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Splitting additive functionals

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1 Theorem Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to [0, \infty[$ an additive functional. Suppose that $\langle a_i\rangle_{i\in I}$, $\langle\alpha_i\rangle_{i\in I}$ are families in $\mathfrak{A}$, $[0, \infty[$ respectively. Then the following are equiveridical:

(i) there is a family $\langle\nu_i\rangle_{i\in I}$ of non-negative additive functionals on $\mathfrak{A}$ such that $\sum_{i\in I} \nu_i \le \nu$ and $\nu a_i = \alpha_i$ for every $i \in I$;

(ii) $\nu(\sup_{i\in J} a_i) \ge \sum_{i\in J} \alpha_i$ for every finite $J \subseteq I$.

proof (a) If (i) is true, $J \subseteq I$ is finite and $a = \sup_{i\in J} a_i$ we must have

$$\nu a \ge \textstyle\sum_{i\in J} \nu_i a \ge \sum_{i\in J} \nu_i a_i = \sum_{i\in J} \alpha_i.$$

Thus (i)⇒(ii); for the rest of the proof I shall assume that (ii) is true and seek to prove (i). Since the case $\mathfrak{A} = \{0\}$ is trivial, it will be enough to deal with the case $\nu 1 = 1$.

(b) Suppose that $\mathfrak{A}$ and I are both finite, every value of ν is rational, and every α_i is rational. Let $m \in \mathbb{N}$ be such that $m_i = m\alpha_i \in \mathbb{N}$ for every $i \in I$ and $m\nu a \in \mathbb{N}$ for every $a \in \mathfrak{A}$. Then there is a Boolean homomorphism $\pi : \mathfrak{A} \to \mathcal{P}m$ such that $\#(\pi a) = m\nu a$ for every $a \in \mathfrak{A}$. Note that

$$\textstyle\sum_{i\in I} m_i = m\sum_{i\in I} \alpha_i \le m\nu(\sup_{i\in I} a_i) \le m.$$

Let $\langle L_i\rangle_{i\in I}$ be a disjoint family of sets with $\#(L_i) = m_i$ for each $i \in I$, and set $L = \bigcup_{i\in I} L_i$. Set

$$R = \textstyle\bigcup_{i\in I} L_i \times \pi a_i \subseteq L \times m.$$

If $K \subseteq L$, set $J = \{i : K \cap L_i \ne \emptyset\}$; then

$$\begin{aligned}\#(K) &\le \sum_{i\in J} \#(L_i) = m\sum_{i\in J} \alpha_i \le m\nu(\sup_{i\in J} a_i)\\ &= \#(\pi(\sup_{i\in J} a_i)) = \#(\bigcup_{i\in J} \pi a_i) = \#(R[K]).\end{aligned}$$

By Hall's Marriage Lemma (Fremlin 02, 3A1K), there is an injective function $f : L \to m$ such that $(l, f(l)) \in R$ for every $l \in L$, that is, $f[L_i] \subseteq \pi a_i$ for every $i \in I$. Set

$$\nu_i a = \frac{1}{m}\#(f[L_i] \cap \pi a)$$

for $i \in I$ and $a \in \mathfrak{A}$. Because π is a Boolean homomorphism, every ν_i is a non-negative additive functional on $\mathfrak{A}$. Because f is injective, $\langle f[L_i]\rangle_{i\in I}$ is disjoint and

$$\textstyle\sum_{i\in I} \nu_i a = \dfrac{1}{m}\#(f[L] \cap \pi a) \le \nu a$$

for every $a \in \mathfrak{A}$. Finally,

$$\nu_i a_i = \frac{1}{m}\#(f[L_i] \cap \pi a_i) = \frac{1}{m}\#(f[L_i]) = \frac{1}{m}\#(L_i) = \alpha_i$$

for every $i \in I$. So (ii) is true in this case.

(c) Now suppose only that I and $\mathfrak{A}$ are both finite. In this case, for each $n \in \mathbb{N}$, we can find an additive functional $\nu^{(n)} : \mathfrak{A} \to [0, 1]$, taking rational values, such that $\nu^{(n)}1 = 1$ and $|\nu^{(n)}a - \nu a| \le 2^{-n}$ for every $a \in \mathfrak{A}$ (we have only to make suitably small adjustments to the value of ν at each atom of $\mathfrak{A}$). Next, we have rational numbers $\alpha_i^{(n)}$ such that $\alpha_i - 2^{-n+1} \le \alpha_i^{(n)} \le \alpha_i - 2^{-n}$ for each $i \in I$, and in this case

