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Filters of countable type

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Following Mauldin Preiss & Weizsäcker 83, I work through the theory of 'filters of countable type' (1A). Results which may be new are that filters of countable type have the Bolzano-Weierstrass property (3C), the density filter is not of countable type (3D), there are $\mathfrak{c}$ isomorphism classes of filters of countable type (5D), and every filter of countable type has coinitiality at most $\mathfrak{d}$ (6A).

1 Basics

1A Definitions (a) A filter on a set X is **of countable type** if it belongs to the smallest class of filters on X containing the principal ultrafilters and closed under the operations of countable intersection and increasing countable union.

(b) If X is a set, $\langle\mathcal{F}_i\rangle_{i\in I}$ is a non-empty family of filters on X, and $\mathcal{F}$ is a filter on I, I will write $\lim_{i\to\mathcal{F}}\mathcal{F}_i$ for the filter $\{A : A\subseteq X,\,\{i : i\in I,\,A\in\mathcal{F}_i\}\in\mathcal{F}\}$.

Note that if $X=I=\mathbb{N}$ and $\mathcal{F}_n$ is the principal filter generated by $\{n\}$ for each n, then $\lim_{n\to\mathcal{F}}\mathcal{F}_n=\mathcal{F}$.

(c) Let $\mathcal{F}$ be a filter on a set X. For $Y\subseteq X$, set

$$\mathcal{F}\restriction Y=\{A\cap Y : A\in\mathcal{F}\}=\{B : B\subseteq Y,\,B\cup(X\setminus Y)\in\mathcal{F}\};$$

I will call $\mathcal{F}\restriction Y$ the **trace** of $\mathcal{F}$ on Y. If $X\setminus Y\in\mathcal{F}$, then $\mathcal{F}\restriction Y=\mathcal{P}Y$; otherwise, $\mathcal{F}\restriction Y$ is a filter on Y.

(d) I will write $\mathcal{F}_{\mathrm{Fr}}$ for the Fréchet filter on $\mathbb{N}$, the smallest free filter on $\mathbb{N}$.

1B Proposition Let $\mathfrak{F}$ be a family of filters on a set X. Then the following are equiveridical:

(i) $\bigcap_{n\in\mathbb{N}}\mathcal{F}_n\in\mathfrak{F}$ for every sequence $\langle\mathcal{F}_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{F}$, and $\bigcup_{n\in\mathbb{N}}\mathcal{F}_n\in\mathfrak{F}$ for every non-decreasing sequence $\langle\mathcal{F}_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{F}$;

(ii) $\lim_{n\to\mathcal{F}_{\mathrm{Fr}}}\mathcal{F}_n\in\mathfrak{F}$ whenever $\langle\mathcal{F}_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{F}$;

(iii) $\lim_{n\to\mathcal{F}}\mathcal{F}_n\in\mathfrak{F}$ whenever $\langle\mathcal{F}_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{F}$ and $\mathcal{F}$ is a filter of countable type on $\mathbb{N}$.

proof (i)⇒(iii) Assume (i). Let $\mathfrak{G}$ be the set of filters $\mathcal{G}$ on $\mathbb{N}$ such that $\lim_{n\to\mathcal{G}}\mathcal{F}_n\in\mathfrak{F}$.

(α) If $\mathcal{G}$ is the principal filter generated by $\{m\}$, then $\lim_{n\to\mathcal{G}}\mathcal{F}_n=\mathcal{F}_m\in\mathfrak{F}$; so $\mathcal{G}\in\mathfrak{G}$.

(β) If $\langle\mathcal{G}_m\rangle_{m\in\mathbb{N}}$ is a sequence in $\mathfrak{G}$ with intersection $\mathcal{G}$, then

$$\begin{aligned}\lim_{n\to\mathcal{G}}\mathcal{F}_n&=\{A : \{n : A\in\mathcal{F}_n\}\in\mathcal{G}\}\\&=\bigcap_{m\in\mathbb{N}}\{A : \{n : A\in\mathcal{F}_n\}\in\mathcal{G}_m\}=\bigcap_{m\in\mathbb{N}}\lim_{n\to\mathcal{G}_m}\mathcal{F}_n\end{aligned}$$

is the intersection of a sequence in $\mathfrak{F}$ so belongs to $\mathfrak{F}$, and $\mathcal{G}\in\mathfrak{G}$.

(γ) If $\langle\mathcal{G}_m\rangle_{m\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{G}$ with union $\mathcal{G}$, set $\mathcal{F}'_m=\lim_{n\to\mathcal{G}_m}\mathcal{G}_n\in\mathfrak{F}$ for each m. If $k\le m$ then

$$\mathcal{F}'_k=\{A : \{n : A\in\mathcal{F}_n\}\in\mathcal{G}_k\}\subseteq\{A : \{n : A\in\mathcal{F}_n\}\in\mathcal{G}_m\}=\mathcal{F}'_m,$$

so $\langle\mathcal{F}'_m\rangle_{m\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{F}$ and its union belongs to $\mathfrak{F}$. Now

$$\begin{aligned}\lim_{n\to\mathcal{G}}\mathcal{F}_n&=\{A : \{n : A\in\mathcal{F}_n\}\in\mathcal{G}\}\\&=\bigcup_{m\in\mathbb{N}}\{A : \{n : A\in\mathcal{F}_n\}\in\mathcal{G}_m\}=\bigcup_{m\in\mathbb{N}}\mathcal{F}'_m\in\mathfrak{F},\end{aligned}$$

so $\mathcal{G} \in \mathfrak{G}$.

($\boldsymbol{\delta}$) Thus every filter on $\mathbb{N}$ of countable type belongs to $\mathfrak{G}$, as required by (iii).

(iii)$\Rightarrow$(ii) All we have to observe is that $\mathcal{F}_{\mathrm{Fr}}$ is a filter of countable type. **P** For each $n \in \mathbb{N}$ let $\mathcal{G}_n$ be the principal ultrafilter generated by $\{n\}$; then $\mathcal{F}_{\mathrm{Fr}} = \bigcup_{m\in\mathbb{N}} \bigcap_{n\ge m} \mathcal{G}_n$ is of countable type. **Q**

(ii)$\Rightarrow$(i) Assume (ii), and let $\langle \mathcal{F}_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{F}$.

($\boldsymbol{\alpha}$) Let $\langle k_n \rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathbb{N}$ with cofinal repetitions. Then $\mathfrak{F}$ contains

$$\begin{aligned}\lim_{n\to\mathcal{F}_{\mathrm{Fr}}} \mathcal{F}_{k_n} &= \{A : \{n : A \in \mathcal{F}_{k_n}\} \in \mathcal{F}_{\mathrm{Fr}}\} = \{A : \exists\, m \in \mathbb{N},\ A \in \mathcal{F}_{k_n} \text{ for every } n \ge m\}\\ &= \{A : A \in \mathcal{F}_n \text{ for every } n \in \mathbb{N}\} = \bigcap_{n\in\mathbb{N}} \mathcal{F}_n.\end{aligned}$$

($\boldsymbol{\beta}$) If $\langle \mathcal{F}_n \rangle_{n\in\mathbb{N}}$ is non-decreasing, then $\mathfrak{F}$ contains

$$\begin{aligned}\lim_{n\to\mathcal{F}_{\mathrm{Fr}}} \mathcal{F}_n &= \{A : \{n : A \in \mathcal{F}_n\} \in \mathcal{F}_{\mathrm{Fr}}\}\\ &= \{A : \exists\, m \in \mathbb{N},\ A \in \mathcal{F}_n \text{ for every } n \ge m\} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n.\end{aligned}$$

So both clauses of (i) are true.

1C Countable-type levels (a) Let X be any non-empty set. Define $\langle \mathfrak{F}_\xi \rangle_{\xi\in\mathrm{On}}$ as follows. $\mathfrak{F}_0$ is the set of principal ultrafilters on X. For ordinals $\xi > 0$, $\mathfrak{F}_\xi$ is the set of filters on X expressible in the form $\lim_{n\to\mathcal{F}_{\mathrm{Fr}}} \mathcal{F}_n$ where $\mathcal{F}_n \in \bigcup_{\eta<\xi} \mathfrak{F}_\eta$ for every $n \in \mathbb{N}$. Since $\mathfrak{F}_0 \subseteq \mathfrak{F}_1$, $\langle \mathfrak{F}_\xi \rangle_{\xi\in\mathrm{On}}$ is non-decreasing.

(b) From 1B we see that $\mathfrak{F}_{\omega_1} = \bigcup_{\xi<\omega_1} \mathfrak{F}_\xi$ is the set of filters on X of countable type. We therefore have a rank function $\mathrm{ct} : \mathfrak{F}_{\omega_1} \to \omega_1$ defined by saying that $\mathrm{ct}\,\mathcal{F} = \min\{\xi : \mathcal{F} \in \mathfrak{F}_\xi\}$ for every filter $\mathcal{F}$ of countable type; I will call $\mathrm{ct}\,\mathcal{F}$ the **countable-type level** of $\mathcal{F}$.

(c) Let X be a topological space. For $A \subseteq X$, define $\langle A^\sim_\xi \rangle_{\xi\in\mathrm{On}}$ as follows: $A^\sim_0 = A$; for $\xi > 0$, $A^\sim_\xi$ is the set of limits of sequences in $\bigcup_{\eta<\xi} A^\sim_\eta$. Thus $A^\sim_{\omega_1} = \bigcup_{\xi<\omega_1} A^\sim_\xi$ is the sequential closure of A.

1D Lemma If $\mathcal{F}$, $\mathcal{G}$ are filters on X, Y respectively, $\mathcal{F}$ is of countable type and $\mathcal{G} \le_{\mathrm{RK}} \mathcal{F}$, then $\mathcal{G}$ is of countable type and $\mathrm{ct}\,\mathcal{G} \le \mathrm{ct}\,\mathcal{F}$.

proof We can induce on $\mathrm{ct}\,\mathcal{F}$, because if $\langle \mathcal{F}_n \rangle_{n\in\mathbb{N}}$ is any sequence of filters on X, $\mathcal{F}$ is any filter on $\mathbb{N}$, and $g : X \to Y$ is a function, then

$$\begin{aligned}g[[\lim_{n\to\mathcal{F}} \mathcal{F}_n]] &= \{A : \{n : g^{-1}[A] \in \mathcal{F}_n\} \in \mathcal{F}\}\\ &= \{A : \{n : [A \in g[[\mathcal{F}_n]]\} \in \mathcal{F}\} = \lim_{n\to\mathcal{F}} g[[\mathcal{F}_n]].\end{aligned}$$

1E Proposition Let $\langle \mathcal{F}_i \rangle_{i\in I}$ be a non-empty family of filters of countable type on a set X, and $\mathcal{F}$ a filter of countable type on I. Then $\lim_{i\to\mathcal{F}} \mathcal{F}_i$ is a filter of countable type.

proof We can induce on $\mathrm{ct}\,\mathcal{F}$, because if $\langle \mathcal{G}_n \rangle_{n\in\mathbb{N}}$ is any sequence of filters on I and $\mathcal{F} = \lim_{n\to\mathcal{F}_{\mathrm{Fr}}} \mathcal{G}_n$, then

$$\begin{aligned}\lim_{i\to\mathcal{F}} \mathcal{F}_i &= \{A : \{i : A \in \mathcal{F}_i\} \in \mathcal{F}\}\\ &= \{A : \{n : \{i : A \in \mathcal{F}_i\} \in \mathcal{G}_n\} \in \mathcal{F}_{\mathrm{Fr}}\}\\ &= \{A : \{n : A \in \lim_{i\to\mathcal{G}_n} \mathcal{F}_i\} \in \mathcal{F}_{\mathrm{Fr}}\} = \lim_{n\to\mathcal{F}_{\mathrm{Fr}}} \lim_{i\to\mathcal{G}_n} \mathcal{F}_i.\end{aligned}$$

1F Proposition (a) Suppose that $\langle \mathcal{F}_n \rangle_{n\in\mathbb{N}}$ is a sequence of filters on a set X, and $Y \subseteq X$ is such that $X \setminus Y$ does not belong to $\mathcal{F} = \lim_{n\to\mathcal{F}_{\mathrm{Fr}}} \mathcal{F}_n$. Let $\langle n_k \rangle_{k\in\mathbb{N}}$ enumerate the infinite set $D = \{n : X \setminus Y \notin \mathcal{F}_n\}$. Then $\mathcal{F}\restriction Y = \lim_{k\to\mathcal{F}_{\mathrm{Fr}}} \mathcal{F}_{n_k}\restriction Y$.

