac{1}{2}$. Let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of Lebesgue measure on $[0,1]$, and set $a_\xi = K_\xi^\bullet$ for $\xi < \omega_1$. If $\mathcal{F}$ is a uniform ultrafilter on ω_1 and $J \in \mathcal{F}$, then $\bigcap_{\xi\in J} K_\xi = \emptyset$, so there is a finite $L \subseteq J$ such that $\bigcap_{\xi\in l} K_\xi = \emptyset$ and $\inf_{\xi\in L} a_\xi = 0$. Thus $\{a_\xi : \xi \in J\}$ is not centered; as J is arbitrary, $\mathcal{F}$ is not measure-centering.

Problem Suppose that $\mathcal{F}$ is an ultrafilter on $\mathbb{N}$ which is graph-linking and strongly submeasure-n-linking for every n. Does it follow that $\mathcal{F}$ is measure-centering?

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