$$\textstyle\sum_{i\in J} \alpha_i^{(n)} \le \sum_{i\in J} \alpha_i - 2^{-n} \le \nu^{(n)}(\sup_{i\in J} a_i)$$

for every non-empty $J \subseteq I$. By (b), there is a family $\langle \nu_i^{(n)} \rangle_{i\in I}$ of non-negative additive functionals on $\mathfrak{A}$ such that $\sum_{i\in I} \nu_i^{(n)} \le \nu^{(n)}$ and $\nu_i^{(n)} a_i = \alpha_i^{(n)}$ for each i. Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$, and set $\nu_i a = \lim_{n\to\mathcal{F}} \nu_i^{(n)} a$ for $a \in \mathfrak{A}$ and $i \in I$. It is easy to check that every ν_i is an additive functional from $\mathfrak{A}$ to $[0,1]$, that $\sum_{i\in I} \nu_i \le \nu$, and that $\nu_i a_i = \alpha_i$ for $i \in I$.

(d) For the general case, for each finite $D \subseteq \mathfrak{A} \cup I$ take $\mathfrak{A}_D$ to be the subalgebra of $\mathfrak{A}$ generated by $D \cup \{a_i : i \in I \cap D\}$. By (c), we have a family $\langle \nu_i^{(D)} \rangle_{i\in I\cap D}$ of non-negative additive functionals on $\mathfrak{A}_D$ such that $\sum_{i\in I\cap D} \nu_i^{(D)} \le \nu{\restriction}\mathfrak{A}_D$ and $\nu_i^{(D)} a_i = \alpha_i$ for $i \in I \cap D$. Let $\mathcal{F}$ be an ultrafilter on $[\mathfrak{A} \cup I]^{<\omega}$ containing all the sets $\{D : d \in D \in [\mathfrak{A} \cup I]^{<\omega}\}$ for $d \in I \cup \mathfrak{A}$, and set $\nu_i a = \lim_{D\to\mathcal{F}} \nu_i^{(D)} a$ for $a \in \mathfrak{A}$, $i \in I$; as at the end of (c), it is easy to check that $\langle \nu_i \rangle_{i\in I}$ satisfies the conditions of (ii).

2 Theorem (Compare Avilés & Kalenda p07, Lemma 26.) Let X, Y be K-analytic Hausdorff spaces, $g : X \to Y$ a continuous surjection, $\langle F_i \rangle_{i\in I}$ a disjoint family of K-analytic subsets of X, ν a Radon measure on Y, and $\langle \alpha_i \rangle_{i\in I}$ a family in $[0,\infty[$. Then the following are equiveridical:

(i) there is a Radon measure μ on X such that ν is the image measure μg^{-1} and $\mu_i F_i \ge \alpha_i$ for every $i \in I$;

(ii) $\nu(\bigcup_{i\in J} g[F_i]) \ge \sum_{i\in J} \alpha_i$ for every finite $J \subseteq I$.

proof (a) As in Theorem 1, (i)$\Rightarrow$(ii) is elementary.

(b) In the other direction, assuming (ii), we see from Theorem 1 that there is a family $\langle \nu_i \rangle_{i\in I}$ of non-negative additive functionals on the Borel algebra $\mathcal{B}(Y)$ of Y such that $\sum_{i\in I} \nu_i \le \nu{\restriction}\mathcal{B}(Y)$ and $\nu_i g[F_i] = \alpha_i$ for each i. For $i \in I$, ν_i is dominated by ν, so is countably additive, locally finite and tight. Let $\tilde{\nu}_i$ be the Radon measure on $g[F_i]$ agreeing with ν_i on the Borel subsets of the K-analytic space $g[F_i]$ (Fremlin 03, 432D). By Fremlin 03, 432G, there is a Radon measure $\tilde{\mu}_i$ on F_i such that $\tilde{\nu}_i = \tilde{\mu}_i (g{\restriction}F_i)^{-1}$. Let μ_i be the Radon measure on X, extending $\tilde{\mu}_i$, such that $\mu_i(X \setminus F_i) = 0$. Next, there is a tight locally finite Borel measure ν' on Y such that $\nu = \nu' + \sum_{i\in I} \nu_i$. **P** Set

$$\nu' E = \sup_{H\subseteq Y \text{ is open}, \nu H<\infty} \bigl(\nu(E \cap H) - \textstyle\sum_{i\in I} \nu_i(E \cap H)\bigr)$$

for Borel sets $E \subseteq Y$. **Q** There is a Radon measure μ' on X such that g is inverse-measure-preserving for μ' and ν'. Set $\mu = \mu' + \sum_{i\in I} \tilde{\mu}_i$ (Fremlin n06, or Fremlin 00, 112Ya); this works.

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