(b) Let $\mathcal{F}$ be a filter on a set X and Y a subset of X such that $X \setminus Y \notin \mathcal{F}$. If $\mathcal{F}$ is of countable type, so is $\mathcal{F}\lceil Y$, and $\mathrm{ct}(\mathcal{F}\lceil Y) \le \mathrm{ct}\,\mathcal{F}$.

proof (a) For $B \subseteq Y$,

$$\begin{aligned} B \in \mathcal{F}\lceil Y &\iff B \cup (X \setminus Y) \in \mathcal{F} \\ &\iff \text{there is an } m \in \mathbb{N} \text{ such that } B \cup (X \setminus Y) \in \mathcal{F}_n \text{ for every } n \ge m \\ &\iff \text{there is an } m \in \mathbb{N} \text{ such that } B \cup (X \setminus Y) \in \mathcal{F}_n \\ &\qquad\qquad \text{whenever } n \in D \text{ and } n \ge m \\ &\iff \text{there is an } m \in \mathbb{N} \text{ such that } B \cup (X \setminus Y) \in \mathcal{F}_{n_k} \text{ for every } k \ge m \\ &\iff \text{there is an } m \in \mathbb{N} \text{ such that } B \in \mathcal{F}_{n_k}\lceil Y \text{ for every } k \ge m \\ &\iff B \in \lim_{k \to \mathcal{F}_{\mathrm{Fr}}} \mathcal{F}_{n_k}\lceil Y. \ \mathbf{Q} \end{aligned}$$

(b) The result is now an easy induction on $\mathrm{ct}\,\mathcal{F}$.

1G Proposition Let X be a countably compact topological space, I a non-empty set and $\mathcal{F}$ a filter on I of countable type. Then $f[[\mathcal{F}]]$ has a cluster point in X for every $f : I \to X$.

proof (a) If $f : I \to X$ is a function, $\langle \mathcal{F}_n \rangle_{n\in\mathbb{N}}$ is a sequence of filters on I, and x_n is a cluster point of $f[[\mathcal{F}_n]]$ for every $n \in \mathbb{N}$, then any cluster point x of $\langle x_n \rangle_{n\in\mathbb{N}}$ will also be a cluster point of $f[[\lim_{n\to\mathcal{F}_{\mathrm{Fr}}} \mathcal{F}_n]]$. $\mathbf{P}$ If G is an open set containing x and $A \in f[[\lim_{n\to\mathcal{F}_{\mathrm{Fr}}} \mathcal{F}_n]]$, then there is an $m \in \mathbb{N}$ such that $f^{-1}[A] \in \mathcal{F}_n$ for every $n \ge m$. Now there is an $n \ge m$ such that $x_n \in G$, in which case $f[f^{-1}[A]] \in f[[\mathcal{F}_n]]$ and

$$\emptyset \ne G \cap f[f^{-1}[A]] \subseteq G \cap A.$$

As G and A are arbitrary, x is a cluster point of $f[[\lim_{n\to\mathcal{F}_{\mathrm{Fr}}} \mathcal{F}_n]]$.

(b) Inducing on $\mathrm{ct}\,\mathcal{F}$ we get the result.

1H Theorem Let X be a topological space and A a subset of X.

(a) If $\xi < \omega_1$ and $x \in A^{\sim}_{\xi}$, then there are an $f : \mathbb{N} \to A$ and a filter $\mathcal{F}$ on $\mathbb{N}$, of countable-type level at most ξ, such that $f[[\mathcal{F}]] \to x$.

(b) If X is sequentially compact, I is a non-empty set, $f : I \to A$ is a function, and $\mathcal{F}$ is a filter on I of countable-type level ξ, then

(i) there is a filter $\mathcal{G}$ on I, of countable-type level at most ξ, such that $\mathcal{G} \supseteq \mathcal{F}$ and $f[[\mathcal{G}]]$ has a limit;

(ii) if $f[[\mathcal{F}]] \to x$ and X is Hausdorff then $x \in A^{\sim}_{\xi}$.

proof (a) Induce on ξ. If $\xi = 0$, then $x \in A$ and we can take f to be the constant function with value x and $\mathcal{F}$ the principal ultrafilter generated by $\{0\}$. For the inductive step to $\xi > 0$, let $\langle x_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\bigcup_{\eta<\xi} A^{\sim}_{\eta}$ converging to x. For each $n \in \mathbb{N}$ let $f_n : \mathbb{N} \to A$ and $\mathcal{F}_n$ be such that $\mathrm{ct}\,\mathcal{F}_n < \xi$ and $f_n[[\mathcal{F}_n]] \to x_n$; set $g_n(i) = 2^n(2i+1) - 1$ for $i \in \mathbb{N}$; set $\mathcal{G}_n = g_n[[\mathcal{F}_n]]$, so that $\mathrm{ct}\,\mathcal{G}_n < \xi$. Set $\mathcal{F} = \lim_{n\to\mathcal{F}_{\mathrm{Fr}}} \mathcal{G}_n$, so that $\mathrm{ct}\,\mathcal{F} \le \xi$; define $f : \mathbb{N} \to A$ by setting $f(g_n(i)) = f_n(i)$ for n, $i \in \mathbb{N}$.

For each $n \in \mathbb{N}$, $f[[\mathcal{G}_n]] = f_n[[\mathcal{F}_n]] \to x_n$. So if H is any open set containing x,

$$\{n : f^{-1}[H] \in \mathcal{G}_n\} \supseteq \{n : x_n \in H\} \in \mathcal{F}_{\mathrm{Fr}},$$

and $f^{-1}[H] \in \mathcal{F}$. Accordingly $f[[\mathcal{F}]] \to x$ and the induction proceeds.

(b) Again induce on ξ. If $\xi = 0$ then $\mathcal{F}$ is the principal ultrafilter on $\mathbb{N}$ generated by $\{i\}$ for some i; take $\mathcal{G} = \mathcal{F}$. Then $f[[\mathcal{G}]]$ is the principal ultrafilter on X generated by $f(i)$, and converges to $f(i)$. If the topology on X is T_1 and $f[[\mathcal{F}]] \to x$, then $x = f(i) \in A = A^{\sim}_0$. So the induction starts.

For the inductive step to $\xi > 0$, express $\mathcal{F}$ as $\lim_{n\to\mathcal{F}_{\mathrm{Fr}}} \mathcal{F}_n$ where $\mathrm{ct}\,\mathcal{F}_n < \xi$ for every n. For each $n \in \mathbb{N}$ we have a filter $\mathcal{G}_n \supseteq \mathcal{F}_n$ such that $\mathrm{ct}\,\mathcal{G}_n < \xi$ and $f[[\mathcal{G}_n]]$ has a limit x_n say. Now $\langle x_n \rangle_{n\in\mathbb{N}}$ has a convergent subsequence; suppose that $\langle n_k \rangle_{k\in\mathbb{N}}$ is a strictly increasing sequence in $\mathbb{N}$ such that $\langle x_{n_k} \rangle_{k\in\mathbb{N}}$ has a limit z in X. Set $\mathcal{G} = \lim_{k\to\mathcal{F}_{\mathrm{Fr}}} \mathcal{G}_{n_k}$; then $\mathrm{ct}\,\mathcal{G} \le \xi$. If $J \in \mathcal{F}$, then there is an $m \in \mathbb{N}$ such that $J \in \mathcal{F}_n$ for every $n \ge m$, and now

$$\{k : J \in \mathcal{G}_{n_k}\} \supseteq \{k : J \in \mathcal{F}_{n_k}\} \supseteq \{k : n_k \geq m\} \in \mathcal{F}_{\mathrm{Fr}},$$

so $J \in \mathcal{G}$. Thus $\mathcal{F} \subseteq \mathcal{G}$. If H is an open set including z, then

$$\{k : f^{-1}[H] \in \mathcal{G}_{n_k}\} \supseteq \{k : x_{n_k} \in H\} \in \mathcal{F}_{\mathrm{Fr}},$$

so $f^{-1}[H] \in \mathcal{G}$; thus $\mathcal{G} \to z$.

This deals with (i). As for (ii), if $f[[\mathcal{F}]] \to x$ then $f[[\mathcal{G}]] \to x$; as X is Hausdorff, $x = z = \lim_{k\to\infty} x_{n_k}$; also the inductive hypothesis now tells us that

$$x_{n_k} = \lim f[[\mathcal{G}_{n_k}]] \in \bigcup_{\eta<\xi} A_\eta^\sim$$

for every k, so $x \in A_\xi^\sim$ and the induction proceeds.

2 Borel filters and the Fatou property

2A Proposition If I is a non-empty set and $\mathcal{F}$ is a filter of countable type on I, then $\mathcal{F}$ is a Baire subset of $\mathcal{P}I$, therefore a Borel subset of $\mathcal{P}I$.

2B Proposition Let $\mathcal{F}$ be a filter on $\mathbb{N}$ which is a Borel subset of $\mathcal{P}\mathbb{N}$, X a set and Σ a σ-algebra of subsets of X.

(a) If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence of Σ-measurable functions from X to $[-\infty,\infty]$, then $\liminf_{n\to\mathcal{F}} f_n$ is Σ-measurable.

(b) If Y is a Polish space and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence of Σ-measurable functions from X to Y, then $\lim_{n\to\mathcal{F}} f_n$ is Σ-measurable and has domain in Σ.

proof (a) Set $f = \liminf_{n\to\mathcal{F}} f_n$. For $x \in X$, $\alpha \in \mathbb{R}$ set $J(x,\alpha) = \{n : x(n) \geq \alpha\}$. Then, for given α, $x \mapsto J(x,\alpha) : X \to \mathcal{P}\mathbb{N}$ is Σ-measurable, so $F_\alpha = \{x : J(x,\alpha) \in \mathcal{F}\}$ belongs to Σ. Now

$$\begin{aligned}\{x : f(x) > \alpha\} &= \{x : \text{ there is an } I \in \mathcal{F} \text{ such that } \inf_{n\in I} f_n(x) > \alpha\}\\ &= \{x : \text{ there are an } I \in \mathcal{F} \text{ and a rational } q > \alpha \text{ such that } I \subseteq J(x,q)\}\\ &= \{x : \text{ there is a rational } q > \alpha \text{ such that } J(x,q) \in \mathcal{F}\}\\ &= \bigcup_{q\in\mathbb{Q},q>\alpha} F_q \in \Sigma.\end{aligned}$$

As α is arbitrary, f is measurable.

(b) Let ρ be a complete metric on Y inducing the topology of Y. Let $\langle y_i\rangle_{i\in\mathbb{N}}$ run over a dense subset of Y. (I am passing over the trivial case $X = Y = \emptyset$. For i, $j \in \mathbb{N}$, $F_{ij} = \{x : \limsup_{n\to\mathcal{F}} \rho(f_n(x), y_i) \leq 2^{-j}\}$ belongs to Σ, by (a), inverted. Setting $f = \lim_{n\to\mathcal{F}} f_n$, $\operatorname{dom} f = \bigcap_{j\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} F_{ij} \in \Sigma$. Moreover, for i, $j \in \mathbb{N}$, $\{x : \rho(f(x), y_i) \leq 2^{-j}\} = F_{ij} \cap \operatorname{dom} f$ belongs to Σ, so f is measurable.

2B Definition I will say that a filter $\mathcal{F}$ on $\mathbb{N}$ has the **Fatou property** if whenever (X,Σ,μ) is a measure space and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-negative functions defined on X, then $\overline{\int} \liminf_{n\to\mathcal{F}} f_n d\mu \leq \liminf_{n\to\mathcal{F}} \overline{\int} f_n d\mu$.

