Version of 21.7.11

Products of gauge integrals

D.H.Fremlin

University of Essex, Colchester, England

1 Strict Saks-Henstock indefinite integrals

1A Tagged-partition structures I repeat some definitions from Fremlin 03, §481, in the form I will use here.

(a) If X is a set, a **gauge** on X is a subset δ of $X \times \mathcal{P}X$. If X is a topological space, a **neighbourhood gauge** on X is a set δ expressible in the form $\{(x, C) : x \in X,\, C \subseteq G_x\}$ where G_x is an open set containing x for each $x \in X$. A **tagged partition** in X is a finite subset $\boldsymbol{t}$ of $X \times \mathcal{P}X$; in this case I will write $W_{\boldsymbol{t}}$ for $\bigcup_{(x,C)\in\boldsymbol{t}} C$. If δ is a gauge on X and $\boldsymbol{t}$ is a tagged partition in X, $\boldsymbol{t}$ is **δ-fine** if $\boldsymbol{t} \subseteq \delta$. A **straightforward set of tagged partitions** on X is a set of the form

$$T = \{\boldsymbol{t} : \boldsymbol{t} \in [Q]^{<\omega},\ C \cap C' = \emptyset \text{ whenever } (x, C),\ (x', C') \text{ are distinct members of } \boldsymbol{t}\}$$

where $Q \subseteq X \times \mathcal{P}X$; I will say that T is **generated** by Q.

(b) $(X, T, \Delta, \{\{\emptyset\}\})$ is a **tagged-partition structure allowing subdivisions**, **witnessed by** $\mathcal{C}$, if

(i) X is a set;

(ii) Δ is a downwards-directed family of gauges on X;

(iii) $\mathcal{C}$ is a family of subsets of X such that whenever $C, C' \in \mathcal{C}$ then $C \cap C' \in \mathcal{C}$ and $C \setminus C'$ is expressible as the union of a disjoint finite subset of $\mathcal{C}$;

(iv) X is expressible as the union of a finite subset of $\mathcal{C}$;

(v) $T \subseteq [X \times \mathcal{C}]^{<\omega}$ is a straightforward set of tagged partitions on X;

(vii) whenever $C \in \mathcal{C}$ and $\delta \in \Delta$ there is a δ-fine tagged partition $\boldsymbol{t} \in T$ such that $W_{\boldsymbol{t}} = C$.

(c) Suppose that $(X, T, \Delta, \{\{\emptyset\}\})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, and $\nu : \mathcal{C} \to \mathbb{R}$ is a functional. For $\boldsymbol{t} \in T$ and $f \in \mathbb{R}^X$, set

$$S_{\boldsymbol{t}}(f, \nu) = \sum_{(x,C)\in\boldsymbol{t}} f(x)\nu C.$$

For $f \in \mathbb{R}^X$ say that $I_\nu(f)$ is defined and equal to γ if for every $\epsilon > 0$ there is a $\delta \in \Delta$ such that $|S_{\boldsymbol{t}}(f, \nu) - \gamma| \le \epsilon$ whenever $\boldsymbol{t} \in T$ is δ-fine and $W_{\boldsymbol{t}} = X$. In this case, f has a unique **Saks-Henstock indefinite integral** $F : \mathcal{E} \to \mathbb{R}$, where $\mathcal{E}$ is the algebra of subsets of X generated by $\mathcal{C}$, such that F is additive and for every $\epsilon > 0$ there is a $\delta \in \Delta$ such that $\sum_{(x,C)\in\boldsymbol{t}} |F(C) - f(x)\nu C| \le \epsilon$ for every δ-fine $\boldsymbol{t} \in T$ (Fremlin 03, 482B); and $F(X) = I_\nu(f)$.

