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Notes on first-return integration

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Throughout this note I shall take it that we are working with Lebesgue measure μ. An undecorated integral $\int f$ will always be the Lebesgue integral $\int_0^1 f d\mu$.

1 Basic facts

1A First-return integrals I will say that a **trajectory** in $[0,1]$ is a sequence $\omega \in [0,1]^{\mathbb{N}}$ such that $\{\omega(n) : n \in \mathbb{N}\}$ is dense in $[0,1]$. Fix a trajectory ω for the time being.

I refer to §481 of Fremlin 03 for basic theory and notation concering gauge integrals. Let $\mathcal{C}$ be the set of non-empty subintervals (open, closed or half-open) of $[0,1]$. For $C \in \mathcal{C}$, write $r_\omega(C) = \omega(n)$ if $n = \min\{i : \omega(i) \in \overline{I}\}$ is defined; otherwise, C must be a singleton, take $r_\omega(C)$ is to be its member. Let Q_ω be $\{(r_\omega(C), C) : C \in \mathcal{C}\}$ and T_ω the straightforward set of tagged partitions generated by Q_ω. Let Δ be the set of neighbourhood gauges on $[0,1]$, and $\mathfrak{R} = \{\{\emptyset\}\}$. Then $(X, T_\omega, \Delta, \mathfrak{R})$ is a tagged-partition structure admitting subdivisions, witnessed by $\mathcal{C}$. Write ω-$\mathrm{F}\!\!\int\!\!\mathrm{R}$ for the corresponding gauge integral. It is elementary to check that if f is Riemann integrable over $[0,1]$ then ω-$\mathrm{F}\!\!\int\!\!\mathrm{R} f$ is defined and equal to $\int f$ (cf. Fremlin 03, 481I).

By Fremlin 03, 482B and 482G we see that if ω-$\mathrm{F}\!\!\int\!\!\mathrm{R} f$ is defined then we have a corresponding Saks-Henstock indefinite integral $F : \mathcal{C} \to \mathbb{R}$, and ω-$\mathrm{F}\!\!\int\!\!\mathrm{R} f \times \chi C$ is defined and equal to $F(C)$ whenever ω-$\mathrm{F}\!\!\int\!\!\mathrm{R} f$ is defined and $C \in \mathcal{C}$.

1B Lemma Suppose that $f : [0,1] \to \mathbb{R}$ and $g : [0,1] \to [0,\infty]$ are functions such that g is Lebesgue integrable and $|f| \le g$, and that there is a non-decreasing sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ of closed subsets of $[0,1]$ such that $f{\restriction}K_n$ is continuous for every n, $g(x) \le n$ for every $n \in \mathbb{N}$ and $x \in K_n$, and $\mu\{x : x \in [0,1] \setminus K_n, g(x) \le n\} \le 2^{-n}$ for every $n \in \mathbb{N}$. Let ω be a trajectory in $[0,1]$ such that whenever $(x, C) \in Q_\omega$, $n \in \mathbb{N}$ and $x \notin K_n$, then $C \cap K_n = \emptyset$. Then ω-$\mathrm{F}\!\!\int\!\!\mathrm{R} f$ is defined, and ω-$\mathrm{F}\!\!\int\!\!\mathrm{R} f \times \chi C = \int_C f$ for every $C \in \mathcal{C}$.

proof (a) Of course f is measurable, because $\bigcup_{n\in\mathbb{N}} K_n$ is conegligible; so the Lebesgue integral $\int f$ is defined.

(b)(i) Take $\epsilon > 0$. Let n be so large that $\int_{[0,1]\setminus K_n}(g+1) \le \epsilon$ and $\sum_{k=n}^{\infty} 2^{-k}(k+1) \le \epsilon$. Let $f_1 : [0,1] \to [-n, n]$ be a continuous function extending $f{\restriction}K_n$. Let $\eta > 0$ be such that $|S_{\boldsymbol{t}}(f_1, \mu) - \int f_1| \le \epsilon$ whenever $\boldsymbol{t} \in T_\omega$ is δ_η-fine and $\{\emptyset\}$-filling, where δ_η is the uniform metric gauge corresponding to η; that is, $\operatorname{diam} C \le \eta$ whenever $(x, C) \in \boldsymbol{t}$ and $W_{\boldsymbol{t}} = \bigcup_{(x,C)\in\boldsymbol{t}} C = [0,1]$.

(ii) $|\int f - \int f_1| \le 2\int_{[0,1]\setminus K_n} g \le 2\epsilon$.

(iii) Suppose that $\boldsymbol{t} \in T_\omega$ is $\{\emptyset\}$-filling. Set

$$h(y) = f(x) \text{ whenever } (x, C) \in \boldsymbol{t} \text{ and } y \in C,$$

$$h_1(y) = f_1(x) \text{ whenever } (x, C) \in \boldsymbol{t} \text{ and } y \in C,$$

so that h and h_1 are step-functions and

$$S_{\boldsymbol{t}}(f, \mu) = \int h, \quad S_{\boldsymbol{t}}(f_1, \mu) = \int h_1.$$

By (b), h and h_1 agree on K_n.

(iv) Set $F_n = \{x : g(x) \le n\}$ for $n \in \mathbb{N}$. Then

$$\int_{[0,1]\setminus K_n} |h_1| \le n\mu([0,1] \setminus K_n) \le n(\mu(F_n \setminus K_n) + \mu([0,1] \setminus F_n)$$

$$\le 2^{-n} n + \int_{[0,1]\setminus K_n} g \le 2\epsilon.$$

(v) If $k \geq n$ and $y \in K_{k+1} \setminus K_k$, let $(x, C) \in \boldsymbol{t}$ be such that $y \in C$. Then $x \in K_{k+1}$ so $|h(y)| = |f(x)| \leq g(x) \leq k+1$. This shows that

$$\begin{aligned}\int_{[0,1]\setminus K_n} |h| &\leq \sum_{k=n}^{\infty}(k+1)\mu(K_{k+1} \setminus K_k) \leq \sum_{k=n}^{\infty}(k+1)(\mu(F_{k+1} \setminus F_k) + \mu(F_k \setminus K_k)) \\ &\leq \int_{[0,1]\setminus K_n}(g+1) + \sum_{k=n}^{\infty} 2^{-k}(k+1) \leq 2\epsilon.\end{aligned}$$

(vi) So if $\boldsymbol{t}$ is also δ_η-fine,

$$\begin{aligned}|S_{\boldsymbol{t}}(f,\mu) - \int f| &\leq |S_{\boldsymbol{t}}(f,\mu) - S_{\boldsymbol{t}}(f_1,\mu)| + |S_{\boldsymbol{t}}(f_1,\mu) - \int f_1| + \int |f_1 - f| \\ &\leq \int |h - h_1| + \epsilon + 2\epsilon \leq \int_{[0,1]\setminus K_n} |h| + \int_{[0,1]\setminus K_n} |h_1| + 3\epsilon \\ &\leq 2\epsilon + 2\epsilon + 3\epsilon = 7\epsilon.\end{aligned}$$

(vii) As ϵ is arbitrary, ω-$\mathrm{F}\!\!\!\int\!\!\mathrm{R} f$ is defined and equal to $\int f$.