2C Lemma If $\mathcal{F}$ is a filter on $\mathbb{N}$, the following are equiveridical:

(i) $\mathcal{F}$ has the Fatou property;

(ii) whenever (X,Σ,μ) is a probability space, $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ, and $X = \bigcup_{I\in\mathcal{F}} \bigcap_{n\in I} E_n$, then $\lim_{n\to\mathcal{F}} \mu E_n = 1$.

proof (i)$\Rightarrow$(ii) is elementary. In the reverse direction, suppose that $\mathcal{F}$ does not have the Fatou property. Let (X,Σ,μ) be a measure space and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of non-negative functions defined on X such that $\overline{\int} \liminf_{n\to\mathcal{F}} f_n d\mu > \liminf_{n\to\mathcal{F}} \overline{\int} f_n d\mu$. Take α such that $\overline{\int} \liminf_{n\to\mathcal{F}} f_n d\mu > \alpha > \liminf_{n\to\mathcal{F}} \overline{\int} f_n d\mu$; set $J = \{n : \overline{\int} f_n d\mu\} \leq \alpha$; then J meets every member of $\mathcal{F}$. For $n \in J$, let $g_n : X \to [0,\infty[$ be a Σ-measurable function such that $f_n \leq_{\text{a.e.}} g_n$ and $\int g_n d\mu \leq \alpha$. Then $G = \{x : \sup_{n\in J} g_n(x) > 0\}$ is a countable union of sets of finite measure, and there is a negligible set $H \in \Sigma$ such that $f_n(x) \leq g_n(x)$ whenever $x \in X \setminus H$ and $n \in J$. Note that this means that $\liminf_{n\to\mathcal{F}} f_n(x) = 0$ whenever $x \notin G \cup H$, so that $\overline{\int}_{G\cup H} \liminf_{n\to\mathcal{F}} f_n d\mu > \alpha$.

Let $\mathcal{G}$ be the filter $\{I \cap J : I \in \mathcal{F}\}$ on J. Then

$$\liminf_{n\to\mathcal{G}} f_n(x) = \sup_{I\in\mathcal{F}} \inf_{n\in I\cap J} f_n(x) \ge \liminf_{n\to\mathcal{F}} f(x)$$

for every x, so

$$\overline{\int}_{G\cup H} \liminf_{n\to\mathcal{G}} g_n d\mu \ge \overline{\int}_{G\cup H} \liminf_{n\to\mathcal{G}} f_n d\mu \ge \overline{\int}_{G\cup H} \liminf_{n\to\mathcal{F}} f_n d\mu > \alpha.$$

Let λ be the product measure on $(G \cup H) \times \mathbb{R}$, and consider the ordinate sets $W_n = \{(x, \alpha) : x \in G \cup H, 0 \le \alpha < g_n(x)\}$ for $n \in J$. Set

$$W = \bigcup_{I\in\mathcal{G}} \bigcap_{n\in I} W_n = \{(x,\alpha) : x \in G \cup H,\ \alpha \ge 0,\ \{n : n \in J,\ \alpha < g_n(x)\} \in \mathcal{G}\};$$

setting $g = \liminf_{n\to\mathcal{G}} g_n$,

$$\{(x, \alpha) : x \in G \cup H,\ 0 \le \alpha < g(x)\} \subseteq W \subseteq \{(x, \alpha) : x \in G \cup H,\ 0 \le \alpha \le g(x)\}.$$

So

$$\lambda^* W = \overline{\int}_{G\cup H} \liminf_{n\to\mathcal{G}} g_n d\mu > \alpha$$

(Fremlin 01, 252Yh).

There is therefore a set $V \subseteq (G \cup H) \times \mathbb{R}$ such that $\lambda V < \infty$ and $\lambda^*(V \cap W) > \alpha$. Let ν be the subspace measure on $V \cap W$. Set

$$\begin{aligned} V_n &= V \cap W \cap W_n \text{ if } n \in J \\ &= V \cap W \text{ if } n \in \mathbb{N} \setminus J. \end{aligned}$$

Then

$$\begin{aligned} \liminf_{n\to\mathcal{F}} \nu V_n &= \sup_{I\in\mathcal{F}} \inf_{n\in I} \nu V_n \le \sup_{n\in J} \nu V_n \\ &\le \sup_{n\in J} \lambda W_n = \sup_{n\in J} \int g_n d\mu \le \alpha. \end{aligned}$$

On the other hand,

$$\bigcup_{I\in\mathcal{F}} \bigcap_{n\in I} V_n = \bigcup_{I\in\mathcal{F}} \bigcap_{n\in I\cap J} V \cap W \cap W_n = V \cap W \cap \bigcup_{I\in\mathcal{G}} \bigcap_{n\in I} W_n = V \cap W$$

and $\nu(V \cap W) = \lambda^*(V \cap W) > \alpha$. Moving to a normalization of ν, we see that (ii) is false.

2D Proposition Let $\langle \mathcal{F}_n \rangle_{n\in\mathbb{N}}$ be a sequence of filters on $\mathbb{N}$ with the Fatou property, and $\mathcal{F}$ a filter with the Fatou property. Then $\mathcal{G} = \lim_{n\to\mathcal{F}} \mathcal{F}_n$ has the Fatou property.

proof (a) The point is that if $\langle t_n \rangle_{n\in\mathbb{N}}$ is any sequence in $[0, \infty]$ then

$$\liminf_{n\to\mathcal{G}} t_n = \liminf_{n\to\mathcal{F}} \liminf_{i\to\mathcal{F}_n} t_i.$$

P For any $\alpha \in [0, \infty[$,

$$\begin{aligned} \liminf_{n\to\mathcal{G}} t_n > \alpha &\iff \exists\, \beta > \alpha,\ \{n : t_n \ge \beta\} \in \mathcal{G} \\ &\iff \exists\, \beta > \alpha,\ A \in \mathcal{F},\ \{i : t_i \ge \beta\} \in \mathcal{F}_n\ \forall\, n \in A \\ &\iff \exists\, \beta > \alpha,\ A \in \mathcal{F},\ \liminf_{i\to\mathcal{F}_n} t_i \ge \beta\ \forall\, n \in A \\ &\iff \liminf_{n\to\mathcal{F}} \liminf_{i\to\mathcal{F}_n} t_i > \alpha. \ \mathbf{Q} \end{aligned}$$

(b) So if we have a measure space (X, Σ, μ) and a sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ of non-negative functions defined on X,

$$\begin{aligned} \overline{\int} \liminf_{n\to\mathcal{G}} f_n &= \overline{\int} \liminf_{n\to\mathcal{F}} \liminf_{i\to\mathcal{F}_n} f_i \le \liminf_{n\to\mathcal{F}} \overline{\int} \liminf_{i\to\mathcal{F}_n} f_i \\ &\le \liminf_{n\to\mathcal{F}} \liminf_{i\to\mathcal{F}_n} \overline{\int} f_i = \liminf_{n\to\mathcal{G}} \overline{\int} f_n. \end{aligned}$$

2E Corollary A filter on $\mathbb{N}$ of countable type has the Fatou property.

proof Principal ultrafilters on $\mathbb{N}$, and the Fréchet filter, have the Fatou property; use 1B(ii).

2E Example There is a filter $\mathcal{F}$ on $\mathbb{N}$, an F_σ subset of $\mathcal{P}\mathbb{N}$, which does not have the Fatou property.

construction Let μ be the usual measure on $X = \{0,1\}^{\mathbb{N}}$. Let $\langle E_n\rangle_{n\in\mathbb{N}}$ enumerate the family of sets of the form $\{x : x \in X,\ x{\restriction}k \in J\}$ where $k \in \mathbb{N}$ and $J \subseteq \{0,1\}^k$ has $k+1$ members. Then $\lim_{n\to\infty} \mu E_n = 0$. For $x \in X$ set $\phi(x) = \{n : x \in E_n\}$; then $\phi : X \to \mathcal{P}\mathbb{N}$ is continuous so $K = \phi[X]$ is compact. Set

$$\mathcal{F} = \bigcup_{k\in\mathbb{N}}\{a \cup (b_0 \cap \ldots \cap b_k \setminus k) : a \subseteq \mathbb{N},\ b_0, \ldots , b_k \in K\};$$

then $\mathcal{F}$ is an F_σ set. If $x_0, \ldots , x_k \in X$, there is an $n \geq k$ such that $x_0, \ldots , x_k$ all belong to E_n; accordingly $\emptyset \notin \mathcal{F}$ and $\mathcal{F}$ is a filter on $\mathbb{N}$.

Since $\mathcal{F}$ contains all cofinite subsets of $\mathbb{N}$,

$$\lim_{n\to\mathcal{F}} \mu E_n = \lim_{n\to\infty} \mu E_n = 0.$$

If $x \in X$, then $\phi(x) \in \mathcal{F}$ and

$$x \in \bigcap_{n\in\phi(x)} E_n \subseteq \bigcup_{I\in\mathcal{F}} \bigcap_{n\in I} E_n.$$

So 2C(ii) is false and $\mathcal{F}$ does not have the Fatou property.

3 The Bolzano-Weierstrass property

3A Definition (Filipów Mrožek Recław & Szuca 07) A filter $\mathcal{F}$ on a set X has the **Bolzano-Weierstrass property** if for every $g : X \to [0,1]$ there is an $I \subseteq X$, meeting every member of $\mathcal{F}$, such that $(g{\restriction}I)[[\mathcal{F}{\restriction}I]]$ is convergent.

3B Basic facts (a) Any filter which is not free has the Bolzano-Weierstrass property. $\mathcal{F}_{\mathrm{Fr}}$ has the Bolzano-Weierstrass property. If $\mathcal{F}$ is a filter on X with the Bolzano-Weierstrass property, Z is a compact metrizable space and $g : X \to Z$ is a function, there is an $I \subseteq X$, meeting every member of $\mathcal{F}$, such that $(g{\restriction}I)[[\mathcal{F}{\restriction}I]]$ is convergent (Filipów Mrožek Recław & Szuca 07, §2.3).

(b) (Fridy 93, or Filipów Mrožek Recław & Szuca 07, §3) Let $\mathcal{F}_d$ be the filter of subsets of $\mathbb{N}$ with asymptotic density 1. Then $\mathcal{F}_d$ does not have the Bolzano-Weierstrass property. **P** Let $g = \langle g(n)\rangle_{n\in\mathbb{N}}$ be a sequence in $[0,1]$ which is equidistributed for Lebesgue measure μ. Then $d^*(I) \leq \mu\overline{g[I]}$ for every $I \subseteq \mathbb{N}$. If $I \subseteq \mathbb{N}$ meets every member of $\mathcal{F}_d$, then $d^*(I) > 0$. Let $m \geq 1$ be such that $\frac{2}{m} < d^*(I)$, and for $k \leq m$ set

$$J_k = \{n : n \in I,\ \tfrac{k-1}{m} \leq g(n) \leq \tfrac{k+1}{m}\}.$$

Then

$$d^*(I \cap J_k) \leq \mu\overline{g[I \cap J_k]} \leq \frac{2}{m} < d^*(I)$$

so $I \cap J_k \notin \mathcal{F}_d{\restriction}I$ for every k. But this means that $(g{\restriction}I)[[\mathcal{F}_d{\restriction}I]]$ cannot contain any interval of the form $[0,1] \cap [\frac{k-1}{m}, \frac{k+1}{m}]$ and cannot be convergent. **Q**

(c) For $I \subseteq \mathbb{N}$ write $d_s^*(I)$ for its **Banach upper density**, that is,

$$d_s^*(I) = \inf_{m\geq 1} \sup_{k\in\mathbb{N}} \frac{1}{m}\#(I \cap [k, k+m[) = \lim_{m\to\infty} \sup_{k\in\mathbb{N}} \frac{1}{m}\#(I \cap [k, k+m[)$$

(Fremlin n05). Let $\mathcal{F}_s$ be the **Banach density filter**, that is, the filter of sets $I \subseteq \mathbb{N}$ such that $d_s^*(\mathbb{N}\setminus N) = 0$. Then $\mathcal{F}_s$ does not have the Bolzano-Weierstrass property. **P** Argue as in (b), but with a well-distributed sequence. **Q**

3C Theorem Let $\mathcal{G}$ be a filter on a set X, and suppose that there is a filter $\mathcal{F}$ on X, of countable type, such that $\mathcal{G} \subseteq \mathcal{F}$. Then $\mathcal{G}$ has the Bolzano-Weierstrass property.

proof Induce on the countable-type level $\mathrm{ct}\,\mathcal{F}$ of $\mathcal{F}$.