1B I come now to the first new idea of this note. Suppose that $(X, T, \Delta, \{\{\emptyset\}\})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, and $\nu : \mathcal{C} \to \mathbb{R}$ is a functional. Write Q for the set $\{(x, C) : \{(x, C)\} \in T\}$ (so that T is the straightforward set of tagged partitions generated by Q). Let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{C}$, and $f : X \to \mathbb{R}$ a function such that $I_\nu(f)$ is defined. I will say that the Saks-Henstock indefinite integral F of f is **moderated** there are a $\delta \in \Delta$ and an additive functional $\lambda : \mathcal{E} \to [0, \infty[$ such that $|F(C) - f(x)\nu C| \le \lambda C$ whenever $(x, C) \in \delta \cap Q$.

1C Lemma Let $\mathcal{C}_1$ be the set of non-empty intervals (open, closed or half-open) in $[0, 1]$, and $\mathcal{E}_1$ the algebra of subsets of $[0, 1]$ generated by $\mathcal{C}_1$. Suppose that $\epsilon \ge 0$, and $\phi : \mathcal{C}_1 \to [0, 1]$ is a function such that $\sum_{i=0}^{n} \phi C_i \le \epsilon$ whenever $C_0, \dots, C_n \in \mathcal{C}_1$ are disjoint. Then there is an additive functional $\lambda : \mathcal{E}_1 \to [0, \epsilon]$ such that $\phi C \le \lambda C$ for every $C \in \mathcal{C}_1$.

much better proof

proof (a) Let $D_0, \dots, D_k \in \mathcal{C}_1$ be disjoint intervals such that $\sup D_i = \inf D_{i+1}$ for $i < k$, $\mathcal{D}$ the subring of $\mathcal{E}_1$ generated by $\{D_i : i \le k\}$, and $\psi : \mathcal{C}_1 \cap \mathcal{D} \to [0, \infty[$ any function; set

$$\gamma = \max\{\sum_{i=0}^n \psi C_i : C_0,\dots,C_n \in \mathcal{C}_1 \cap \mathcal{D} \text{ are disjoint}\}.$$

Then there is an additive functional $\lambda : \mathcal{D} \to [0,\gamma]$ such that $\lambda C \ge \psi C$ for every $C \in \mathcal{C}_1 \cap \mathcal{D}$. **P** Induce on k. If $k = 0$ we have only to set $\lambda\emptyset = 0$, $\lambda D_0 = \psi D_0$. For the inductive step to $k+1$, let $\mathcal{D}'$ be the subring of $\mathcal{E}_1$ generated by $D_0,\dots,D_k$ and

$$\gamma' = \max\{\sum_{i=0}^n \psi C_i : C_0,\dots,C_n \in \mathcal{C}_1 \cap \mathcal{D}' \text{ are disjoint}\}, \quad \alpha = \gamma - \gamma'.$$

Note that if $C_0,\dots,C_n \in \mathcal{C}_1 \cap \mathcal{D}'$ are disjoint then $C_0,\dots,C_n,D_{k+1} \in \mathcal{C}_1 \cap \mathcal{D}$ are disjoint, so $\psi D_{k+1} + \sum_{i=0}^n \psi C_i \le \gamma$; as $C_0,\dots,C_n$ are arbitrary, $\psi D_{k+1} \le \alpha$.

For $C \in \mathcal{C}_1 \cap \mathcal{D}'$ set

$$\begin{aligned}\psi' C &= \max(\psi C, \psi(C \cup D_{k+1}) - \alpha) \text{ if } D_k \subseteq C\\ &= \psi C \text{ if } D_k \not\subseteq C.\end{aligned}$$

If $C_0,\dots,C_n \in \mathcal{C}_1 \cap \mathcal{D}'$ are disjoint, with $\sup C_i \le \inf C_{i+1}$ for every $i < n$, then either $\sum_{i=0}^n \psi' C_i = \sum_{i=0}^n \psi C_i \le \gamma'$, or $D_k \subseteq C_n$ and $\psi' C_n = \psi(C_n \cup D_{k+1}) - \alpha$. In the latter case, $C_0,\dots,C_{n-1},C_n \cup D_{k+1}$ are disjoint members of $\mathcal{C}_1 \cap \mathcal{D}$, so

$$\sum_{i=0}^n \psi' C_i = \psi(C_n \cup D_{k+1}) - \alpha + \sum_{i=0}^{n-1} \psi C_i \le \gamma - \alpha = \gamma'.$$