(c) As for the indefinite integral, observe that if ϵ, n and η are as in (b), and $\boldsymbol{s} \in T_\omega$ is δ_η-fine, then $\boldsymbol{s}$ can be extended to a δ_η-fine $\{\emptyset\}$-filling $\boldsymbol{t}$. Defining h from $\boldsymbol{t}$ as in (c-iii), we have

$$\begin{aligned}\sum_{(x,C)\in\boldsymbol{s}} |\int_C f - f(x)\mu C| &\leq \sum_{(x,C)\in\boldsymbol{t}} |\int_C f - f(x)\mu C| = \int |f - h| \\ &\leq \int_{[0,1]\setminus K_n} |f| + |h| \leq 2\epsilon + \int_{[0,1]\setminus K_n} |g| \leq 3\epsilon.\end{aligned}$$

But this means that $C \mapsto \int_C f$ satisfies the defining property of the Saks-Henstock indefinite integral (Fremlin 03, 482B), and $\int_C f = \omega$-$\mathrm{F}\!\!\!\int\!\!\mathrm{R} f \times \chi C$ for every $C \in \mathcal{C}$.

1C Theorem Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence of Lebesgue integrable real-valued functions defined everywhere on $[0,1]$, and $E \subseteq [0,1]$ a negligible set. Then there is a trajectory $\omega \in ([0,1] \setminus E)^{\mathbb{N}}$ such that ω-$\mathrm{F}\!\!\!\int\!\!\mathrm{R} f_n \times \chi C$ is defined and equal to $\int_C f_n$ for every $n \in \mathbb{N}$ and $C \in \mathcal{C}$.

proof Let $\langle \epsilon_n \rangle_{n\in\mathbb{N}}$ be a sequence of strictly positive real numbers such that $\sum_{n=0}^{\infty} \epsilon_n \int |f_n|$ is finite. Set $g(x) = \sum_{n=0}^{\infty} \epsilon_n |f_n(x)|$ for every x. Since μ is inner regular with respect to $\{K : f_n\restriction K$ is continuous for every $n \in \mathbb{N}\}$ (Fremlin 03, 418J), we can find a non-decreasing sequence $\langle K_n \rangle_{n\in\mathbb{N}}$ of compact subsets of $[0,1] \setminus E$ such that $f_m\restriction K_n$ is continuous for every m and n, $g(x) \leq n$ for every $n \in \mathbb{N}$ and $x \in K_n$, and $\mu\{x : x \in [0,1] \setminus K_n,\ g(x) \leq n\} \leq 2^{-n}$ for every $n \in \mathbb{N}$. Next, we can find a trajectory ω satisfying the condition of Lemma 1B. **P** Enumerate $\mathbb{Q} \cap [0,1]$ as $\langle q_n \rangle_{n\in\mathbb{N}}$. Choose $\langle \omega(n) \rangle_{n\in\mathbb{N}}$ inductively, as follows. Start with $\omega(0) = \max K_l$ where l is the first integer such that $K_l \neq \emptyset$. Given $\langle \omega(i) \rangle_{i<n}$, then for $k \in \mathbb{N}$ let C_{nk} be the component of $[0,1] \setminus K_k$ containing q_n, if there is one, and otherwise set $C_{nk} = \{q_n\}$. If there is a first k such that one of the endpoints of C_{nk} belongs to $K_k \setminus \{\omega(i) : i < n\}$, take such an endpoint for $\omega(n)$; otherwise set $\omega(n) = \omega(0)$. In this case, if $(x, C) \in Q_\omega$, $m \in \mathbb{N}$ and $x \notin K_m$, we have $x = r_\omega(C) = \omega(n)$ for some n. **?** If $C \cap K_m \neq \emptyset$ let C' be the component of $[0,1] \setminus K_m$ containing $\omega(n)$; one of the endpoints of C' must belong to $C \cap K_m$. We surely do not have $n = 0$, because $K_m \neq \emptyset$ so $m \geq l$ and $x \notin K_l$. So $\omega(n)$ is an endpoint of C_{nk} for the first k such that ∂C_{nk} meets $K_k \setminus \{\omega(i) : i < n\}$, and we must have $k > m$. In particular, $C' \neq C_{nm}$ and $C' \cap \overline{C}_{nk} = \emptyset$, which is absurd, because $\omega(n) \in \overline{C}_{nk}$. **X**

I had better explain why $\{\omega(n) : n \in \mathbb{N}\}$ is dense; this is because if $C \subseteq [0,1[$ is a non-empty open interval, then there is a first k such that $C \cap K_k \neq \emptyset$. If C' is a component of $[0,1] \setminus K_k$ meeting C, and $n_0 < \ldots < n_{2k}$ are such that $q_{n_i} \in C' \cap C$ for every $i \leq 2k$, then there must be an $i \leq 2k$ such that $\omega(n_i) \in C$.

Thus ω has the required properties. **Q**

Now Lemma 2B tells us that ω-$\mathrm{F}\!\!\!\int\!\!\mathrm{R} \epsilon_n f_n \times \chi C$ is defined and equal to $\int_C \epsilon_n f_n$ whenever $n \in \mathbb{N}$ and $C \in \mathcal{C}$, so ω-$\mathrm{F}\!\!\!\int\!\!\mathrm{R} f_n \times \chi C$ is defined and equal to $\int_C f_n$ whenever $n \in \mathbb{N}$ and $C \in \mathcal{C}$.

1D The following fact will be useful.

Lemma Suppose that $\omega \in [0,1]^{\mathbb{N}}$ is a trajectory. Let Q'_ω be the set of pairs $(r_\omega(C), C)$ such that $C \in \mathcal{C}$ and $\partial C \subseteq \mathbb{Q} \cup \omega[\mathbb{N}]$; let T'_ω be the straightforward set of tagged partitions generated by Q'_ω. Suppose that $f : [0,1] \to \mathbb{R}$ is Lebesgue integrable function, and η, $\epsilon > 0$ are such that $\sum_{(x,C)\in \boldsymbol{t}} |\int_C f - f(x)\mu C| \le \epsilon$ whenever $\boldsymbol{t} \in T'_\omega$ is δ_η-fine. Then $\sum_{(x,C)\in \boldsymbol{t}} |\int_C f - f(x)\mu C| \le 2\epsilon$ whenever $\boldsymbol{t} \in T_\omega$ is δ_η-fine.

proof If $C \in \mathcal{C}$ has endpoints $\alpha < \beta$, where $\alpha \notin \omega[\mathbb{N}]$, then a small adjustment of α will not change $r_\omega(C)$ and will therefore have a small effect on $|\int_C f - f(r_\omega(C))\mu C$; and the same will apply at the upper end. So for each $(x,C) \in \boldsymbol{t}$ we can fractionally contract C, if necessary, to get a $C' \subseteq C$ such that $(x, C') \in T'_\omega$, and arrange that

$$\textstyle\sum_{(x,C)\in \boldsymbol{t}} |\int_C f - f(x)\mu C| \le \sum_{(x,C)\in \boldsymbol{t}} |\int_{C'} f - f(x)\mu C'| + \epsilon \le 2\epsilon$$

because $\{(x,C') : (x,C) \in \boldsymbol{t}\}$ belongs to T'_ω and is δ_η-fine.

2 An example

Answering a question of B.Bongiorno, I give an example of a function $f : [0,1] \to \mathbb{R}$ and a natural trajectory in $[0,1]$ such that the first-return integral $\omega\text{-}\mathrm{FR}\!\!\int f \times \chi C$ and the Henstock integral $\mathrm{H}\!\!\int_C f$ are defined and equal for every $C \in \mathcal{C}$, but f is not Lebesgue integrable.