(a) If $\mathcal{F}$ is the principal filter generated by $\{x\}$, then $\mathcal{G}$ is not free, so has the Bolzano-Weierstrass property.

(b) If $\operatorname{ct}\mathcal{F} = \xi > 0$, there is a sequence $\langle\mathcal{F}_n\rangle_{n\in\mathbb{N}}$ of filters on X, all of countable-type level less than ξ, such that $\mathcal{F} = \lim_{n\to\mathcal{F}_{\mathrm{Fr}}}\mathcal{F}_n$. Let $g : X \to [0,1]$ be any function.

(i) Suppose that there are a set $I \subseteq X$ and an $n \in \mathbb{N}$ such that I meets every member of $\mathcal{F}_n$ and $\mathcal{G}\lceil I \subseteq \mathcal{F}_n\lceil I$. In this case, $\mathcal{G}\lceil I \ne \mathcal{P}I$, so $\mathcal{G}\lceil I$ is a filter, while $\operatorname{ct}(\mathcal{F}_n\lceil I) \le \operatorname{ct}\mathcal{F}_n < \xi$ (1Fb). So $\mathcal{G}\lceil I$ has the Bolzano-Weierstrass property, by the inductive hypothesis. Applying this to the function $g\restriction I$, we see that there is a $J \subseteq I$, meeting every member of $\mathcal{G}\lceil I$, such that $(g\restriction J)[[(\mathcal{G}\lceil I)\lceil J]]$ converges; but in this case J meets every member of $\mathcal{G}$, $\mathcal{G}\lceil J = (\mathcal{G}\lceil I)\lceil J$ and $(g\restriction J)[[\mathcal{G}\lceil J]]$ converges.

(ii) Otherwise, choose $\langle I_k\rangle_{k\in\mathbb{N}}$, $\langle J_k\rangle_{k\in\mathbb{N}}$ and $\langle n_k\rangle_{k\in\mathbb{N}}$ inductively, as follows. $I_0 = \mathbb{N}$. Given that I_k meets every set in $\mathcal{F}$, there must be an n_k, greater than n_i for any $i < k$, such that I_k meets every member of $\mathcal{F}_{n_k}$. In this case, since (i) is false, there is a $J_k \in (\mathcal{G}\lceil I_k) \setminus (\mathcal{F}_{n_k}\lceil I_k)$. As $J_k \cup (X \setminus I_k) \in \mathcal{G} \subseteq \mathcal{F}$, $J_k = I_k \cap (J_k \cup (X \setminus I_k))$ meets every member of $\mathcal{F}$; let $I_{k+1} \subseteq J_k$ be a set meeting every member of $\mathcal{F}$ such that $\operatorname{diam} g[I_{k+1}] \le 2^{-k-1}$. Continue.

At the end of the induction, set $I = \bigcup_{k\in\mathbb{N}} I_k\setminus J_k$. As $I_{k+1} \subseteq J_k \subseteq I_k$ for every k, $I\cap I_{k+1} = (I\cap I_k)\setminus(I_k\setminus J_k)$ for every k. Now, for each k, $J_k \notin \mathcal{F}_{n_k}\lceil I_k$, so $I_k \setminus J_k$ meets every member of $\mathcal{F}_{n_k}\lceil I_k$ and I meets every member of $\mathcal{F}_{n_k}$. Consequently I meets every member of $\mathcal{F} \supseteq \mathcal{G}$. On the other hand, $\mathbb{N} \setminus (I_k \setminus J_k) \in \mathcal{G}$ for every k, so (inducing on k) $I \cap I_k \in \mathcal{G}\lceil I$ for every k. Since $\operatorname{diam} g[I_k] \le 2^{-k}$ for every k, $(g\lceil I)[[\mathcal{G}\lceil I]]$ is Cauchy, therefore convergent.

(iii) Thus in either case the condition of Definition 3A is satisfied. As g is arbitrary, $\mathcal{G}$ has the Bolzano-Weierstrass property, and the induction continues.

3D Corollary Neither the density filter nor the Banach density filter is of countable type.

3E Proposition $\mathcal{F}_d$ has the Fatou property.

proof (a) Let (X,Σ,μ) be a probability space and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence in Σ such that $\bigcup_{I\in\mathcal{F}_d}\bigcap_{n\in I} E_n = X$; set $\alpha = \liminf_{n\to\mathcal{F}_d}\mu E_n$. Take any $\epsilon > 0$. Set $I = \{n : \mu E_n \le \alpha + \epsilon\}$; then $d^*(I) > 0$. Take any $\eta \in \left]0, \frac{1}{2}d^*(I)\right]$. Set $M = \{m : m > 0, \#(I\cap m) \ge m(d^*(I) - \eta)\}$; then $M \subseteq \mathbb{N}$ is infinite. For $k \in \mathbb{N}$ set

$$H_k = \bigcap_{m\in M\setminus k}\{x : x \in X, \#(\{n : n \in I\cap m, x \in E_n\}) \ge m(d^*(I) - 2\eta)\}.$$

Then $\langle H_k\rangle_{k\in\mathbb{N}}$ is a non-decreasing sequence in Σ and $\bigcup_{k\in\mathbb{N}} H_k = X$. **P** If $x \in X$, there is a $J \in \mathcal{F}_d$ such that $x \in E_n$ for every $n \in J$. Now $d^*(\mathbb{N}\setminus J) = 0$ so there is a $k \in \mathbb{N}$ such that $\#(m\setminus J) \le \eta m$ for every $m \ge k$. If $m \in M$ and $m \ge k$, then

$$\#(\{n : n \in I\cap m, x \in E_n\}) \ge \#(m\cap I\cap J) \ge \#(m\cap I) - \eta m \ge m(d^*(I) - 2\eta)\},$$

so $x \in H_k$. **Q**

(b) There is therefore an $m \in M$ such that $\mu H_m \ge 1 - \epsilon$ and $\#(I\cap m) \le m(d^*(I) + \eta)$. In this case

$$\begin{aligned} m(1-\epsilon)(d^*(I) - 2\eta) \le m(d^*(I) - 2\eta)\mu H_m &\le \int \#(\{n : n \in I\cap m, x \in E_n\})\mu(dx) \\ &= \sum_{n\in I\cap m}\mu E_n \le (\alpha+\epsilon)\#(I\cap m) \le m(\alpha+\epsilon)(d^*(I) + \eta). \end{aligned}$$

So $(1-\epsilon)(d^*(I) - 2\eta) \le (\alpha+\epsilon)(d^*(I) + \eta)$. As η and ϵ are arbitrary, and $d^*(I) > 0$, $\alpha = 1$. By 2C, $\mathcal{F}$ has the Fatou property.

3F Proposition $\mathcal{F}_s$ has the Fatou property.

proof (a) Let (X,Σ,μ) be a probability space and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence in Σ such that $\bigcup_{I\in\mathcal{F}_s}\bigcap_{n\in I} E_n = X$; set $\alpha = \liminf_{n\to\mathcal{F}_s}\mu E_n$. Take any $\epsilon > 0$. Set $I = \{n : \mu E_n \le \alpha + \epsilon\}$; then $d_s^*(I) > 0$. Take any $\eta \in \left]0, \frac{1}{2}d_s^*(I)\right]$. Then there is a sequence $\langle k_m\rangle_{m\in\mathbb{N}}$ such that $k_m + m \le k_{m+1}$ and $\#(I\cap [k_m, k_m + m[) \ge m(d_s^*(I) - \eta)$ for every $m \in \mathbb{N}$; set $K_m = [k_m, k_m + m[$ for each m. For $l \in \mathbb{N}$ set

$$H_l = \bigcap_{m \geq l} \{x : x \in X,\, \#(\{n : n \in I \cap K_m,\, x \in E_n\}) \geq m(d_s^*(I) - 2\eta)\}.$$

Then $\langle H_l \rangle_{l \in \mathbb{N}}$ is a non-decreasing sequence in Σ and $\bigcup_{l \in \mathbb{N}} H_l = X$. **P** If $x \in X$, there is a $J \in \mathcal{F}_s$ such that $x \in E_n$ for every $n \in J$. Now $d_s^*(\mathbb{N} \setminus J) = 0$ so there is an $l \in \mathbb{N}$ such that $\#(K_m \setminus J) \leq \eta m$ for every $m \geq l$. If $m \geq l$, then

$$\#(\{n : n \in I \cap K_m,\, x \in E_n\}) \geq \#(I \cap J \cap K_m) \geq \#(I \cap K_m) - \eta m \geq m(d_s^*(I) - 2\eta)\},$$

so $x \in H_l$. **Q**

(b) There is therefore an $m \geq 1$ such that $\mu H_m \geq 1 - \epsilon$ and $\#(I \cap K_m) \leq m(d_s^*(I) + \eta)$. In this case

$$\begin{aligned} m(1-\epsilon)(d_s^*(I) - 2\eta) \leq m(d^*(I) - 2\eta)\mu H_m &\leq \int \#(\{n : n \in I \cap K_m,\, x \in E_n\})\mu(dx) \\ &= \sum_{n \in I \cap K_m} \mu E_n \leq (\alpha + \epsilon)\#(I \cap K_m) \leq m(\alpha + \epsilon)(d_s^*(I) + \eta). \end{aligned}$$

So $(1-\epsilon)(d^*(I) - 2\eta) \leq (\alpha + \epsilon)(d_s^*(I) + \eta)$. As η and ϵ are arbitrary, and $d_s^*(I) > 0$, $\alpha = 1$. By 2C, $\mathcal{F}_s$ has the Fatou property.

4 Medial limits

4A Recall that a **medial limit** is a non-negative additive functional θ on $\mathbb{PN}$ such that $\int w\,d\theta = \lim_{n \to \infty} w(n)$ whenever $w \in \mathbb{R}^{\mathbb{N}}$ is a convergent sequence and $\iint f_n(x)\theta(dn)\mu(dx)$ is defined and equal to $\iint f_n(x)\mu(dx)\theta(dn)$ whenever (X, Σ, μ) is a probability space and $\langle f_n \rangle_{n \in \mathbb{N}}$ is a uniformly bounded sequence of integrable real-valued functions on X; here $\int \ldots d\theta$ is defined as in Fremlin 02, 363L. See Fremlin n02 or Hoffman-Jørgensen 78 for the basic theory of medial limits, in particular, for the proof that if $\mathfrak{p} = \mathfrak{c}$ then there is a medial limit.

If $\mathcal{F}$ is a filter on $\mathbb{N}$ a medial limit θ **refines** $\mathcal{F}$ if $\liminf_{n \to \mathcal{F}} w(n) \leq \int w\,d\theta$ for every $w \in \ell^\infty$.

4B Proposition Let $\mathfrak{F}$ be the set of filters $\mathcal{F}$ on $\mathbb{N}$ such that there is an additive functional $\theta : \mathcal{P}\mathbb{N} \to [0,1]$ such that

(†) $\iint f_n(x)\theta(dn)\mu(dx)$ is defined and equal to $\iint f_n(x)\mu(dx)\theta(dn)$ whenever (X, Σ, μ) is a probability space and $\langle f_n \rangle_{n \in \mathbb{N}}$ is a uniformly bounded sequence of integrable real-valued functions on X,

(‡) $\liminf_{n \to \mathcal{F}} w(n) \leq \int w\,d\theta$ for every $w \in \ell^\infty$.

Then $\lim_{n \to \mathcal{F}} \mathcal{F}_n \in \mathfrak{F}$ whenever $\langle \mathcal{F}_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\mathfrak{F}$ and $\mathcal{F} \in \mathfrak{F}$.

proof (a) Write $\mathcal{G}$ for $\lim_{n \to \mathcal{F}} \mathcal{F}_n$. For each $n \in \mathbb{N}$ let θ_n witness that $\mathcal{F}_n \in \mathfrak{F}$, and let θ witness that $\mathcal{F} \in \mathfrak{F}$. Set $\theta^*(a) = \int \theta_n(a)\theta(dn)$ for $a \subseteq \mathbb{N}$; then $\theta^* : \mathcal{P}\mathbb{N} \to [0,1]$ is an additive functional.