By the inductive hypothesis, there is an additive functional $\lambda' : \mathcal{D}' \to [0,\gamma']$ such that $\lambda' C \ge \psi' C$ for every $C \in \mathcal{C}_1 \cap \mathcal{D}'$. Define $\lambda : \mathcal{D} \to [0,\gamma]$ by setting

$$\begin{aligned}\lambda E &= \alpha + \lambda'(E \setminus D_{k+1}) \text{ if } D_{k+1} \subseteq E\\ &= \lambda' E \text{ otherwise;}\end{aligned}$$

then λ is additive. If $C \in \mathcal{C}_1 \cap \mathcal{D}$, then

$$\begin{aligned}\lambda C &= \alpha \ge \psi C \text{ if } C = D_{k+1},\\ &= \alpha + \lambda'(C \setminus D_{k+1}) \ge \alpha + \psi'(C \setminus D_{k+1}) \ge \alpha + \psi C - \alpha = \psi C\\ &\qquad\qquad \text{if } C \supseteq D_k \cup D_{k+1},\\ &= \lambda' C \ge \psi' C \ge \psi C \text{ if } C \cap D_{k+1} = \emptyset.\end{aligned}$$

(The point is that these three alternatives are exhaustive.) So λ witnesses that the induction proceeds. **Q**

(b) It follows that for each subring $\mathcal{D}$ of $\mathcal{E}_1$ generated by a finite set of contiguous intervals, we have an additive functional $\lambda_{\mathcal{D}} : \mathcal{D} \to [0,\epsilon]$ such that $\lambda_{\mathcal{D}} C \ge \phi C$ for every $C \in \mathcal{C}_1 \cap \mathcal{D}$. Taking a cluster point of these as $\mathcal{D}$ increases, we get a suitable functional on the whole of $\mathcal{E}_1$.

1D Corollary Let $([0,1], T, \Delta, \{\{\emptyset\}\})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}_1$, and $\nu : \mathcal{C}_1 \to \mathbb{R}$ a functional. If $f : [0,1] \to \mathbb{R}$ is such that $I_\nu(f)$ is defined, then the Saks-Henstock indefinite integral of f is moderated.

proof Let $F : \mathcal{E}_1 \to \mathbb{R}$ be the Saks-Henstock indefinite integral of f. Then there is a $\delta \in \Delta$ such that $\sum_{(x,C)\in \boldsymbol{t}} |F(C) - f(x)\nu C| \le 1$ for every δ-fine $\boldsymbol{t} \in T$. Set $Q = \{(x,C) : \{(x,C)\} \in T\}$. For $C \in \mathcal{C}_1$, set

$$\phi(C) = \sup\{|F(C) - f(x)\nu C| : (x,C) \in Q \cap \delta\},$$

interpreting $\sup\emptyset$ as 0. Then $\sum_{i=0}^n \phi(C_i) \le 1$ whenever $C_0,\dots,C_n \in \mathcal{C}_1$ are disjoint. **P** It is enough to consider the case in which $\phi C_i > 0$ for each i. For any $\eta > 0$, we can find $x_0,\dots,x_k$ such that $(x_i,C_i) \in Q \cap \delta$ and $\phi C_i \le |F(C_i) - f(x_i)\mu_1 C_i| + \eta$. Now $\boldsymbol{t} = \{(x_i,C_i) : i \le k\}$ belongs to T and is δ-fine, so

$$\sum_{i=0}^k \phi C_i \le (k+1)\eta + \sum_{i=0}^k |F(C_i) - f(x_i)\mu_1 C_i| \le (k+1)\eta + 1;$$

as η is arbitrary, we have the result. **Q** By Lemma 1C, there is an additive functional $\lambda : \mathcal{E}_1 \to [0,1]$ such that $\phi(C) \le \lambda C$ for every $C \in \mathcal{C}_1$, and this is more than we need.