2A Construction, part 1 (a) Define $h : \mathbb{R} \to [0,1]$ by setting

$$\begin{aligned} h(x) &= 0 \text{ if } <x> \in [0, \tfrac{1}{4}[\\ &= \tfrac{1}{2} \text{ if } <x> = \tfrac{1}{4} \\ &= 1 \text{ if } <x> \in]\tfrac{1}{4}, \tfrac{1}{2}[\\ &= 0 \text{ if } <x> = \tfrac{1}{2} \\ &= -1 \text{ if } <x> \in]\tfrac{1}{2}, \tfrac{3}{4}[\\ &= -\tfrac{1}{2} \text{ if } <x> = \tfrac{3}{4} \\ &= 0 \text{ if } <x> \in]\tfrac{3}{4}, 1[, \end{aligned}$$

where $<x> = x - \lfloor x \rfloor \in [0,1[$ is the fractional part of x, for each $x \in \mathbb{R}$.

Observe that h is an odd function with period 1, and that $\int_0^1 h = 0$.

(b) For $n \in \mathbb{N}$, $x \in \mathbb{R}$ set

$$g_n(x) = \textstyle\sum_{i=0}^{n-1} h(4^i x).$$

Then each g_n is an odd function with period 1; $g_n(0) = g_n(\frac{1}{2}) = 0$; $g_n(\frac{1}{4}) = \frac{1}{2}$, $g_n(\frac{3}{4}) = -\frac{1}{2}$. Observe also that

$$g_{n+1}(x) = h(x) + g_n(4x)$$

for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$, and that

$$\int_k^{k+1} g_n = \textstyle\sum_{i=0}^{n-1} 4^{-i} \displaystyle\int_{4^i k}^{4^i(k+1)} h = 0$$

for every $n \in \mathbb{N}$ and $k \in \mathbb{Z}$.

(c) $0 \le \int_0^a g_n \le \frac{1}{2}\min(a, 1-a)$ for every $a \in [0,1]$. **P** Induce on n. $g_0(x) = 0$ for every x, so the start is trivial. For the inductive step to $n+1$,

$$\int_0^a g_{n+1} = \int_0^a h(x) + g_n(4x)dx = \int_0^a h + \frac{1}{4}\int_0^{4a} g_n$$
$$= \int_0^a h + \frac{1}{4}\int_{\lfloor 4a\rfloor}^{4a} g_n = \int_0^a h + \frac{1}{4}\int_0^{<4a>} g_n \geq 0$$

for every a. Next, using the inductive hypothesis to bound integrals of the form $\int_0^b g_n$,

$$\int_0^a g_{n+1} = \frac{1}{4}\int_0^{4a} g_n \leq \frac{1}{4}\cdot\frac{1}{2}\cdot 4a = \frac{1}{2}a$$
$$\text{if } 0 \leq a \leq \frac{1}{4},$$
$$= a - \frac{1}{4} + \frac{1}{4}\int_0^{4a-1} g_n \leq a - \frac{1}{4} + \frac{1}{8}(1-(4a-1)) = \frac{1}{2}a$$
$$\text{if } \frac{1}{4} \leq a < \frac{1}{2},$$
$$= \frac{3}{4} - a + \frac{1}{4}\int_0^{4a-2} g_n \leq \frac{3}{4} - a + \frac{1}{8}(4a-2) = \frac{1}{2}(1-a)$$
$$\text{if } \frac{1}{2} \leq a < \frac{3}{4},$$
$$= \frac{1}{4}\int_0^{4a-3} g_n \leq \frac{1}{8}(4-4a) = \frac{1}{2}(1-a)$$
$$\text{if } \frac{3}{4} \leq a < 1,$$
$$= 0 = \frac{1}{2}(1-a) \text{ if } a = 1. \ \mathbf{Q}$$

(d) $|\int_a^b g_n| \leq b - a$ if $0 \leq a \leq \frac{1}{2} \leq b \leq 1$. **P** Induce on n. If $n = 0$, this is trivial. For the inductive step to $n+1$, argue as follows.

(i) If $a = \frac{1}{2} \leq b \leq \frac{3}{4}$, then

$$\int_a^b g_{n+1} = -(b-\frac{1}{2}) + \int_a^b g_n(4x)dx = -(b-\frac{1}{2}) + \frac{1}{4}\int_2^{4b} g_n = -(b-\frac{1}{2}) + \frac{1}{4}\int_0^{4b-2} g_n$$

lies between $-(b-\frac{1}{2})$ and

$$-(b-\frac{1}{2}) + \frac{1}{8}(4b-2) = -\frac{1}{2}b + \frac{1}{4} \leq 0,$$

so has modulus at most $b - \frac{1}{2} = b - a$.

(ii) If $a = \frac{1}{2}$ and $\frac{3}{4} \leq b \leq 1$, then

$$\int_a^b g_{n+1} = -\frac{1}{4} + \frac{1}{4}\int_0^{4b-3} g_n$$

lies between $-\frac{1}{4}$ and

$$-\frac{1}{4} + \frac{1}{8}(4-4b) = \frac{1}{4} - \frac{1}{2}b \leq 0,$$

so has modulus at most $\frac{1}{4} \leq b - a$.

(iii) If $0 \leq a \leq \frac{1}{2} = b$ then, because $g_{n+1}(1-x) = -g_{n+1}(-x) = -g_{n+1}(x)$ for every x,

$$|\int_a^b g_{n+1}| = |\int_{1-b}^{1-a} g_{n+1}| \leq (1-a) - (1-b) = b - a$$

by (i) and (ii).

(iv) Finally, in general, if $0 \le a \le \frac{1}{2} \le b \le 1$,

$$|\int_a^b g_{n+1}| \le |\int_a^{1/2} g_{n+1}| + |\int_{1/2}^b g_{n+1}| \le (\frac{1}{2} - a) + (b - \frac{1}{2}) = b - a. \textbf{ Q}$$

(e) $|\int_a^b g_n - \frac{1}{2}(b-a)| \le b - a$ if $n \in \mathbb{N}$ and $0 \le a \le \frac{1}{4} \le b \le \frac{1}{2}$. **P** Once again, induce on n. If $n = 0$ the result is trivial, of course. For the inductive step to $n+1$, again look at the special cases first.

(i) If $a = \frac{1}{4} \le b \le \frac{1}{2}$ then

$$\int_a^b g_{n+1} - \frac{1}{2}(b-a) = (b - \frac{1}{4}) + \int_a^b g_n(4x)dx - \frac{1}{2}(b-a) = \frac{1}{2}(b - \frac{1}{4}) + \frac{1}{4}\int_0^{4b-1} g_n$$

lies between $\frac{1}{2}(b - \frac{1}{4})$ and $\frac{1}{2}(b - \frac{1}{4}) + \frac{1}{8}(4b-1) = b - \frac{1}{4}$, so has modulus at most $b - \frac{1}{4} = b - a$.

(ii) If $0 \le a \le \frac{1}{4} = b$ then

$$\int_a^b g_{n+1} - \frac{1}{2}(b-a) = \frac{1}{4}\int_{4a}^1 g_n - \frac{1}{2}(\frac{1}{4} - a) = -\frac{1}{4}\int_0^{1-4a} g_n - \frac{1}{2}(\frac{1}{4} - a)$$

lies between $-\frac{1}{2}(\frac{1}{4} - a)$ and $-\frac{1}{2}(\frac{1}{4} - a) - \frac{1}{8}(1 - 4a) = -(\frac{1}{4} - a)$ so has modulus at most $\frac{1}{4} - a = b - a$.