(b) θ^* has the property (†). **P** If (X, Σ, μ) is a probability space and $\langle f_i \rangle_{i \in \mathbb{N}}$ is a uniformly bounded sequence of integrable real-valued functions on X,

$$\begin{aligned} \iint f_i(x)\theta^*(di)\mu(dx) &= \iiint f_i(x)\theta_n(di)\theta(dn)\mu(dx) = \iiint f_i(x)\theta_n(di)\mu(dx)\theta(dn) \\ &= \iiint f_i(x)\mu(dx)\theta_n(di)\theta(dn) = \iiint f_i(x)\mu(dx)\theta^*(di). \text{ } \mathbf{Q} \end{aligned}$$

(c) For any $w \in \ell^\infty$,

$$\liminf_{i \to \mathcal{G}} w(i) \leq \liminf_{n \to \mathcal{F}} \liminf_{i \to \mathcal{F}_n} w(i).$$

P Suppose that $\gamma > \liminf_{n \to \mathcal{F}} \liminf_{i \to \mathcal{F}_n} w(i)$, and that $A \in \mathcal{G}$. Then $B = \{n : A \in \mathcal{F}_n\}$ belongs to $\mathcal{F}$, so there is an $n \in B$ such that $\liminf_{i \to \mathcal{F}_n} w(i) \leq \gamma$. Now $A \in \mathcal{F}_n$, so

$$\inf_{i \in A} w(i) \leq \liminf_{i \to \mathcal{F}_n} w(i) \leq \gamma.$$

As A is arbitrary, $\liminf_{i \to \mathcal{G}} w(i) \leq \gamma$. **Q**

Accordingly,

$$\liminf_{i\to\mathcal{G}} w(i) \le \liminf_{n\to\mathcal{F}} \liminf_{i\to\mathcal{F}_n} w(i) \le \int \liminf_{i\to\mathcal{F}_n} w(i)\theta(dn)$$
$$\le \iint w\,d\theta_n\theta(dn) = \int w\,d\theta^*.$$

So θ^* has property (‡), and witnesses that $\mathcal{G} \in \mathfrak{F}$.

4C Corollary If there is any medial limit, and $\mathcal{F}$ is a free filter on $\mathbb{N}$ of countable type, then there is a medial limit refining $\mathcal{F}$.

proof In 4B, it is easy to check that $\mathfrak{F}$ contains all principal ultrafilters on $\mathbb{N}$; if $\mathcal{F}$ is generated by $\{k\}$, set $\theta a = \chi a(k)$ for $a \subseteq \mathbb{N}$; then θ witnesses that $\mathcal{F} \in \mathfrak{F}$. By definition, a medial limit is a witness that $\mathcal{F}_{\mathrm{Fr}} \in \mathfrak{F}$. So an induction on the countable-type level of $\mathcal{F}$ shows that every filter on $\mathbb{N}$ of countable type belongs to $\mathfrak{F}$. If $\mathcal{F}$ is a free filter and belongs to $\mathfrak{F}$, then a witness that $\mathcal{F} \in \mathfrak{F}$ is a medial limit refining $\mathcal{F}$.

4D Proposition Suppose that $\mathcal{F}$ is a filter on $\mathbb{N}$ such that whenever $I \subseteq \mathbb{N}$ meets every member of $\mathcal{F}$ there is a medial limit refining

$$\mathcal{F} \vee \{I\} = \{A : A \subseteq \mathbb{N},\ A \cup (\mathbb{N} \setminus I) \in \mathcal{F}\},$$

the filter generated by $\mathcal{F} \cup \{I\}$. Then $\mathcal{F}$ has the Fatou property.

proof Let J be any set meeting every member of $\mathcal{F}$, and θ a medial limit refining $\mathcal{F} \vee \{J\}$. Let (X, Σ, μ) be a probability space and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence in Σ such that $X = \bigcup_{I\in\mathcal{F}} \bigcap_{n\in I} E_n$, that is, $\liminf_{n\to\mathcal{F}} \chi E_n(x) = 1$ for every $x \in X$. Then

$$1 = \int \liminf_{n\to\mathcal{F}} \chi E_n(x)\mu(dx) \le \iint \chi E_n(x)\theta(dn)\mu(dx)$$

(since θ also refines $\mathcal{F}$)

$$= \iint \chi E_n(x)\mu(dx)\theta(dn) = \int \mu E_n\theta(dn) \le \limsup_{n\to\mathcal{F}\vee\{J\}} \mu E_n \le \sup_{n\in J} \mu E_n.$$

As J is arbitrary, $\liminf_{n\to\mathcal{F}} \mu E_n = 1$. As (X, Σ, μ) and $\langle E_n\rangle_{n\in\mathbb{N}}$ are arbitrary, $\mathcal{F}$ has the Fatou property.

4E Remark The filter $\mathcal{F}$ of Example 2E is a Borel subset of $\mathcal{P}\mathbb{N}$ but is not refined by any medial limit. **P?** Otherwise, follow the argument of 4D with $J = \mathbb{N}$; we should get $1 \le \limsup_{n\to\mathcal{F}} \mu E_n = 0$. **XQ**

4F Proposition If there is a medial limit, there is a medial limit refining the density filter.

proof Let θ be a medial limit, and define θ^* by setting

$$\theta^*(a) = \int \tfrac{1}{n+1}\#(a \cap (n+1))\theta(dn)$$

for $a \subseteq \mathbb{N}$. Then

$$\int w\,d\theta^* = \int \tfrac{1}{n+1}\textstyle\sum_{i=0}^n w(i)\theta(dn)$$

for $w \in \ell^\infty$. So if (X, Σ, μ) is a probability space and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a uniformly bounded sequence of real-valued functions on X,

$$\iint f_n(x)\theta^*(dn)\mu(dx) = \iint \frac{1}{n+1}\sum_{i=0}^n f_i(x)\theta(dn)\mu(dx) = \iint \frac{1}{n+1}\sum_{i=0}^n f_i(x)\mu(dx)\theta(dn)$$
$$= \int \frac{1}{n+1}\sum_{i=0}^n \int f_i(x)\mu(dx)\theta(dn) = \iint f_n(x)\mu(dx)\theta(dn).$$

Thus θ^* is a medial limit. Now $\theta^*(a) \le d^*(a)$ for every $a \subseteq \mathbb{N}$; in particular, $\theta^*(a) = 0$ whenever $d^*(a) = 0$ and $\theta^*(a) = 1$ for every $a \in \mathcal{F}_d$; it follows that $\liminf_{n\to\mathcal{F}_d} w(n) \le \int w\,d\theta^*$ for every $w \in \ell^\infty$, and θ^* refines $\mathcal{F}_d$.

5 Isomorphism classes of filters of countable type

5A Homogeneous and critical filters (a) I will say that a filter $\mathcal{F}$ on a set X is **homogeneous** if $\mathcal{F}$ is isomorphic to its trace $\mathcal{F}\lceil Y$ whenever $Y \subseteq X$ and $X \setminus Y \notin \mathcal{F}$. Observe that in this case $\mathcal{F}$ must contain every set $A \subseteq X$ such that $\#(X \setminus A) < \#(X)$; in particular, $\mathcal{F}$ must be free, unless X is a singleton. An ultrafilter on X is homogeneous iff it is uniform.

(b) I will say that a filter $\mathcal{F}$ of countable type on a set I is **critical** if there are a sequentially compact Hausdorff space X and a function $f : I \to X$ such that $f[[\mathcal{F}]]$ converges to a point of $X \setminus \bigcup_{\eta<\mathrm{ct}\,\mathcal{F}} f[I]^\sim_\eta$.

5B Lemma (a) Let X be a discrete space. Then there is a locally countable sequentially compact Hausdorff space Y with an open set homeomorphic to X.

(b) Let $(X, \mathfrak{T})$ be a Hausdorff space, and suppose that whenever $\langle A_i\rangle_{i\in I}$ is a countable family of countably infinite subsets of X and $x \in X$ then

either there are an $i \in I$ and a sequence of distinct points in A_i convergent to a point of X

or there are distinct i, $j \in I$ such that $A_i \cap A_j$ is infinite

or there is a disjoint family $\langle G_i\rangle_{i\in I}$ of open sets such that $A_i \setminus G_i$ is finite for every i and $x \notin \overline{\bigcup_{i\in I} G_i}$.

Suppose that we are also given a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in X with no convergent subsequence in X. Then X can be embedded as an open set in a sequentially compact Hausdorff space Z such that $\langle x_n\rangle_{n\in\mathbb{N}}$ is convergent in Z to a point z^* such that no sequence in $Z \setminus (\{z^*\} \cup \{x_n : n \in \mathbb{N}\})$ converges to z^*.

proof (a)(i) We can suppose that X is a cardinal κ; let $\lambda > \kappa$ be a cardinal of uncountable cofinality such that the cardinal power λ^ω is equal to λ, and let $\langle A_\xi\rangle_{\kappa\le\xi<\lambda}$ enumerate $[\lambda]^\omega$. Define $\langle B_\xi\rangle_{\xi<\lambda}$ inductively by setting

$$\begin{aligned} B_\xi &= A_\xi \text{ if } \kappa \le \xi < \lambda \text{ and } A_\xi \cap B_\eta \text{ is finite for every } \eta < \xi \\ &= \emptyset \text{ otherwise.} \end{aligned}$$

Observe that $B_\xi \cap B_\eta$ is finite for all distinct ξ, $\eta < \lambda$, and that if $C \in [\lambda]^\omega$ there is a $\xi < \lambda$ such that $C \cap B_\xi$ is infinite.

On $Y = \lambda$, let $\mathfrak{T}$ be the topology

$$\{G : G \subseteq Y,\ B_\xi \setminus G \text{ is finite for every } \xi \in G\}.$$

(ii) Since $B_\xi = \emptyset$ for $\xi < \kappa$, every subset of X is an open subset of Y, and X, with its discrete topology, is a subspace of Y.

(iii) If $\langle \xi_n\rangle_{n\in\mathbb{N}}$ is a sequence in Y, then either it has a constant subsequence, which is surely convergent, or $C = \{\xi_n : n \in \mathbb{N}\}$ is infinite. In this case, let $\xi < \lambda$ be such that $B_\xi \cap C$ is infinite; then $\langle \xi_n\rangle_{n\in\mathbb{N}}$ has a subsequence converging to ξ.

Thus Y is sequentially compact.

(iv) If α, β are distinct points of Y, enumerate $\mathbb{N} \times \mathbb{N}$ as $\langle (i_n, j_n)\rangle_{n\in\mathbb{N}}$ in such a way that $i_n \le n$ for every n, and define $\langle (U_n, V_n)\rangle_{n\in\mathbb{N}}$ as follows. $U_0 = \{\alpha\}$, $V_0 = \{\beta\}$. Given that U_n and V_n are disjoint countable sets such that $B_\xi \cap U_n$ is finite for every $\xi \in \lambda \setminus U_n$, while $B_\xi \cap V_n$ is finite for every $\xi \in \lambda \setminus V_n$, let $\langle \xi_{nj}\rangle_{j\in\mathbb{N}}$ run over U_n and $\langle \eta_{nj}\rangle_{j\in\mathbb{N}}$ run over V_n. Set

$$U_{n+1} = U_n \cup (B_{\xi_{i_n,j_n}} \setminus V_n), \qquad V_{n+1} = V_n \cup (B_{\eta_{i_n,j_n}} \setminus U_{n+1})$$

and continue. At the end of the induction, set $G = \bigcup_{n\in\mathbb{N}} U_n$ and $H = \bigcup_{n\in\mathbb{N}} V_n$; then $G = \{\xi_{ij} : i, j \in \mathbb{N}\}$ and $H = \{\eta_{ij} : i, j \in \mathbb{N}\}$ are disjoint and open, $\alpha \in G$ and $\beta \in H$.

Thus Y is Hausdorff.

(v) Note that in the construction of (iv), both G and H are necessarily countable; it follows at once that Y is locally countable.

(b)(i) Let $\mathcal{C}$ be the family of all infinite subsets C of X such that there is no convergent sequence of distinct points of C. Note that $A^* = \{x_n : n \in \mathbb{N}\}$ belongs to $\mathcal{C}$. Let $\mathcal{A} \subseteq \mathcal{C}$ be a maximal set, containing A^*, such that $A \cap B$ is finite for all distinct A, B in $\mathcal{A}$. Let $\langle z_A \rangle_{A \in \mathcal{A}}$ be a family of distinct points not in X, and set $Z_0 = \{z_A : A \in \mathcal{A}\}$.