1E Example For each $n \in \mathbb{N}$, let X_n be the set of doubleton subsets of $n+2$, and set $C_{ni} = \{a : i \in a \in X_n\}$ for $i \le n+1$; let μ_n be the uniform probability measure on X_n, so that $\mu_n C_{ni} = \frac{2}{n+2}$. Set

$X=\prod_{n\in\mathbb{N}}X_n$ and let ν be the product measure on X, so that ν is an atomless Radon measure. Let $\mathcal{C}$ be the family of subsets of X of the form $\{x:x\restriction J=z\}$ where $J\in[\mathbb{N}]^{<\omega}$ and $z\in\prod_{n\in J}X_n$; let T be the straightforward set of tagged partitions generated by $Q=\{(x,C):x\in C\in\mathcal{C}\}$. For $n\in\mathbb{N}$ and $i\le n+1$ set $C'_{ni}=\{x:x(n)\in C_{ni}\}$, and fix a point y_{ni} of C'_{ni}. For $n\in\mathbb{N}$ let δ_n be

$$\{(x,C):x\in C\in\mathcal{C},\,x\ne y_{ni}\text{ for all }n,\,i\}\cup\{(y_{mi},C'_{mi}):C\in\mathcal{C},\,m\ge n,\,i\le m+1\}.$$

Set $\Delta=\{\delta_n:n\in\mathbb{N}\}$. Then $(X,T,\Delta,\{\{\emptyset\}\})$ is a tagged-partition structure allowing subdivisions.

Define $f:X\to\{0,1\}$ by setting $f(y_{ni})=\sqrt{n}$ for $i\le n\in\mathbb{N}$, $f(x)=0$ for other x. Then $I_\nu(f)=0$. **P** If $n\in\mathbb{N}$ and $\boldsymbol{t}\in T$ is δ_n-fine, then there is at most one pair (m,i) such that $(y_{mi},C'_{mi})\in\boldsymbol{t}$, and in this case $m\ge n$, so $|S_{\boldsymbol{t}}(f,\nu)|\le\frac{2\sqrt{n}}{n+2}$. **Q** The same argument shows that the constant function with value 0 is the Saks-Henstock indefinite integral of f. **?** If this is moderated, let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{C}$, and take $n\in\mathbb{N}$ and an additive functional $\lambda:\mathcal{E}\to[0,\infty[$ such that $|f(x)\nu C|\le\lambda C$ whenever $(x,C)\in Q\cap\delta_n$. Let $m\ge n$ be such that $m>(\lambda X)^2$. Then $(y_{mi},C'_{mi})\in Q\cap\delta_n$ so $\lambda C'_{mi}\ge\frac{2\sqrt{m}}{m+2}$ whenever $i\le m+1$. But since every point of X belongs to C'_{mi} for exactly two different $i\le m+1$, $\lambda X\ge\sqrt{m}$. **X**

Thus we have a Saks-Henstock indefinite integral which is not moderated.

2 Product gauge integrals

2A Theorem Let $(X_1,\mathfrak{T}_1,Q_1,T_1,\Delta_1,\mathcal{C}_1,\mathcal{E}_1,\nu_1,I_{\nu_1})$ and $(X_2,\mathfrak{T}_2,Q_2,T_2,\Delta_2,\mathcal{C}_2,\mathcal{E}_2,\nu_2,I_{\nu_2})$ be such that, for each i,

$(X_i,\mathfrak{T}_i)$ is a topological space,

Δ_i is the set of all neighbourhood gauges on X_i,

$\mathcal{C}_i\subseteq\mathcal{P}X_i$, $Q_i\subseteq X_i\times\mathcal{C}_i$ and T_i is the straightforward set of tagged partitions generated by Q_i,

$(X_i,T_i,\Delta_i,\{\{\emptyset\}\})$ is a tagged-partition structure allowing subdivisions witnessed by $\mathcal{C}_i$,

$\mathcal{E}_i$ is the algebra of subsets of X_i generated by $\mathcal{C}_i$, and $\nu_i:\mathcal{E}_i\to[0,\infty[$ is an additive function,

I_{ν_i} is the gauge integral defined from $(X_i,T_i,\Delta_i,\{\{\emptyset\}\})$ and ν_i.