(iii) As in (d), we can now fit these together to give the result as stated.

(f) $|\int_a^b g_n + \frac{1}{2}(b-a)| \le b - a$ if $\frac{1}{2} \le a \le \frac{3}{4} \le b \le 1$. **P**

$$|\int_a^b g_n + \frac{1}{2}(b-a)| = |-\int_{1-b}^{1-a} g_n + \frac{1}{2}((1-a) - (1-b))| \le (1-a) - (1-b) = b - a$$

by (e).

(g) Suppose that $t = 2^{-m}j$ where $m \in \mathbb{N}$ and $j \in \mathbb{Z}$, and $t - 2^{-m} \le a \le t \le b \le t + 2^{-m}$. Then $|\int_a^b g_n - (b-a)g_n(t)| \le b - a$ for every $n \in \mathbb{N}$. **P**

(i) If $m = 0$ then $j - 1 \le a \le j \le b \le j + 1$. Now

$$|\int_j^b g_n| = |\int_0^{b-j} g_n| \le \frac{1}{2}(b - j)$$

by (c), while

$$|\int_a^j g_n| = |\int_{a-j}^0 g_n| = |\int_0^{j-a} g_n| \le \frac{1}{2}(j - a);$$

putting these together,

$$|\int_a^b g_n - (b-a)g_n(j)| = |\int_a^b g_n| \le \frac{1}{2}(b-a) \le b - a.$$

(ii) If $m = 1$ and j is even then (i) gives the result at once. If $m = 1$ and j is odd then set $c = j - \frac{1}{2}$; we shall have $0 \le a - c \le \frac{1}{2} \le b - c \le 1$, so

$$|\int_a^b g_n - (b-a)g_n(t)| = |\int_a^b g_n| = |\int_{a-c}^{b-c} g_n| \le b - a$$

by (d).

(iii) If $m = 2$ and j is even then (ii) is enough. If $m = 2$ and $j = 4k + 1$ for some $k \in \mathbb{Z}$, then

$$g_n(t) = g_n(k + \frac{1}{4}) = g_n(\frac{1}{4}) = \frac{1}{2}$$

and $0 \le a - k \le \frac{1}{4} \le b \le \frac{1}{2}$, so

$$|\int_a^b g_n - (b-a)g_n(t)| = |\int_{a-k}^{b-k} g_n - \frac{1}{2}((b-k) - (a-k))| \le |(b-k) - (a-k)| = b - a$$

by (e). If $m = 2$ and $j = 4k + 3$ for some $k \in \mathbb{Z}$, then $g_n(t) = -\frac{1}{2}$ and $\frac{1}{2} \le a - k \le \frac{3}{4} \le b - k \le 1$, so

$$|\int_a^b g_n - (b-a)g_n(t)| = |\int_{a-k}^{b-k} g_n + \frac{1}{2}((b-k)-(a-k))| \le b-a$$

by (f). Thus I have dealt with the cases $m \le 2$.

(iv) If $n = 0$ the result is trivial.

(v) Now induce on $m+n$. For the inductive step to $m \ge 3$ and $n+1 \ge 1$, if j is even we can move at once to $m-1$ and $j/2$, as in (ii) and (iii) above; so suppose that j is odd. In this case, we have $2^{-m}(j-1) \le a \le b \le 2^{-m}(j+1)$ so h is constant on $]a,b[\cup\{t\}$ and

$$\begin{aligned}|\int_a^b g_{n+1} - (b-a)g_{n+1}(t)| &= |\int_a^b g_n(4x)dx - (b-a)g_n(4t)| \\ &= \frac{1}{4}|\int_{4a}^{4b} g_n - (4b-4a)g_n(4t)|.\end{aligned}$$

Now

$$2^{-m+2}(j-1) \le 4a \le 4t = 2^{-m+2}j \le 4b \le 2^{-m+2}(j+1)$$

so the inductive hypothesis tells us that

$$|\int_{4a}^{4b} g_n - (4b-4a)g_n(4t)| \le 4b-4a$$

and $|\int_a^b g_{n+1} - (b-a)g_{n+1}(t)| \le b-a$. Thus the induction proceeds. **Q**

(h) $\lim_{n\to\infty}\int_0^1 |g_n| = \infty$. **P** Consider $\langle h_i\rangle_{i\in\mathbb{N}}$ where $h_i(x) = h(4^i x)$ for $x \in [0,1]$. This is an independent sequence in $\mathcal{L}^2$ and $\mathbb{E}(h_i) = 0$, $\mathrm{Var}(h_i) = \frac{1}{2}$ for every i. So the distribution of $\sqrt{\frac{2}{n}}g_n\restriction[0,1]$ converges to the distribution of a standard normal random variable Z as $n \to \infty$, by the Central Limit Theorem; and

$$\liminf_{n\to\infty}\mathbb{E}(|\sqrt{\tfrac{2}{n}}g_n\restriction[0,1]|) \ge \mathbb{E}(|Z|) = \frac{2}{\sqrt{2\pi}}. \quad \mathbf{Q}$$

2B Construction, part 2 (a) Let $\langle n_k\rangle_{k\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ such that $\int_0^1 |g_{n_k}| \ge 2^{k+1}$ for every $k \in \mathbb{N}$. Define $f : [0,1] \to \mathbb{R}$ by setting $f(0) = 0$ and

$$f(x) = g_{n_k}(2^{k+1}x) = g_{n_k}(2^{k+1}x - 1)$$

whenever $k \in \mathbb{N}$ and $2^{-k-1} \le x \le 2^{-k}$. Observe that $f(2^{-k}) = 0$ for every k.

(b) Suppose that $t = 2^{-m}j$ where $m \in \mathbb{N}$ and $0 < j \le 2^m$, and $a > 0$, $b \le 1$ are such that $t - 2^{-m} \le a \le t \le b$. Then $|\int_a^b f - (b-a)f(t)| \le b-a$. **P** Of course it is enough to consider the case in which j is odd. Suppose that $k \in \mathbb{N}$ is such that $2^{-k-1} < t < 2^{-k}$. In this case, $m > k+1$ and

$$2^{-k-1} \le t - 2^{-m}, \quad t + 2^{-m} \le 2^{-k}.$$

So

$$\begin{aligned}|\int_a^b f - (b-a)f(t)| &= |\int_a^b g_{n_k}(2^{k+1}x)dx - (b-a)g_{n_k}(2^{k+1}t)| \\ &= |2^{-k-1}\int_{a'}^{b'} g_{n_k} - (b-a)g_{n_k}(t')| \\ &= 2^{-k-1}|\int_{a'}^{b'} g_{n_k} - (b'-a')g_{n_k}(t')|\end{aligned}$$

where $a' = 2^{k+1}a$, $b' = 2^{k+1}b$ and $t' = 2^{k+1}t$. Since $t' = 2^{-m+k+1}j$ and

$$t' - 2^{-m+k+1} \le a' \le t' \le b' \le t' + 2^{-m+k+1},$$

2Ag tells us that

$$|\int_{a'}^{b'} g_{n_k} - (b-a)g_{n_k}(t')| \le b'-a' = 2^{k+1}(b-a)$$

and $|\int_a^b f - (b-a)f(t)| \le b-a$.