(ii) By (a), we can construct a locally countable sequentially compact Hausdorff space $(Z_1, \mathfrak{T}_1)$, disjoint from X, such that $Z_0 \subseteq Z_1$ and the subspace topology on Z_0 is discrete. Set $Z = Z_1 \cup X$, and give Z the topology

$$\begin{aligned}\mathfrak{S} = \{G : G \subseteq Z,\, G \cap Z_1 \in \mathfrak{T}_1,\, G \cap X \in \mathfrak{T},\\ A \setminus G \text{ is finite whenever } A \in \mathcal{A} \text{ and } z_A \in G\}.\end{aligned}$$

(iii) X is an open subset of Z and $\mathfrak{T}$ is the subspace topology induced by $\mathfrak{S}$ on X.

(iv) If $\langle t_n \rangle_{n \in \mathbb{N}}$ is a sequence in Z, then

either it has a constant subsequence, which is surely convergent;

or it has a subsequence in Z_1, which must in turn have a subsequence converging in Z_1 and in Z;

or it has a convergent subsequence in X, which will converge in Z;

or $C = \{t_n : n \in \mathbb{N}\} \cap X$ belongs to $\mathcal{C}$, and there is an $A \in \mathcal{A}$ such that $C \cap A$ is infinite, in which case $\langle t_n \rangle_{n \in \mathbb{N}}$ has a subsequence convergent to z_A.

So Z is sequentially compact.

(v) Z is Hausdorff. **P** Suppose that w, z are distinct points of Z.

(α) If w, z both belong to X, they are separated by disjoint open sets in X, which are still open in Z.

(β) If $w \in X$ and $z \in Z_1$, let V be a countable neighbourhood of z in Z_1, and consider the countable set $\mathcal{D} = \{A : A \in \mathcal{A},\, z_A \in V\}$. By the hypothesis on X, there is a family $\langle H_A \rangle_{A \in \mathcal{D}}$ of open sets in X such that $A \setminus H_A$ is finite for every $A \in \mathcal{D}$ and $w \notin \overline{\bigcup_{A \in \mathcal{D}} H_A}$. Set $G = X \setminus \overline{\bigcup_{A \in \mathcal{D}} H_A}$, $H = V \cup \bigcup_{A \in \mathcal{D}} H_A$; then G and H are open subsets of Z separating w from z.

(γ) Similarly, w and z can be separated if $w \in Z_1$ and $z \in X$.

(δ) Suppose that w, z both belong to Z_1. Let U, $V \subseteq Z_1$ be disjoint countable sets, both open in Z_1, containing w, z respectively. Set $\mathcal{D} = \{A : A \in \mathcal{A},\, z_A \in U \cup V\}$; let $\langle G_A \rangle_{A \in \mathcal{D}}$ be a disjoint family of open subsets in X such that $A \setminus G_A$ is finite for every $A \in \mathcal{D}$; set $G = U \cup \{G_A : A \in \mathcal{D},\, z_A \in U\}$, $H = V \cup \{G_A : A \in \mathcal{D},\, z_A \in V\}$. Then G and H are open sets in Z separating w from z. **Q**

(vi) Setting $z^* = z_{A^*}$, we see that $A^* \setminus G$ is finite for every neighbourhood G of z^*, so that $z^* = \lim_{n\to\infty} x_n$. The construction ensured that $\{z^*\}$ is an open set in Z_1, so that $X \cup \{z^*\}$ is an open set in Z, as required. If $\langle z_n \rangle_{n \in \mathbb{N}}$ is a sequence in $Z \setminus (\{z^*\} \cup \{x_n : n \in \mathbb{N}\})$, then

either it has a subsequence in Z_1, which cannot converge to z^* because z^* is isolated in Z_1,

or it has a subsequence which is a convergent sequence in X, and does not converge to z^*,

or $C = \{z_n : n \in \mathbb{N}\} \cap X$ belongs to $\mathcal{C}$, so meets some $A \in \mathcal{A}$ in an infinite set; as $A \neq A^*$, $\langle z_n \rangle_{n \in \mathbb{N}}$ has a subsequence converging to some $z_A \neq z^*$.

In any case, we see that $\langle z_n \rangle_{n \in \mathbb{N}}$ does not converge to z^*, which is what we needed to know.

5C Lemma Suppose that $\langle \mathcal{F}_n \rangle_{n \in \mathbb{N}}$ is a sequence of critical filters of countable type on $\mathbb{N}$ such that $\langle \operatorname{ct} \mathcal{F}_n \rangle_{n \in \mathbb{N}}$ is non-decreasing. Let $\langle I_n \rangle_{n \in \mathbb{N}}$ be a partition of $\mathbb{N}$ into infinite sets, and $g_n : \mathbb{N} \to I_n$ a bijection for each n; let $\mathcal{F}$ be $\lim_{n \to \mathcal{F}_{\mathrm{Fr}}} g_n[[\mathcal{F}_n]]$. Then $\mathcal{F}$ is a critical filter, and $\operatorname{ct} \mathcal{F} = \sup_{n \in \mathbb{N}} (\operatorname{ct} \mathcal{F}_n + 1)$.

proof (a) Setting $\xi = \sup_{n \in \mathbb{N}} (\operatorname{ct} \mathcal{F}_n + 1)$, then of course

$$\operatorname{ct} \mathcal{F} \le \sup_{n \in \mathbb{N}} (\operatorname{ct} g_n[[\mathcal{F}_n]] + 1) \le \xi$$

by 1D.

(b) For each $n \in \mathbb{N}$, let X_n be a sequentially compact Hausdorff space and $f_n : \mathbb{N} \to X_n$ a function such that $\lim f_n[[\mathcal{F}_n]]$ is defined and does not belong to $\bigcup_{\eta<\mathrm{ct}\,\mathcal{F}_n} f[\mathbb{N}]^{\sim}_{\eta}$. Set $Z_n = X_n \times \{n\}$ for each n, $X = \bigcup_{n\in\mathbb{N}} X_n$ with the disjoint union topology. Then X satisfies the conditions of 5Bb. **P** Let $\langle A_i\rangle_{i\in I}$ be a countable family of countably infinite subsets of X such that $A_i \cap A_j$ is finite for all $i \neq j$ and there is no non-trivial convergent sequence made up of points from any single A_i. As every X_n is sequentially compact, this means that $A_i \cap Z_n$ is finite for every i. Take any $(x, k) \in X$.

If I is finite, there is an $n > k$ such that $A_i \cap A_j$ does not meet Z_m for any $m \geq n$ and any $i \neq j$. In this case, for $m \geq n$, $\bigcup_{i\in I} A_i \cap Z_m$ is finite, so we can find disjoint open sets $G_{im} \subseteq X_m$ such that $A_i \cap Z_m \subseteq G_{im} \times \{m\}$ for each i; setting $G_i = \bigcup_{m\geq n} G_{im} \times \{m\}$, we have a disjoint family $\langle G_i\rangle_{i\in I}$ of open sets in X such that $A_i \setminus G_i$ is finite for every i, and $(x, k) \notin \overline{\bigcup_{i\in I} G_i}$.

If I is infinite, we may take it that $I = \mathbb{N}$. In this case, let $\langle n_i\rangle_{i\in\mathbb{N}}$ be a strictly increasing sequence, starting with $n_0 > k$, such that $A_i \cap A_j \cap Z_m$ is empty whenever $i < j$ and $m \geq n_j$; set $A'_i = A_i \setminus \bigcup_{m<n_i} Z_m$. Then $A_i \setminus A'_i$ is finite and $A'_i \cap A'_j$ is empty whenever $i < j$ in $\mathbb{N}$. This time, $\bigcup_{i\in\mathbb{N}} A_i \cap Z_m$ is finite for every $m \in \mathbb{N}$, so we can use the same method as before to find G_{im}, for i, $m \in \mathbb{N}$, and G_m, for $m \in \mathbb{N}$, such that $A'_i \subseteq G_i$ for every i and $\langle G_i\rangle_{i\in\mathbb{N}}$ is a disjoint family of open sets. If we take the elementary precaution of setting $G_{ik} = \emptyset$ for every i, then $(x, k) \notin \overline{\bigcup_{i\in\mathbb{N}} G_i}$. **Q**

(c) By Lemma 5Bb, we can embed X in a sequentially compact Hausdorff space Z such that $\langle(x_n, n)\rangle_{n\in\mathbb{N}}$ converges to a point z^* of Z and no sequence in $Z\setminus(\{z^*\}\cup\{(x_n, n) : n \in \mathbb{N}\})$ converges to z^*. Let $f : \mathbb{N} \to Z$ be such that $f(g_n(i)) = (f_n(i), n)$ for all i, $n \in \mathbb{N}$. Then $f[[\mathcal{F}]] \to z^*$. **P** If G is any neighbourhood of z^*, then

$$\begin{aligned}\{n : \{j : f(j) \in G\} \in g_n[[\mathcal{F}_n]]\} &= \{n : \{i : f(g_n(i)) \in G\} \in \mathcal{F}_n\}\\ &= \{n : \{i : (f_n(i), n) \in G\} \in \mathcal{F}_n\}\\ &\supseteq \{n : (x_n, i) \in G\} \in \mathcal{F}_{\mathrm{Fr}},\end{aligned}$$

so $f^{-1}[G] \in \mathcal{F}$. **Q**

If $n \in \mathbb{N}$ and $\eta < \mathrm{ct}\,\mathcal{F}_n$, then $(x_n, n) \notin f[\mathbb{N}]^{\sim}_{\eta}$. **P** Because Z_n is open in X and Z,

$$\begin{aligned}Z_n \cap f[\mathbb{N}]^{\sim}_{\eta} &= (Z_n \cap f[\mathbb{N}])^{\sim}_{\eta} = f[I_n]^{\sim}_{\eta}\\ &= (f_n[\mathbb{N}] \times \{n\})^{\sim}_{\eta} = f_n[\mathbb{N}]^{\sim}_{\eta} \times \{n\}\end{aligned}$$

(calculating $f_n[\mathbb{N}]^{\sim}_{\eta}$ in X_n, of course); as $x_n \notin f_n[\mathbb{N}]^{\sim}_{\eta}$, $(x_n, n) \notin f[\mathbb{N}]^{\sim}_{\eta}$. **Q**

? If $\zeta < \xi$ and $z^* \in f[\mathbb{N}]^{\sim}_{\zeta}$, then (as z^* surely does not belong to $f[\mathbb{N}]$) there is a sequence $\langle z_i\rangle_{i\in\mathbb{N}}$ in $\bigcup_{\eta<\zeta} f[\mathbb{N}]^{\sim}_{\eta}$ converging to z^*, with no z_i equal to z^*. Now $\langle z_i\rangle_{i\in\mathbb{N}}$ must have a subsequence in common with $\langle(x_n, n)\rangle_{n\in\mathbb{N}}$, so there must be infinitely many n such that $(x_n, n) \in \bigcup_{\eta<\zeta} f[\mathbb{N}]^{\sim}_{\eta}$. However, there is an $m \in \mathbb{N}$ such that $\zeta \leq \mathrm{ct}\,\mathcal{F}_n$ for every $n \geq m$, in which case $(x_n, n) \notin \bigcup_{\eta<\zeta} f_n[\mathbb{N}]^{\sim}_{\eta}$; which is impossible. **X**

Thus we know that $\mathrm{ct}\,\mathcal{F} \leq \xi$, that $f[[\mathcal{F}]] \to z^*$, and that $z^* \notin \bigcup_{\zeta<\xi} f[\mathbb{N}]^{\sim}_{\zeta}$; we conclude that $\mathrm{ct}\,\mathcal{F} = \xi$ and that $f : \mathbb{N} \to Z$ witnesses that $\mathcal{F}$ is critical.

5D Theorem For every $\xi < \omega_1$ there is a critical filter on $\mathbb{N}$ with countable-type level ξ.

proof Induce on ξ. For $\xi = 0$, take a principal ultrafilter. For the step to $\xi > 0$, let $\langle\xi_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence such that $\xi = \sup_{n\in\mathbb{N}}(\xi_n + 1)$. For each $n \in \mathbb{N}$, there is a critical filter $\mathcal{F}_n$ on $\mathbb{N}$ such that $\mathrm{ct}\,\mathcal{F}_n = \xi_n$; now the construction of Lemma 5C gives a critical filter with countable-type level ξ.