(a) Set

$X=X_1\times X_2$, with the product topology,

Δ the set of neighbourhood gauges on X,

$Q=\{((x,y),C\times D):(x,C)\in Q_1,\,(y,D)\in Q_2\}$,

T the straightforward set of tagged partitions generated by Q,

$\nu(C\times D)=\nu_1C\cdot\nu_2D$ for $C\in\mathcal{C}_1$, $D\in\mathcal{D}_2$.

Then $(X,T,\Delta,\{\{\emptyset\}\})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}=\{C\times D:C\in\mathcal{C}_1,\,D\in\mathcal{C}_2\}$.

(b) Let I_ν be the gauge integral defined by $(X,T,\Delta,\{\{\emptyset\}\})$ and ν. Suppose that $f:X_1\to\mathbb{R}$ and $g:X_2\to\mathbb{R}$ are functions such that $I_{\nu_1}(f)$ and $I_{\nu_2}(g)$ are defined, and f has a moderated Saks-Henstock indefinite integral. Set $(f\otimes g)(x,y)=f(x)g(y)$ for $x\in X$ and $y\in Y$. Then $I_\nu(f\otimes g)$ is defined and equal to $I_{\nu_1}(f)\cdot I_{\nu_2}(g)$.

proof (a) See 482M in Fremlin 03.

(b)(i) For each i let $\mathcal{E}_i$ be the algebra of subsets of X_i generated by $\mathcal{C}_i$; let $F:\mathcal{E}_1\to\mathbb{R}$ and $G:\mathcal{E}_2\to\mathbb{R}$ be the Saks-Henstock indefinite integrals of f, g respectively. Let $\delta_1\in\Delta_1$ and $\lambda:\mathcal{E}_1\to[0,\infty[$ be such that λ is additive and $|F(C)-f(x)\nu_1C|\le\lambda C$ whenever $(x,C)\in Q_1\cap\delta_1$.

Let $\epsilon>0$. Then for each $n\in\mathbb{N}$ there are $\delta_{1n}\in\Delta_1$, $\delta_{2n}\in\Delta_2$ such that

$$\sum_{(x,C)\in\boldsymbol{s}}|F(C)-f(x)\nu_1C|\le 2^{-n}\epsilon\text{ whenever }\boldsymbol{s}\in T_1\text{ is }\delta_{1n}\text{-fine,}$$

$$\sum_{(y,D)\in\boldsymbol{t}}|G(D)-g(y)\nu_2D|\le 2^{-n}\epsilon\text{ whenever }\boldsymbol{t}\in T_2\text{ is }\delta_{2n}\text{-fine.}$$

Because Δ_1 is downwards-directed, we may suppose that $\delta_{1n}\subseteq\delta_1$ for every n.

For each $n\in\mathbb{N}$, let $\langle G^{(n)}_x\rangle_{x\in X_1}$ and $\langle H^{(n)}_y\rangle_{y\in X_2}$ be families of open sets such that

$$\delta_{1n}=\{(x,C):x\in X_1,\,C\subseteq G^{(n)}_x\},\quad\delta_{2n}=\{(y,D):y\in X_2,\,D\subseteq H^{(n)}_y\}.$$

For $(x,y)\in X$, set $m(x,y)=\lceil |f(x)|+|g(y)|\rceil$, and set

$$\delta=\{((x,y),A):x\in X_1,\,y\in X_2,\,A\subseteq G_x^{(m(x,y))}\times H_y^{(m(x,y))}\}\in\Delta.$$