I still have to consider the case in which $t = 2^{-k}$ for some $k \in \mathbb{N}$. In this case $f(t) = 0$ and we take the intervals $[a,t]$, $[t,b]$ separately; the same calculations (with g_{n_k} for $[a,t]$, $g_{n_{k-1}}$ for $[t,b]$ if $t < b$) show that

$$|\int_a^t f| \le t-a, \quad |\int_t^b f| \le b-t,$$

so again we have the result.

(c) In particular, $|\int_a^{2^{-k}} f| \le 2^{-k}$ whenever $k \in \mathbb{N}$ and $0 < a \le 2^{-k}$; consequently $\lim_{a\downarrow 0}\int_a^1 f$ is defined. Since

$$\int_{2^{-k-1}}^{2^{-k}} f = \int_{2^{-k-1}}^{2^{-k}} g_{n_k}(2^{k+1}x)dx = 2^{-k-1}\int_1^2 g_{n_k} = 0$$

for every k, the improper Riemann integral $\int_0^1 f$ is defined, and this is also of course the Henstock integral $\text{H}\!\!\!\!\int_0^1 f$. On the other hand,

$$\int_{2^{-k-1}}^{2^{-k}} |f| = 2^{-k-1}\int_1^2 |g_{n_k}| \ge 1$$

for every k, so f is not Lebesgue integrable.

(d) Note that as $\int_{2^{-k}}^1 f = 0$ for every k, $\text{H}\!\!\!\!\int_0^{2^{-k}} f = 0$ for every k.

2C The trajectory (a) Let ω enumerate the dyadic rationals in $]0,1]$ in such a way that

$$\omega(0) = 1,\ \omega(1) = \frac{1}{2},\ \omega(2) = \frac{1}{4},\ \omega(3) = \frac{3}{4},$$

$$\omega(4) = \frac{1}{8},\ \omega(5) = \frac{3}{8},\ \omega(6) = \frac{5}{8},\ \omega(7) = \frac{7}{8},$$

$$\omega(8) = \frac{1}{16}, \dots .$$

(That is, $\omega(2^k+j) = 2^{-k-1}(2j+1)$ for $k \in \mathbb{N}$, $j < 2^k$.) Take the corresponding machinery for the first-return gauge integral defined by ω as in §1A.

(b) If $C \in \mathcal{C}$, then $|f(r_\omega(C))\mu C - \text{H}\!\!\!\!\int_C f| \le \mu C$. **P** If $\mu C = 0$ this is trivial. If $0 < a = \inf C < b = \sup C$ and $r_\omega(C) = t$, then t is of the form $2^{-m}(2j+1)$ where $t - 2^{-m} < a \le t \le b < t + 2^{-m}$; so we have only to apply 2Bb. If $0 = \inf C < b = \sup C$, then $t = r_\omega(C)$ is of the form 2^{-m} where $2^{-m} \le b < 2^{-m+1}$, so

$$|f(r_\omega(C)) - \text{H}\!\!\!\!\int_C f| = |f(t) - \int_t^b f| \le b - t \le \mu C$$

by 2Bb again.

(c) Suppose that $k \in \mathbb{N}$ and $\boldsymbol{t} \in T_\omega$ is $\{\emptyset\}$-filling, that is, $\bigcup_{(x,C)\in\boldsymbol{t}} C = [0,1]$. Then $|S_{\boldsymbol{t}}(f \times \chi[0,2^{-k}],\mu)| \le 2^{-k}$. **P**

$$|S_{\boldsymbol{t}}(f \times \chi[0,2^{-k}],\mu)| = |\sum_{(x,C)\in\boldsymbol{t}, x\le 2^{-k}} f(x)\mu C - \text{H}\!\!\!\!\int_0^{2^{-k}} f|$$

$$= |\sum_{(x,C)\in\boldsymbol{t}, x\le 2^{-k}} f(x)\mu(C \cap [0,2^{-k}]) - \text{H}\!\!\!\!\int_0^{2^{-k}} f|$$

(because $f(2^{-k}) = 0$ and if $(x,C) \in \boldsymbol{t}$ and $x < 2^{-k}$ then $2^{-k} \notin \overline{C}$)

$$\le \sum_{(x,C)\in\boldsymbol{t}, x\le 2^{-k}} |f(x)\mu(C\cap[0,2^{-k}]) - \text{H}\!\!\!\!\int_{C\cap[0,2^{-k}]} f|$$

$$\le \sum_{(x,C)\in\boldsymbol{t}} \mu(C \cap [0,2^{-k}])$$

(by (b), because if $(x,C) \in \boldsymbol{t}$ and $x \le 2^{-k}$ then $x = r_\omega(C \cap [0,2^{-k}])$)

$\leq 2^{-k}$. **Q**

(d) ω-$\text{F}\!\!\!\!\int\!\! f\times\chi[a,b]$ is defined and equal to $⨍_a^b f$ whenever $0\leq a\leq b\leq 1$. **P** If $a>0$ then both are equal to the Riemann integral of $f\times\chi[a,b]$. If $b=0$ the result is trivial. If $0=a<b$, take any $\epsilon>0$. Let $k\in\mathbb{N}$ be such that $2^{-k}\leq\min(b,\epsilon)$, and let $\eta>0$ be such that $|S_{\boldsymbol{t}}(f\times\chi\,]2^{-k},b]\,,\mu)-\omega\text{-}\text{F}\!\!\!\!\int\!\! f\times\chi\,]2^{-k},b]\,|\leq\epsilon$ whenever $\boldsymbol{t}\in T_\omega$ is δ_η-fine and $\{\emptyset\}$-filling, where $\delta_\eta\in\Delta$ is the uniform metric gauge corresponding to η. Now observe that (c) tells us that $|S_{\boldsymbol{t}}(f\times\chi[0,2^{-k}])|\leq 2^{-k}$ for any such $\boldsymbol{t}$, so that $|S_{\boldsymbol{t}}(f\times\chi[0,b])-⨍_0^b f|\leq 2\epsilon$. As ϵ is arbitrary, ω-$\text{F}\!\!\!\!\int_0^b f$ is defined and equal to $⨍_0^b f$.

(d) Thus for this trajectory, at least, there is a first-return integrable function which is Henstock integrable, with the same indefinite integral, but is not Lebesgue integrable.

The trajectory is as natural as one could wish, and the function is also very straightforward, though perhaps demanding a little ingenuity in its construction.

3 Integrating trajectories

3A A filter on $[0,1]^{\mathbb{N}}$ **(a)** Let $\Omega\subseteq[0,1]^{\mathbb{N}}$ be the set of trajectories; this is a G_δ set and is conegligible for the product measure on $[0,1]^{\mathbb{N}}$. For any Lebesgue integrable $f:[0,1]\to\mathbb{R}$ let Ω_f be the set of those trajectories such that the corresponding first-return integral of f is defined, and has Saks-Henstock indefinite integral $C\mapsto\int_C f:\mathcal{C}\to\mathbb{R}$. From Theorem 1C we see that if $E\subseteq[0,1]$ is conegligible then $E^{\mathbb{N}}\cap\bigcap_{n\in\mathbb{N}}\Omega_{f_n}$ is non-empty for any sequence of integrable functions $f_n:[0,1]\to\mathbb{R}$. So we have an ω_1-complete filter $\mathcal{F}$ on $[0,1]^{\mathbb{N}}$ generated by sets of the form Ω_f, for Lebesgue integrable $f:[0,1]\to\mathbb{R}$, and $E^{\mathbb{N}}$, for conegligible $E\subseteq[0,1]$.