5E Lemma For every $n \in \mathbb{N}$ there is a homogeneous critical filter $\mathcal{H}_n$ on $\mathbb{N}$ such that $\mathrm{ct}\,\mathcal{H}_n = n + 1$.

proof Induce on n.

(a) Start with $\mathcal{H}_0 = \mathcal{F}_{\mathrm{Fr}}$. To see that $\mathcal{F}_{\mathrm{Fr}}$ is critical, take $X = \mathbb{N} \cup \{\infty\}$ to be the one-point compactification of $\mathbb{N}$, and $f : \mathbb{N} \to X$ the identity function.

(b) For the inductive step to $n + 1$, let $\langle I_k\rangle_{k\in\mathbb{N}}$ be a partition of $\mathbb{N}$ into infinite sets, and let g_k be the increasing enumeration of I_k for each k; set $\mathcal{H}_{n+1} = \lim_{k\to\mathcal{F}_{\mathrm{Fr}}} g_k[[\mathcal{H}_n]]$. By 5C, $\mathcal{H}_{n+1}$ is a critical filter and $\mathrm{ct}\,\mathcal{H}_{n+1} = n + 2$.

(c)(i) For each $k \in \mathbb{N}$, set $\mathcal{G}_k = g_k[[\mathcal{H}_n]]\lceil I_k$, so that $\mathcal{G}_k$ is isomorphic to $\mathcal{H}_n$ and is homogeneous; observe that

$$\mathcal{H}_{n+1} = \{A : A \subseteq \mathbb{N},\, A \cap I_k \in \mathcal{G}_k\} \text{ for all but finitely many } k\}.$$

(ii) Suppose that $A \subseteq \mathbb{N}$ is infinite, and that $Y \subseteq \bigcup_{k\in A} I_k$ is such that $I_k \setminus Y \notin \mathcal{G}_k$ for every $k \in A$. Of course $\mathbb{N} \setminus Y \notin \mathcal{H}_{n+1}$. For $k \in A$, $\mathcal{G}_k$ is isomorphic to its trace $\mathcal{G}'_k = \mathcal{G}_k\lceil(Y \cap I_k)$; let $h_k : I_k \to I_k \cap Y$ be a corresponding isomorphism. Enumerate A in ascending order as $\langle k_i\rangle_{i\in\mathbb{N}}$, and let $h : \mathbb{N} \to Y$ be the bijection defined by setting

$$h(j) = h_{k_i} g_{k_i} g_i^{-1}(j) \text{ if } i \in \mathbb{N} \text{ and } j \in I_i.$$

For $B \subseteq \mathbb{N}$,

$$\begin{aligned}
B \in \mathcal{H}_{n+1} &\iff B \cap I_i \in \mathcal{G}_i \text{ for all but finitely many } i\\
&\iff g_i^{-1}[B \cap I_i] \in \mathcal{H}_n \text{ for all but finitely many } i\\
&\iff g_{k_i}[g_i^{-1}[B \cap I_i]] \in \mathcal{G}_{k_i} \text{ for all but finitely many } i\\
&\iff h_{k_i}[g_{k_i}[g_i^{-1}[B \cap I_i]]] \in \mathcal{G}'_{k_i} \text{ for all but finitely many } i\\
&\iff h[B] \cap I_{k_i} \in \mathcal{G}'_{k_i} \text{ for all but finitely many } i\\
&\iff h[B] \cap I_k \in \mathcal{G}'_k \text{ for all but finitely many } k \in A\\
&\iff \exists\, C \subseteq \mathbb{N},\, h[B] = Y \cap C,\, C \cap I_k \in \mathcal{G}_k \text{ for all but finitely many } k \in A\\
&\iff \exists\, C \subseteq \mathbb{N},\, h[B] = Y \cap C,\, C \cap I_k \in \mathcal{G}_k \text{ for all but finitely many } k \in \mathbb{N}\\
&\iff \exists\, C \in \mathcal{H}_{n+1},\, h[B] = Y \cap C\\
&\iff B \in \mathcal{H}_{n+1}\lceil Y.
\end{aligned}$$

So h is an isomorphism between $\mathcal{H}_{n+1}$ and its trace on Y.

(iii) Now let Y be any subset of $\mathbb{N}$ such that $\mathbb{N}\setminus Y \notin \mathcal{H}_{n+1}$; set $A = \{k : Y \cap I_k \notin \mathcal{G}_k\}$, $Y' = Y \cap \bigcup_{k\in A} I_k$ and $Y'' = Y \cap \bigcup_{k\in A}(I_k \setminus \{\min(I_k \cap Y)\})$. By (ii), $\mathcal{H}_{n+1}$ is isomorphic to its trace on Y'. But observe that $J = Y \setminus Y''$ is an infinite set such that $\mathbb{N} \setminus J$ belongs to $\mathcal{H}_{n+1}$, and that the same is true of $J' = Y' \setminus Y''$. So the traces of $\mathcal{H}_{n+1}$ on Y and Y' are isomorphic, since one is mapped to the other by any bijection between J and J'; and the trace of $\mathcal{H}_{n+1}$ on Y is therefore isomorphic to $\mathcal{H}_{n+1}$.

Thus $\mathcal{H}_{n+1}$ is homogeneous.

5F Lemma Suppose that $K \subseteq \mathbb{N}$. Then there is a filter $\mathcal{F}$ on $\mathbb{N}$ such that $\operatorname{ct}\mathcal{F} = \omega$ and

$$\begin{aligned}
K = \{n : n \in \mathbb{N} &\text{ and there is an } A \subseteq \mathbb{N} \text{ such that}\\
&\mathcal{F}\lceil A \text{ is a homogeneous filter and } \operatorname{ct}(\mathcal{F}\lceil A) = n+2\}.
\end{aligned}$$

proof (a) Let $\langle n(k)\rangle_{k\in\mathbb{N}}$ be an unbounded sequence in $\mathbb{N}$ such that

$$K = \{n : n \in \mathbb{N},\, \{k : k \in \mathbb{N},\, n(k) = n\} \text{ is infinite}\}.$$

Let $\langle I_k\rangle_{k\in\mathbb{N}}$ be a partition of $\mathbb{N}$ into infinite sets, and $g_k : \mathbb{N} \to I_k$ a bijection for each k; let $\langle \mathcal{H}_n\rangle_{n\in\mathbb{N}}$ be the sequence of filters constructed in Lemma 5E, and set $\mathcal{F} = \lim_{k\to\mathcal{F}_{\mathrm{Fr}}} g_k[[\mathcal{H}_{n(k)}]$. Since

$$\operatorname{ct} g_k[[\mathcal{H}_{n(k)}]] \le \operatorname{ct}\mathcal{H}_{n(k)} < \omega$$

for every k, $\operatorname{ct}\mathcal{F} \le \omega$.

(b) Suppose that $\langle k_i\rangle_{i\in\mathbb{N}}$ is a strictly increasing sequence in $\mathbb{N}$ such that $\langle n(k_i)\rangle_{i\in\mathbb{N}}$ is non-decreasing. Set $V = \bigcup_{i\in\mathbb{N}} I_{k_i}$. Then

$$\mathcal{F}\lceil V = \lim\nolimits_{i\to\mathcal{F}_{\mathrm{Fr}}} g_{k_i}[[\mathcal{H}_{n(k_i)}]]\lceil V$$

(1Fa), so $\operatorname{ct}(\mathcal{F}\lceil V) = \sup_{i\in\mathbb{N}} n(k_i) + 1$ (Lemma 5C, transferred to the partition $\langle I_{k_i}\rangle_{i\in\mathbb{N}}$ of A). In particular, since $\langle n(k)\rangle_{k\in\mathbb{N}}$ is unbounded, we can find a strictly increasing sequence $\langle k_i\rangle_{i\in\mathbb{N}}$ such that $\langle n(k_i)\rangle_{i\in\mathbb{N}}$ is also strictly increasing, in which case $\operatorname{ct}\mathcal{F} \ge \operatorname{ct}(\mathcal{F}\lceil V) \ge \omega$, and $\operatorname{ct}\mathcal{F}$ must be exactly ω.

(c) If $n \in K$, there is a strictly increasing sequence $\langle k_i \rangle_{i\in\mathbb{N}}$ such that $n(k_i) = n$ for every $i \in \mathbb{N}$. In this case, setting $V = \bigcup_{i\in\mathbb{N}} I_{k_i}$,

$$\mathcal{F}\lceil V = \lim_{i\to\mathcal{F}_{\mathrm{Fr}}} g_{k_i}[[\mathcal{H}_n]]\lceil V \cong \mathcal{H}_{n+1}$$

by the construction in the proof of 5E. By 5E, $\mathcal{F}\lceil V$ is homogeneous with countable-type level $n+2$.

(d) Now suppose that $A \subseteq \mathbb{N}$ is such that $\mathbb{N} \setminus A \notin \mathcal{F}$ and $\mathcal{F}\lceil A$ is homogeneous with finite countable-type level. Set

$$J = \{k : k \in \mathbb{N},\ \mathbb{N} \setminus A \notin g_k[[\mathcal{H}_{n(k)}]]\},$$

so that J is infinite; let $\langle k_i \rangle_{i\in\mathbb{N}}$ be a strictly increasing sequence in J such that $\langle n(k_i) \rangle_{i\in\mathbb{N}}$ is either strictly increasing or constant. For each $i \in \mathbb{N}$, set $\mathcal{G}_i = g_{k_i}[[\mathcal{H}_{n(k_i)}]]\lceil I_{k_i}$, so that $\mathcal{G}_i \cong \mathcal{H}_{n(k_i)}$ is homogeneous and $I_{k_i} \setminus A \notin \mathcal{G}_i$; let $h_i : I_{k_i} \to I_{k_i} \cap A$ be an isomorphism between $(I_{k_i}, \mathcal{G}_i)$ and $(I_{k_i} \cap A, \mathcal{G}_i\lceil I_{k_i} \cap A)$. Set $V = \bigcup_{i\in\mathbb{N}} I_{k_i}$, and define $h : V \to V \cap A$ by setting $h(j) = h_i(j)$ if $i \in \mathbb{N}$ and $j \in I_{k_i}$. Just as in part (c-ii) of the proof of 5E, h is an isomorphism between $\mathcal{F}\lceil V$ and $\mathcal{F}\lceil(V \cap A)$. But $\mathcal{F}\lceil(V \cap A)$ must be isomorphic to $\mathcal{F}\lceil A$, so $\mathcal{F}\lceil V$ also is.

(e) Looking back at (b), we see that because $\mathcal{F}\lceil V$ has finite countable-type level, $\langle n(k_i) \rangle_{i\in\mathbb{N}}$ must be bounded, and therefore constant, with value belonging to K. In the latter case, (c) tells us that

$$\mathrm{ct}(\mathcal{F}\lceil A) = \mathrm{ct}(\mathcal{F}\lceil V) = n + 2.$$

(f) So

$$K \subseteq \{n : n \in \mathbb{N} \text{ and there is an } A \subseteq \mathbb{N} \text{ such that } \mathcal{F}\lceil A \text{ is a homogeneous filter and } \mathrm{ct}(\mathcal{F}\lceil A) = n+2\}$$

(by (c))

$$\subseteq K$$

by (e), and we're done.

5G Theorem There are $\mathfrak{c}$ non-isomorphic filters of countable-type level ω.

proof For every $K \subseteq \mathbb{N}$ we have a corresponding filter $\mathcal{F}_K$ as constructed in 5F. But the formula

$$K = \{n : n \in \mathbb{N} \text{ and there is an } A \subseteq \mathbb{N} \text{ such that } \mathcal{F}_K\lceil A \text{ is a homogeneous filter and } \mathrm{ct}(\mathcal{F}_K\lceil A) = n+2\}$$

tells us that the $\mathcal{F}_K$ are all non-isomorphic, so there are at least $\mathfrak{c}$ isomorphism classes of these filters. In the other direction, of course, $\mathcal{P}\mathbb{N}$ has only $\mathfrak{c}$ Borel sets, so there can be at most $\mathfrak{c}$ filters on $\mathbb{N}$ of countable type.