Let $\boldsymbol{u}\in T$ be δ-fine. I seek to estimate

$$\begin{aligned}\sum_{((x,y),C\times D)\in\boldsymbol{u}}&|F(C)G(D)-f(x)g(y)\nu(C\times D)|\\&\le\sum_{((x,y),C\times D)\in\boldsymbol{u}}|g(y)|\nu_2D|F(C)-f(x)\nu_1C|\\&\qquad\qquad+|F(C)||G(D)-g(y)\nu_2D|\\&=\sum_{n=0}^{\infty}\sum_{((x,y),C\times D)\in\boldsymbol{u_n}}|g(y)|\nu_2D|F(C)-f(x)\nu_1C|\\&\qquad\qquad+|F(C)||G(D)-g(y)\nu_2D|,\end{aligned}$$

where $\boldsymbol{u}_n=\{((x,y),C\times D):((x,y),C\times D)\in\boldsymbol{u},\,m(x,y)=n\}$ for each n.

(ii) For each $n\in\mathbb{N}$,

$$\textstyle\sum_{((x,y),C\times D)\in\boldsymbol{u}_n}|F(C)||G(D)-g(y)\nu_2(D)|\le 2^{-n}(n\nu_1X_1+\lambda X)\epsilon.$$

P For $z\in X_1$, set

$$h(z)=\sum_{\substack{((x,y),C\times D)\in\boldsymbol{u}_n\\ z\in C}}|G(D)-g(y)\nu_2(D)|=\sum_{(y,D)\in\boldsymbol{t}_z}|G(D)-g(y)\nu_2(D)|$$

(where $\boldsymbol{t}_z=\{(y,D):((x,y),C\times D)\in\boldsymbol{u}_n,\,z\in C\}$)

$$\le 2^{-n}\epsilon,$$

because if $((x,y),C\times D)$ and $((x',y'),C'\times D')$ are distinct members of $\boldsymbol{u}_n$ such that $z\in C\cap C'$, then we must have $D\cap D'=\emptyset$, so $\boldsymbol{t}_z\in T_2$; while also $((x,y),C\times D)\in\delta$ and $m(x,y)=n$, so $D\subseteq H_y^{(n)}$ and $(y,D)\in\delta_{2n}$, so $\boldsymbol{t}_z$ is δ_{2n}-fine.

Set $\lambda_nE=n\nu_1E+\lambda E$ for $E\in\mathcal{E}_1$, so that λ_n is a non-negative finitely additive functional and $\lambda_nX\le n\nu_1X_1+\lambda X$. Since h is constant on each of the atoms of the finite algebra generated by $\{C:((x,y),C\times D)\in\boldsymbol{u}_n\}$, we can speak of $\int h\,d\lambda_n$, interpreted as in Fremlin 02, 363L, and we shall have

$$\int h\,d\lambda_n\le 2^{-n}(n\nu_1X_1+\lambda X)\epsilon.$$

For $((x,y),C\times D)\in\boldsymbol{u}_n$, $|f(x)|\le n$ and $(x,C)\in\delta_{10}$, so

$$|F(C)|\le|f(x)|\nu_1C+|F(C)-f(x)\nu_1C|\le n\nu_1C+\lambda C=\lambda_nC.$$

Accordingly

$$\begin{aligned}\sum_{((x,y),C\times D)\in\boldsymbol{u}_n}&|F(C)||G(D)-g(y)\nu_2(D)|\\&\le\sum_{((x,y),C\times D)\in\boldsymbol{u}_n}|G(D)-g(y)\nu_2(D)|\lambda_nC\\&=\int\sum_{((x,y),C\times D)\in\boldsymbol{u}_n}|G(D)-g(y)\nu_2(D)|\chi C\,d\lambda_n\\&=\int h\,d\lambda_n\le 2^{-n}(n\nu_1X_1+\lambda X)\epsilon,\end{aligned}$$

as required. **Q**

(iii) Similarly, for each $n\in\mathbb{N}$,

$$\textstyle\sum_{((x,y),C\times D)\in\boldsymbol{u}_n} |g(y)|\nu_2 D|F(C)-f(x)\nu_1 C| \le 2^{-n}n\epsilon\nu_2 X_2.$$