(b) If $f:[0,1]\to\mathbb{R}$ is Borel measurable and Lebesgue integrable, then Ω_f is a Borel set. **P** For $\omega\in\Omega$ and $k\in\mathbb{N}$, let $Q_\omega^{(k)}$ be the set of pairs $(r_\omega(C),C)$ such that $C\in\mathcal{C}$ and $\partial C\subseteq\mathbb{Q}\cup\omega[k]$, and $T_\omega^{(k)}$ the straightforward set of tagged partitions generated by $Q_\omega^{(k)}$. Then 1D tells us that $\omega\in\Omega_f$ iff

for every $m\in\mathbb{N}$ there is an $n\in\mathbb{N}$ such that $\sum_{(x,C)\in\boldsymbol{t}}|\int_C f-f(x)\mu C|\leq 2^{-m}$ whenever $k\in\mathbb{N}$ and $\boldsymbol{t}\in T_\omega^{(k)}$ is $\delta_{2^{-n}}$-fine.

Now if m, n and k belong to $\mathbb{N}$, then

$$\{\omega:\textstyle\sum_{(x,C)\in\boldsymbol{t}}|\int_C f-f(x)\mu C|\leq 2^{-m}\text{ whenever }\boldsymbol{t}\in T_\omega^{(k)}\text{ is }\delta_{2^{-n}}\text{-fine}\}$$

is a Borel set, since we can enumerate $T_\omega^{(k)}$ as a sequence of Borel measurable functions of ω. **Q**

(c) If $f:[0,1]\to\mathbb{R}$ is Lebesgue integrable, there are a Borel measurable $g:[0,1]\to\mathbb{R}$ and a conegligible Borel set $E\subseteq[0,1]$ such that $g\restriction E=f\restriction E$; in which case $\Omega_f\cap E^{\mathbb{N}}\subseteq\Omega_g$. Accordingly $\mathcal{F}$ is generated by sets of the form Ω_g, for Borel measurable Lebesgue integrable $g:[0,1]\to\mathbb{R}$, and $E^{\mathbb{N}}$, for conegligible Borel sets $E\subseteq[0,1]^{\mathbb{N}}$; which are all Borel subsets of $[0,1]^{\mathbb{N}}$.

3B Example There is a closed set $F\subseteq[0,1]$ such that $\Omega_{\chi F}$ is negligible for the product measure λ on $[0,1]^{\mathbb{N}}$.

proof (a)(i) For $m\in\mathbb{N}$, $\omega\in[0,1]^{\mathbb{N}}$ and $x\in[0,1]$ define $\gamma(\omega,x,m)$ and $s(\omega,x,m)$ as follows. $\gamma(\omega,x,0)=\min(x,1-x)$. Given $\gamma(\omega,x,m)$, then if $|x-\omega(i)|\geq\gamma(\omega,x,m)$ for every i, stop; otherwise, take $s(\omega,x,m)$ to be $\omega(i)$ where $i=\min\{j:|x-\omega(j)|<\gamma(\omega,x,m)\}$, and set $\gamma(\omega,x,m+1)=|x-s(\omega,x,m)|$. Observe that if $\omega\in\Omega$ and $x\in\,]0,1[\,\setminus\{\omega(i):i\in\mathbb{N}\}$, as is true for almost every pair $(\omega,x)\in[0,1]^{\mathbb{N}}\times[0,1]$, $s(\omega,x,m)$ is defined for every m. Moreover, the functions γ and s are Borel measurable.

(ii) Fix $x\in\,]0,1[$, and set $h_n(\omega)=\gamma(\omega,x,n)$ for those $\omega\in[0,1]^{\mathbb{N}}$ for which this is defined. Then the ratios $U_n=h_{n+1}/h_n$ are independent and uniformly distributed in $[0,1]$.[1] Since $\ln U_n$ has expectation -1, $\frac{1}{n+1}\sum_{i=0}^n\ln U_n\to-1$ a.e., by the strong law of large numbers; that is, $\langle\frac{1}{n+1}\ln(h_{n+1}/h_0)\rangle_{n\in\mathbb{N}}$ and $\langle\frac{1}{n}\ln h_n\rangle_{n\geq1}$ converge to -1, $\langle h_n^{1/n}\rangle_{n\geq1}$ converges to $\frac{1}{e}$ almost everywhere. Consequently $\{n:h_n(\omega)\leq 3^{-n}\}$ is finite for almost every ω.

[1] I hope this is 'obvious'.

(b) For $r \in \mathbb{N}$, set $k_r = 3^{3^r}$ and set

$$G_r = \bigcup_{i<k_r} \Big]\frac{i}{k_r}, \frac{i}{k_r} + \frac{1}{2^{r+2}k_r}\Big[,$$

so that G_r is an open set of measure 2^{-r-2}; set $F = [0,1] \setminus \bigcup_{r\in\mathbb{N}} G_r$, so that $\mu F \ge \frac{1}{2}$. Now for any $x \in]0,1[$, $\{n : s(\omega,x,n) \notin F\}$ is infinite for almost every $\omega \in [0,1]^{\mathbb{N}}$. **P** For $n \in \mathbb{N}$ set

$$V_n = \{\omega : \omega \in \Omega,\ \omega(i) \ne x \text{ for every } i \in \mathbb{N},\ \gamma(\omega,x,i) \ge 3^{-i} \text{ for every } i \ge n\};$$

if $n \le i \le 3^r$, set

$$W_{nri} = \{\omega : \gamma(\omega,x,j) \ge \frac{1}{k_r} \text{ for } n \le j \le i,\ s(\omega,x,j) \in F \text{ for } n \le j < i\}.$$

Then $\mu W_{n,r,i+1} \le (1 - \frac{1}{2^{r+3}})\mu W_{nri}$ for $n \le i < 3^r$. (The point is that if we look along the values $\omega(0), \dots$, we confirm at a certain point that $\omega \in W_{nri}$, and from that point we are waiting for the next value of ω to appear in a certain open interval C of length at least $\dfrac{2}{k_r}$; and the probability that this value lies in G_r will be at least $\dfrac{1}{2^{r+3}}$, depending on how the intervals of length $\dfrac{1}{k_r}$ are arranged in C.) So

$$\mu\{\omega : \omega \in V_n,\ s(\omega,x,i) \in F \text{ for } n \le i < 3^r\} \le (1 - \frac{1}{2^{r+3}})^{3^r-n} \le \exp(-\frac{3^r-n}{2^{r+3}}).$$

Letting $r \to \infty$,

$$\mu\{\omega : \omega \in V_n,\ s(\omega,x,i) \in F \text{ for } i \ge n\} = 0$$

for every n. But (a-ii) tells us that $\bigcup_{n\in\mathbb{N}} V_n$ is conegligible, and of course $\langle V_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, so $\{\omega : s(\omega,x,i) \in F \text{ for } i \ge n\}$ is negligible for every $n \in \mathbb{N}$. **Q**

(c) Thus

$$W = \{(\omega,x) : \omega \in \Omega,\ s(\omega,x,n) \text{ is defined for every } n,$$
$$s(\omega,x,n) \notin F \text{ for infinitely many } n\}$$

is conegligible in $[0,1]^{\mathbb{N}} \times [0,1]$, and $\Omega' = \{\omega : W[\{\omega\}] \text{ is conegligible in } [0,1]\}$ is conegligible in $[0,1]^{\mathbb{N}}$.