6 The coinitiality of filters of countable type

6A Theorem If $\mathcal{F}$ is a filter on $\mathbb{N}$ of countable type, $\mathrm{ci}\,\mathcal{F} \le \mathrm{cf}\,\mathbb{N}^{\mathbb{N}}$.

proof (a) Let $\mathfrak{F}$ be the set of filters $\mathcal{F}$ on $\mathbb{N}$ such that there are a function $f : \mathcal{F} \to \mathbb{N}^{\mathbb{N}}$ and a family $\langle A_{\alpha\beta} \rangle_{\alpha,\beta\in\mathbb{N}^{\mathbb{N}}}$ in $\mathcal{F}$ such that

$A_{\alpha\gamma} \subseteq A_{\alpha\beta}$ whenever α, β, $\gamma \in \mathbb{N}^{\mathbb{N}}$ and $\beta \le \gamma$,
whenever $A \in \mathcal{F}$ and $f(A) \le \alpha$ in $\mathbb{N}^{\mathbb{N}}$, there is a $\beta \in \mathbb{N}^{\mathbb{N}}$ such that $A_{\alpha\beta} \subseteq A$.

(b) Principal ultrafilters belong to $\mathfrak{F}$. **P** If $\mathcal{F}$ is generated by $\{n\}$, set $f(A) = \mathbf{0}$ for every $A \in \mathcal{F}$ and $A_{\alpha\beta} = \{n\}$ for α, $\beta \in \mathbb{N}^{\mathbb{N}}$. **Q**

(c) If $\langle \mathcal{F}_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{F}$, then $\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ belongs to $\mathfrak{F}$. **P** For each $n \in \mathbb{N}$, let f_n, $\langle A^{(n)}_{\alpha\beta} \rangle_{\alpha,\beta\in\mathbb{N}^{\mathbb{N}}}$ witness that $\mathcal{F}_n \in \mathfrak{F}$. If $A \in \mathcal{F}$, set $f(A) = f_n(A)$ where $n = \min\{i : A \in \mathcal{F}_i\}$. If α, $\beta \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$, define $<n>^\frown\beta \in \mathbb{N}^{\mathbb{N}}$ by setting

$$(<n>^\frown\beta)(i) = n \text{ if } i = 0$$
$$= \beta(i-1) \text{ if } i > 0,$$

and set

$$A_{\alpha,<n>^\frown\beta} = \bigcap_{k\le n} A^{(k)}_{\alpha\beta} \in \mathfrak{F}.$$

If m, $n \in \mathbb{N}$ and β, $\gamma \in \mathbb{N}^{\mathbb{N}}$ are such that $<m>^\frown\beta \le <n>^\frown\gamma$, then $m \le n$ and $\beta \le \gamma$, so

$$A_{\alpha,<n>^\frown\gamma} = \bigcap_{k\le n} A^{(k)}_{\alpha\gamma} \subseteq \bigcap_{k\le n} A^{(k)}_{\alpha\beta} \subseteq \bigcap_{k\le m} A^{(k)}_{\alpha\beta} = A_{\alpha,<m>^\frown\beta}.$$

If $A \in \mathcal{F}$ and $\alpha \in \mathbb{N}^{\mathbb{N}}$ are such that $f(A) \le \alpha$, then there is an $n \in \mathbb{N}$ such that $A \in \mathcal{F}_n$ and $f_n(A) \le \alpha$. Let $\beta \in \mathbb{N}$ be such that $A^{(n)}_{\alpha\beta} \subseteq A$; then

$$A_{\alpha,<n>^\frown\beta} \subseteq A^{(n)}_{\alpha\beta} \subseteq A.$$

So f and $\langle A_{\alpha\beta}\rangle_{\alpha,\beta\in\mathbb{N}^{\mathbb{N}}}$ witness that $\mathcal{F} \in \mathfrak{F}$. **Q**

(d) If $\langle \mathcal{F}_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathfrak{F}$, then $\mathcal{F} = \bigcap_{n\in\mathbb{N}} \mathcal{F}_n$ belongs to $\mathfrak{F}$. **P** Let f_n, $\langle A^{(n)}_{\alpha\beta}\rangle_{\alpha,\beta\in\mathbb{N}^{\mathbb{N}}}$ witness that $\mathcal{F}_n \in \mathfrak{F}$. Define $f : \mathcal{F} \to \mathbb{N}^{\mathbb{N}}$ by setting

$$f(A)(i) = \max_{n\le i} f_n(A)(i)$$

for $A \in \mathcal{F}$ and $i \in \mathbb{N}$. For $\alpha \in \mathbb{N}^{\mathbb{N}}$ and $k \in \mathbb{N}$ set $(\alpha \vee k)(i) = \max(\alpha(i), k)$ for every $i \in \mathbb{N}$. For α, $\gamma \in \mathbb{N}^{\mathbb{N}}$ and sequences $\langle\beta_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{N}^{\mathbb{N}}$, set

$$C(\alpha, \gamma, \langle\beta_n\rangle_{n\in\mathbb{N}}) = \bigcup_{n\in\mathbb{N}} \bigcap_{k\le\gamma(n)} A_{\alpha\vee k,\beta_n} \in \mathcal{F}.$$

If α, γ, $\gamma' \in \mathbb{N}^{\mathbb{N}}$ and two sequences $\langle\beta_n\rangle_{n\in\mathbb{N}}$, $\langle\beta'_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{N}^{\mathbb{N}}$ are such that $\gamma \le \gamma'$ and $\beta_n \le \beta'_n$ for every n, then

$$C(\alpha, \gamma', \langle\beta'_n\rangle_{n\in\mathbb{N}}) = \bigcup_{n\in\mathbb{N}} \bigcap_{k\le\gamma'(n)} A_{\alpha\vee k,\beta'_n} \subseteq \bigcup_{n\in\mathbb{N}} \bigcap_{k\le\gamma(n)} A_{\alpha\vee k,\beta'_n}$$
$$\subseteq \bigcup_{n\in\mathbb{N}} \bigcap_{k\le\gamma(n)} A_{\alpha\vee k,\beta_n} = C(\alpha, \gamma, \langle\beta_n\rangle_{n\in\mathbb{N}}).$$

If $A \in \mathcal{F}$ and $\alpha \in \mathbb{N}^{\mathbb{N}}$ are such that $f(A) \le \alpha$, then for each $n \in \mathbb{N}$ we have a $\gamma(n) \in \mathbb{N}$ such that $f_n(A) \le \alpha \vee \gamma(n)$, and now there is a $\beta_n \in \mathbb{N}^{\mathbb{N}}$ such that $A \supseteq A_{\alpha\vee\gamma(n),\beta_n}$; so that

$$A \supseteq \bigcup_{n\in\mathbb{N}} A_{\alpha\vee\gamma(n),\beta_n} \supseteq \bigcup_{n\in\mathbb{N}} \bigcap_{k\le\gamma(n)} A_{\alpha\vee k,\beta_n} = C(\alpha, \gamma, \langle\beta_n\rangle_{n\in\mathbb{N}}).$$

Since $\mathbb{N}^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ is isomorphic, as partially ordered set, to $\mathbb{N}^{\mathbb{N}}$, this means that f and C, suitably re-coded, witness that $\mathcal{F} \in \mathfrak{F}$.

(e) Thus every filter on $\mathbb{N}$ of countable type belongs to $\mathfrak{F}$. On the other hand, every filter in $\mathfrak{F}$ has coinitiality at most $\mathfrak{d} = \operatorname{cf} \mathbb{N}^{\mathbb{N}}$. **P** Let $D \subseteq \mathbb{N}^{\mathbb{N}}$ be a cofinal set of cardinal $\mathfrak{d}$. If $\mathcal{F} \in \mathfrak{F}$, let f and $\langle A_{\alpha\beta}\rangle_{\alpha,\beta\in\mathbb{N}^{\mathbb{N}}}$ witness this. If $A \in \mathcal{F}$, there is an $\alpha \in D$ such that $f(A) \le \alpha$; now there are a $\beta \in \mathbb{N}^{\mathbb{N}}$ such that $A \supseteq A_{\alpha\beta}$ and a $\gamma \in D$ such that $\beta \le \gamma$, in which case $A \supseteq A_{\alpha\gamma}$. So $\{A_{\alpha\gamma} : \alpha, \gamma \in D\}$ is coinitial with $\mathcal{F}$ and witnesses that $\operatorname{ci} \mathcal{F} \le \mathfrak{d}$. **Q**

Accordingly every filter on $\mathbb{N}$ of countable type has coinitiality at most $\mathfrak{d}$, as claimed.

6B Remark In the notation of Fremlin 08?, §522, the sequential composition $(\mathbb{N}^{\mathbb{N}}, \le, \mathbb{N}^{\mathbb{N}}) \ltimes (\mathbb{N}^{\mathbb{N}}, \le, \mathbb{N}^{\mathbb{N}})$ is the supported relation $(\mathbb{N}^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}^{\mathbb{N}}}, R, \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}})$, where R is the relation

$$\{((\alpha, h), (\beta, \gamma)) : \alpha,\ \beta,\ \gamma \in \mathbb{N}^{\mathbb{N}},\ h \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}^{\mathbb{N}}},\ \alpha \le \beta \text{ and } h(\beta) \le \gamma\}.$$

Suppose that $\mathcal{F}$ is a filter in the family $\mathfrak{F}$ of the proof of 6A, and $\mathcal{I}$ is the corresponding ideal $\{\mathbb{N}\setminus A : A \in \mathcal{F}\}$. Let f, $\langle A_{\beta\gamma}\rangle_{\beta,\gamma\in\mathbb{N}^{\mathbb{N}}}$ witness that $\mathcal{F} \in \mathfrak{F}$. For $E \in \mathcal{I}$, choose $h_E \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}^{\mathbb{N}}}$ such that $\mathbb{N}\setminus E \supseteq A_{\beta,h_E(\beta)}$ whenever $f(\mathbb{N} \setminus E) \le \beta \in \mathbb{N}^{\mathbb{N}}$. Define $\phi : \mathcal{I} \to \mathbb{N}^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}^{\mathbb{N}}}$ and $\psi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathcal{I}$ by setting

$$\phi(E) = (f(\mathbb{N} \setminus E), h_E), \quad \psi(\beta, \gamma) = \mathbb{N} \setminus A_{\beta\gamma}$$

for $E \in \mathcal{I}$ and β, $\gamma \in \mathbb{N}^{\mathbb{N}}$. Now suppose that $E \in \mathcal{I}$ and β, $\gamma \in \mathbb{N}^{\mathbb{N}}$ are such that $(\phi(E), (\beta, \gamma)) \in R$. Then $f(\mathbb{N} \setminus E) \leq \beta$ and $h_E(\beta) \leq \gamma$, so

$$\mathbb{N} \setminus E \supseteq A_{\beta, h_E(\beta)} \supseteq A_{\beta\gamma}$$

and $E \subseteq \psi(\beta, \gamma)$. But this means that (ϕ, ψ) is a Galois-Tukey connection from $(\mathcal{I}, \subseteq, \mathcal{I})$ to $(\mathbb{N}^{\mathbb{N}}, \leq, \mathbb{N}^{\mathbb{N}}) \ltimes (\mathbb{N}^{\mathbb{N}}, \leq, \mathbb{N}^{\mathbb{N}})$. This implies in particular that

$$\begin{aligned} \operatorname{ci} \mathcal{F} = \operatorname{cf} \mathcal{I} = \operatorname{cov}(\mathcal{I}, \subseteq, \mathcal{I}) &\leq \operatorname{cov}((\mathbb{N}^{\mathbb{N}}, \leq, \mathbb{N}^{\mathbb{N}}) \ltimes (\mathbb{N}^{\mathbb{N}}, \leq, \mathbb{N}^{\mathbb{N}})) \\ &= \operatorname{cov}(\mathbb{N}^{\mathbb{N}}, \leq, \mathbb{N}^{\mathbb{N}}) \cdot \operatorname{cov}(\mathbb{N}^{\mathbb{N}}, \leq, \mathbb{N}^{\mathbb{N}}) \end{aligned}$$

(the cardinal product)

$$= \operatorname{cf} \mathbb{N}^{\mathbb{N}} \cdot \operatorname{cf} \mathbb{N}^{\mathbb{N}} = \operatorname{cf} \mathbb{N}^{\mathbb{N}}.$$

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