P Use the same method, but without the refinement giving bounds for $|F(C)-f(x)\nu_1 C|$. **Q**

(iv) So if $W_{\boldsymbol{u}} = X$, we shall have

$$\begin{aligned}|F(X_1)G(X_2) - &\sum_{((x,y),C\times D)\in\boldsymbol{u}} f(x)g(y)\nu(C\times D)| \\ &\le \sum_{((x,y),C\times D)\in\boldsymbol{u}} |F(C)G(D) - f(x)g(y)\nu(C\times D)|\end{aligned}$$

(because F and G are additive, so $F(X_1)G(X_2) = \sum_{(x,y),C\times D)\in\boldsymbol{u}} F(C)G(D))$

$$\le \sum_{n=0}^{\infty} 2^{-n}(n\nu_1 X_1 + n\nu_2 X_2 + \lambda X)\epsilon = (4\nu_1 X_1 + 4\nu_2 X_2 + 2\lambda X)\epsilon.$$

As ϵ is arbitrary, $I_\nu(f\otimes g)$ is defined and equal to $F(X_1)G(X_2) = I_{\nu_1}(f)\cdot I_{\nu_2}(g)$.

2B Remark Note that in the context of 2A, we have a version of Fubini's theorem: if $h : X \to \mathbb{R}$ is such that $I_\nu(h)$ is defined, and $f : X_1 \to \mathbb{R}$ is such that $f(x) = I_{\nu_2}(h_x)$ whenever the latter is defined (where $h_x(y) = h(x,y)$ for $(x,y)\in X$), then $I_{\nu_1}(f)$ is defined and equal to $I_\nu(h)$ (Fremlin 03, 482M).

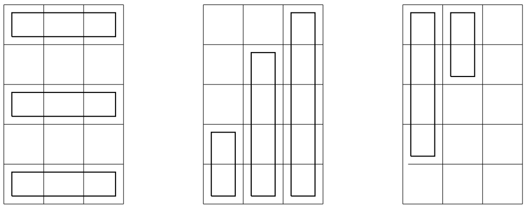
2C The multidimensional Henstock integral Suppose that $r \ge 1$ is an integer. Let $\mathcal{C}_r$ be the set of non-empty intervals in $[0,1]^r$, that is to say, products of non-empty intervals (open, half-open or closed) in $[0,1]$, and T_r the straightforward set of tagged partitions generated by $\{(x,C) : C\in\mathcal{C}_r,\ x\in\overline{C}\}$ in the sense of Fremlin 03, 481B. If Δ_r is the set of neighbourhood gauges on $[0,1]^r$, then $([0,1]^r, T_r, \Delta_r, \{\{\emptyset\}\})$ is a tagged-partition structure allowing subdivisions (Fremlin 03, 481G and 481P). Let μ_r be Lebesgue measure on $[0,1]^r$. I will call the gauge integral corresponding to $([0,1]^r, T_r, \Delta_r, \{\{\emptyset\}\})$ and μ_r the **Henstock integral** on $[0,1]^r$. It is elementary that for any r, $s\ge 1$, the product structure on $[0,1]^r\times[0,1]^s$, as set up in 2A, matches the structure on $[0,1]^{r\times s}$. By Fremlin 03, 482F, $I_{\nu_r}(f)$ is defined and equal to $\int f d\mu_r$ whenever $f : [0,1]^r \to \mathbb{R}$ is μ_r-integrable.

Putting 1D and 2A together, we see that $f\otimes g : [0,1]^{r+1} \to \mathbb{R}$ is Henstock integrable, with the right integral, whenever $f : [0,1] \to \mathbb{R}$ and $g : [0,1]^r \to \mathbb{R}$ are Henstock integrable (compare Henstock 83). I do not know whether the corresponding result is true for Henstock integrable f, $g : [0,1]^2 \to \mathbb{R}$.