Now $\Omega' \cap \Omega_{\chi F} = \emptyset$. **P?** If $\omega \in \Omega' \cap \Omega_{\chi F}$, let $\eta > 0$ be such that $\sum_{(x,C)\in \boldsymbol{t}} |\mu(F \cap C) - \chi F(x)\mu C| \le \frac{1}{3}\mu F$ for every δ_η-fine $\boldsymbol{t} \in T_\omega$. Let $\mathcal{I}$ be $\{\overline{C} : C \in \mathcal{C},\ 0 < \operatorname{diam} C \le \eta,\ r_\omega(C) \notin F\}$. If $x \in W[\{\omega\}]$ and $0 < \eta' \le \eta$, then (because $\omega \in \Omega$) there is certainly an n such that $\gamma(\omega,x,n) \le \dfrac{1}{2}\eta'$, and now there is an $m \ge n$ such that $s(\omega,x,m) \notin F$; but in this case

$$C = [x - \gamma(\omega,x,m+1), x + \gamma(\omega,x,m+1)] \in \mathcal{I}$$

because $r_\omega(C) = s(\omega,x,m)$, while $\operatorname{diam} C \le \eta'$. By Vitali's theorem, there are disjoint $C_0, \dots, C_n \in \mathcal{I}$ such that $\mu(\bigcup_{j\le n} C_j) \ge 1 - \frac{1}{3}\mu F$, in which case $\sum_{j=0}^{n} \mu(C_j \cap F) \ge \frac{2}{3}\mu F$. But in this case $\boldsymbol{t} = \{(r_\omega(C_j), C_j) : j \le n\}$ belongs to T_ω and is δ_η-fine, and

$$\sum_{(x,C)\in \boldsymbol{t}} |\mu(F \cap C) - \chi F(x)\mu C| = \sum_{(x,C)\in \boldsymbol{t}} \mu(F \cap C) \ge \frac{2}{3}\mu F > \frac{1}{3}\mu F. \ \mathbf{XQ}$$

(d) So $\Omega_{\chi F}$ is negligible, as claimed.

3C Thus $\mathcal{F}$ contains a negligible set. It is elementary to see that $\Omega_{\chi F}$ is meager for any nowhere dense set F of non-zero measure, so $\mathcal{F}$ contains many meager sets. Of course $\mathcal{F}$ is invariant under operations like reflection about $\frac{1}{2}$ and rearrangements of finite dissections of $[0,1]$ into intervals. What can we say about cardinal invariants of $\mathcal{F}$ and its complementary ideal $\mathcal{I}_{\mathrm{FR}} = \{[0,1]^{\mathbb{N}} \setminus W : W \in \mathcal{F}\}$?

3D Theorem $\mathcal{I}_{\mathrm{FR}} \preccurlyeq_{\mathrm{T}} \mathcal{N}$, where $\mathcal{N}$ is the null ideal of μ.

Remark For the basic theory of the relations $\preccurlyeq_{\mathrm{T}}$ and $\preccurlyeq_{\mathrm{GT}}$, see Fremlin 08?, chaps. 51 and 52.

proof (a) Let L^1 be the L-space of equivalence classes of Lebesgue integrable real-valued functions on $[0,1]$. Let $\langle u_k\rangle_{k\in\mathbb{N}}$ enumerate a norm-dense subset of L^1, and for $n\in\mathbb{N}$ let $\langle v_{ni}\rangle_{i\in\mathbb{N}}$ enumerate a dense subset of $\{u : u\in L^1,\ u\ge 0,\ \|u\|\le 4^{-n}\}$. Then for any $u\in L^1$ there is a sequence $\langle i(n)\rangle_{n\in\mathbb{N}}$ such that for every $n\in\mathbb{N}$ there is a $k\in\mathbb{N}$ such that

$$2^n|u-u_k|\le\sum_{m=n}^\infty v_{m,i(m)}.$$

P Choose $\langle i(n)\rangle_{n\in\mathbb{N}}$, $\langle k(n)\rangle_{n\in\mathbb{N}}$ inductively, as follows. The inductive hypothesis will be that, setting

$$w_{mn}=(2^m|u-u_{k(m)}|-\sum_{j=m}^{n-1}v_{j,i(j)})^+,$$

$\|w_{mn}\|\le\frac{1}{4^n(n+1)}$ whenever $m<n$. Now choose $k(n)$ such that $2^n\|u-u_{k(n)}\|\le\frac{1}{4^n(n+1)}$, and set $w_{nn}=2^n\|u-u_{k(n)}\|$. Then we shall have $\|\sum_{m=0}^n w_{mn}\|\le 4^{-n}$; let $i(n)$ be such that $\|v_{n,i(n)}-\sum_{m=0}^n w_{mn}\|\le\frac{1}{4^{n+1}(n+2)}$. This will ensure that

$$\begin{aligned}w_{m,n+1}&=(2^m|u-u_{k(m)}|-\sum_{j=m}^n v_{j,i(j)})^+\\&=(w_{mn}-v_{n,i(n)})^+\le\|v_{n,i(n)}-\sum_{m=0}^n w_{mn}\|,\end{aligned}$$

so that $\|w_{m,n+1}\|\le\frac{1}{4^{n+1}(n+2)}$ for every $m\le n$, and the induction proceeds.

At the end of the induction,

$$(2^m|u-u_{k(m)}|-\sum_{j=m}^\infty v_{j,i(j)})^+=\lim_{n\to\infty}w_{mn}=0,$$

so $2^m|u-u_{k(m)}|\le\sum_{j=m}^\infty v_{j,i(j)}$, for every $m\in\mathbb{N}$. **Q**

For m, $i\in\mathbb{N}$ let $g_{mi}:[0,1]\to[0,\infty[$ be an integrable function such that $g_{mi}^\bullet=v_{mi}$ in L^1; for $k\in\mathbb{N}$ let $f_k:[0,1]\to\mathbb{R}$ be an integrable function such that $f_k^\bullet=u_k$ in L^1.

(b) Let $(\mathbb{N}^\mathbb{N},\subseteq^*,\mathcal{S})$ be the $\mathbb{N}$-localization relation (Fremlin 08?, 522K). Let $\mathcal{L}^1$ be the linear space of Lebesgue integrable functions from $[0,1]$ to $\mathbb{R}$. (Note that I insist here on functions defined everywhere on $[0,1]$.) For $f\in\mathcal{L}^1$, choose $\phi_1(f)\in\mathbb{N}^\mathbb{N}$ and $\phi_2(f)\in\mathcal{N}$ as follows. First, by (a), we can find $\phi_1(f)$ such that for every $n\in\mathbb{N}$ there is a $k\in\mathbb{N}$ such that

$$2^n|f^\bullet-u_k|\le\sum_{m=n}^\infty v_{m,\phi_1(f)(m)}$$

in L^1, for every $n\in\mathbb{N}$. Next, let $\phi_2(f)\in\mathcal{N}$ be such that for every $n\in\mathbb{N}$ there is a $k\in\mathbb{N}$ such that

$$2^n|f(x)-f_k(x)|\le\sum_{m=n}^\infty g_{m,\phi_1(f)(m)}(x)$$

for every $x\in[0,1]\setminus\phi_2(f)$.

Next, for $S\in\mathcal{S}$ and $F\in\mathcal{N}$, define $\psi(S,F)\in\mathcal{I}_{\mathrm{FR}}$ as follows. Start by setting $g=\sum_{(n,i)\in S}g_{ni}$; since $\int g_{ni}\le 4^{-n}$ whenever $(n,i)\in S$, and $\#(S[\{n\}])\le 2^n$ for every n, g will be an integrable function from $[0,1]$ to $[0,\infty]$. (It will be helpful later to note that $\int g\le 2$.) Set $E=\{x : x\in[0,1],\ g(x)<\infty\}\setminus F$ and $g'=g\times\chi E$. Now set

$$\psi(S,F)=[0,1]^\mathbb{N}\setminus(E^\mathbb{N}\cap\Omega_{g'}\cap\bigcap_{n\in\mathbb{N}}\Omega_{f_n})\in\mathcal{I}_{\mathrm{FR}}.$$

(c) If $f\in\mathcal{L}^1$, $\phi_1(f)\subseteq^* S\in\mathcal{S}$ and $\phi_2(f)\subseteq F\in\mathcal{N}$, then $[0,1]^\mathbb{N}\setminus\psi(S,F)\subseteq\Omega_f$. **P** Take any $\omega\in[0,1]^\mathbb{N}\setminus\psi(S,F)$ and $\epsilon>0$. Define g, E and g' from S and F as in (b) just above. Let $n\in\mathbb{N}$ be such that $2^{-n}\le\epsilon$ and $(m,\phi_1(f)(m))\in S$ for every $m\ge n$. Then there is a $k\in\mathbb{N}$ such that

$$2^n|f(x)-g_k(x)|\le\sum_{m=n}^\infty g_{m,\phi_1(f)(m)}(x)\le g(x)$$

for every $x\in[0,1]\setminus\phi_2(f)\supseteq E$. Next, there is an $\eta>0$ such that

$$\sum_{(x,C)\in\boldsymbol{t}}|\int_C g_k-g_k(x)\mu C|\le\epsilon,$$

$$\sum_{(x,C)\in\boldsymbol{t}}|\int_C g'-g'(x)\mu C|\le 1$$

whenever $\boldsymbol{t} \in T_\omega$ is δ_η-fine. In this case, for any such t,

$$\sum_{(x,C)\in \boldsymbol{t}} g'(x)\mu C \le 1 + \sum_{(x,C)\in \boldsymbol{t}} \int_C g' \le 1 + \int g' \le 3,$$

$$\sum_{(x,C)\in \boldsymbol{t}} |\int_C f - f(x)\mu C| \le \sum_{(x,C)\in \boldsymbol{t}} |\int_C f - \int_C g_k| + \sum_{(x,C)\in \boldsymbol{t}} |\int_C g_k - g_k(x)\mu C| + \sum_{(x,C)\in \boldsymbol{t}} 2^{-n} g'(x)\mu C$$

(because $x \in E$, so $|f(x) - g_k(x)| \le 2^{-n} g'(x)$, whenever $(x, C) \in \boldsymbol{t}$)

$$\le \int |f - g_k| + \epsilon + 3\epsilon \le 5\epsilon.$$

As ϵ is arbitrary, $\omega \in \Omega_f$. **Q**

(d) Now recall that $(\mathbb{N}^\mathbb{N}, \subseteq^*, \mathcal{S}) \preccurlyeq_{\mathrm{GT}} (\mathcal{N}, \subseteq, \mathcal{N})$ (Fremlin 08?, 522N); consequently

$$(\mathbb{N}^\mathbb{N}, \subseteq^*, \mathcal{S}) \times (\mathcal{N}, \subseteq, \mathcal{N}) \preccurlyeq_{\mathrm{GT}} (\mathcal{N}, \subseteq, \mathcal{N}) \times (\mathcal{N}, \subseteq, \mathcal{N}) \equiv_{\mathrm{GT}} (\mathcal{N}, \subseteq, \mathcal{N})$$

(Fremlin 08?, 512H and 513Eh). We therefore have functions $\phi : \mathbb{N}^\mathbb{N} \times \mathcal{N} \to \mathcal{N}$, $\psi_1 : \mathcal{N} \to \mathcal{S}$ and $\psi_2 : \mathcal{N} \to \mathcal{N}$ such that whenever $h \in \mathbb{N}^\mathbb{N}$, E, $F \in \mathcal{N}$, and $\phi(h, E) \subseteq F$ then $h \subseteq^* \psi_1(F)$ and $E \subseteq \psi_2(F)$. Set $\tilde{\phi}(f) = \phi(\phi_1(f), \phi_2(f)) \in \mathcal{N}$ for $f \in \mathcal{L}^1$ and

$$\tilde{\psi}(F) = \psi(\psi_1(F), \psi_2(F)) \cup ([0, 1]^\mathbb{N} \setminus ([0, 1] \setminus F)^\mathbb{N}) \in \mathcal{I}_{\mathrm{FR}}$$

for $F \in \mathcal{N}$. Now if $f \in \mathcal{L}^1$ and $F \in \mathcal{N}$ are such that $\tilde{\phi}(f) \subseteq F$, we shall have $[0, 1]^\mathbb{N} \setminus \Omega_f \subseteq \tilde{\psi}(F)$. **P** As $\phi(\phi_1(f), \phi_2(f)) \subseteq F$, $\phi_1(f) \subseteq^* \psi_1(F)$ and $\phi_2(f) \subseteq \psi_2(F)$, so (c) gives the result. **Q**

(e) Finally, define $\bar{\phi} : \mathcal{I}_{\mathrm{FR}} \to \mathcal{N}$ as follows. For any $A \in \mathcal{I}_{\mathrm{FR}}$, choose a sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{L}^1$ and a conegligible $E \subseteq [0, 1]$ such that A does not meet $E^\mathbb{N} \cap \bigcap_{n\in\mathbb{N}} \Omega_{f_n}$. Set

$$\bar{\phi}(A) = ([0, 1] \setminus E) \cup \bigcup_{n\in\mathbb{N}} \tilde{\phi}(f_n).$$

If $A \in \mathcal{I}_{\mathrm{FR}}$ and $F \in \mathcal{N}$ are such that $\bar{\phi}(A) \subseteq F$, then $A \subseteq \tilde{\psi}(F)$. **P** Taking E and $\langle f_n \rangle_{n\in\mathbb{N}}$ from A as in the definition of $\bar{\phi}$, we have $\tilde{\phi}(f_n) \subseteq F$, so $[0, 1]^\mathbb{N} \setminus \Omega_{f_n} \subseteq \tilde{\psi}(F)$, by (c). At the same time, $[0, 1] \setminus E \subseteq F$ so $[0, 1]^\mathbb{N} \setminus E^\mathbb{N} \subseteq \tilde{\psi}F$. As

$$A \subseteq \bigcup_{n\in\mathbb{N}} ([0, 1]^\mathbb{N} \setminus \Omega_{f_n}) \cup ([0, 1]^\mathbb{N} \setminus E^\mathbb{N}),$$

we have the result. **Q**

(f) Thus $(\bar{\phi}, \tilde{\psi})$ is a Galois-Tukey connection from $(\mathcal{I}_{\mathrm{FR}}, \subseteq, \mathcal{I}_{\mathrm{FR}})$ to $(\mathcal{N}, \subseteq, \mathcal{N})$, and $\mathcal{I}_{\mathrm{FR}} \preccurlyeq_{\mathrm{T}} \mathcal{N}$ (Fremlin 08?, 513Ea).

3E Corollary $\operatorname{add} \mathcal{I}_{\mathrm{FR}} \ge \operatorname{add} \mathcal{N}$ and $\operatorname{cf} \mathcal{I}_{\mathrm{FR}} \le \operatorname{cf} \mathcal{N}$.

proof Fremlin 08?, 513Ee.

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References

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