2D Remark The example in 1E is not relevant to Henstock integration. However Lemma 1C does not carry over to two dimensions in the exact form given. The following example is due to Mircea Petrarche[1]. Take any $n\ge 1$. Suppose that, in $[0,1]^2$, we set $Q_{ij} = \left[\frac{i}{2n-1}, \frac{i+1}{2n-1}\right[\times \left[\frac{j}{2n-1}, \frac{j+1}{2n-1}\right[$ for $i<n$, $j<2n-1$. Let $\mathcal{D}\subseteq\mathcal{C}_2$ be the family of rectangles of the forms

$$\begin{aligned}C_{0k} &= \bigcup_{i<n} Q_{i,2k} \text{ for } k<n,\\ C_{1k} &= \bigcup_{j\le 2k+1} Q_{k,j} \text{ for } k<n-1,\\ C'_{1k} &= \bigcup_{2k+1\le j<2n-1} Q_{k,j} \text{ for } k<n-1,\\ C_2 &= \bigcup_{j\le 2n-2} Q_{n-1,j}.\end{aligned}$$

[1] E-mail of 1.7.11.



The eight sets when $n = 3$

Then any disjoint $\mathcal{D}_0 \subseteq \mathcal{D}$ has at most n members. **P** $\mathcal{D}_0$ can contain at most one of C_{1k}, C'_{1k} for each k, and having chosen $r \leq n-1$ of these, there are at most $n-r$ possible elements of the form C_{0k} which can be in $\mathcal{D}_0$, all of which are excluded if $C_2 \in \mathcal{D}_0$. **Q** So if we set

$$\begin{aligned} \phi C &= \frac{1}{n} \text{ if } C \in \mathcal{D}, \\ &= 0 \text{ for other } C \in \mathcal{C}_2, \end{aligned}$$

$\sum_{C\in\mathcal{D}_0} \phi C \leq 1$ for every disjoint $\mathcal{D}_0 \subseteq \mathcal{C}_2$. On the other hand, no point belongs to more than two members of $\mathcal{D}$, so if λ is any non-negative additive functional on $\mathcal{E}_2$ such that $\lambda C \geq \phi C$ for every C,

$$\frac{3n-1}{n} \leq \sum_{C\in\mathcal{D}} \sum_{Q_{ij}\subseteq C} \lambda Q_{ij} \leq 2\sum_{i<n,j<2n} \lambda Q_{ij} \leq 2\lambda[0,1]^2,$$

and $\lambda[0,1]^2 \geq \frac{3n-1}{2n}$.

Of course this leaves open the possibility that there is some constant M such that

if $\phi : \mathcal{C}_2 \to [0,1]$ is a function such that $\sum_{i=0}^{n} \phi C_i \leq 1$ whenever $C_0, \dots, C_n \in \mathcal{C}_2$ are disjoint, then there is an additive functional $\nu : \mathcal{E}_2 \to [0,M]$ such that $\phi C \leq \nu C$ for every $C \in \mathcal{C}_2$,

and this would be enough to make Saks-Henstock indefinite integrals moderated for Henstock integration on $[0,1]^2$.

Acknowledgments Conversations with M.Laczkovich and S.Bianchini; hospitality of M.Csörnyei's seminar, University College, London; hospitality of SISSA, Trieste; correspondence with W.F.Pfeffer and M.Petrarche.

References

Fremlin D.H. [02] *Measure Theory, Vol. 3: Measure Algebras.* Torres Fremlin, 2002.

Fremlin D.H. [03] *Measure Theory, Vol. 4: Topological Measure Spaces.* Torres Fremlin, 2003.

Henstock R. [83] 'A problem in two-dimensional integration', J. Australian Math. Soc. (A) 35 (1983) 386-404.