Version of 11.12.06

Sums of measures

D.H.Fremlin

University of Essex, Colchester, England

1 General sums

1A Definition If $\langle\mu_i\rangle_{i\in I}$ is a family of measures on a set X, their **sum** is $\mu:\bigcap_{i\in I}\operatorname{dom}\mu_i\to[0,\infty]$ defined by setting $\mu E=\sum_{i\in I}\mu_i E$ if $E\in\bigcap_{i\in I}\operatorname{dom}\mu_i$.

1B Proposition μ is a measure.

1C Proposition Suppose that every μ_i is complete.

(a) μ is complete and $\mathcal{N}(\mu)=\bigcap_{i\in I}\mathcal{N}(\mu_i)$.

(b) Let f be a function from a subset of X to $[-\infty,\infty]$. Then $\int f d\mu$ is defined in $[-\infty,\infty]$ iff $\int f d\mu_i$ is defined in $[-\infty,\infty]$ for every i and one of $\sum_{i\in I}f^+d\mu_i$, $\sum_{i\in I}f^-d\mu_i$ is finite, and in this case $\int f d\mu=\sum_{i\in I}\int f d\mu_i$.

proof Write $\Sigma_i=\operatorname{dom}\mu_i$, $\Sigma=\bigcap_{i\in I}\Sigma_i=\operatorname{dom}\mu$.

(b)(i) Note first that under either hypothesis, $\operatorname{dom}f$ is conegligible both for μ and for every μ_i, so that if we extend f to X by giving it the value 0 on $X\setminus\operatorname{dom}f$ then neither $\int f d\mu$ nor $\sum_{i\in I}\int f d\mu_i$ is affected. So let us assume from now on that f is defined everywhere on X. Again, either hypothesis ensures that f is Σ-measurable.

(ii) Suppose that f is non-negative. For $n\in\mathbb{N}$ set $f_n(x)=\sum_{k=1}^{4^n}2^{-n}\chi\{x:f(x)\ge 2^{-n}k\}$, so that $\langle f_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum f. Now

$$\begin{aligned}\int f_n d\mu&=\sum_{k=1}^{4^n}2^{-n}\mu\{x:f(x)\ge 2^{-n}k\}\\&=\sum_{k=1}^{4^n}\sum_{i\in I}2^{-n}\mu_i\{x:f(x)\ge 2^{-n}k\}=\sum_{i\in I}\int f_n d\mu_i\end{aligned}$$

for every n, so

$$\begin{aligned}\int f d\mu&=\sup_{n\in\mathbb{N}}\int f_n d\mu=\sup_{n\in\mathbb{N}}\sup_{J\in[I]^{<\omega}}\sum_{i\in J}\int f_n d\mu_i\\&=\sup_{J\in[I]^{<\omega}}\lim_{n\to\infty}\sum_{i\in J}\int f_n d\mu_i=\sup_{J\in[I]^{<\omega}}\sum_{i\in J}\lim_{n\to\infty}\int f_n d\mu_i=\sum_{i\in I}\int f d\mu_i.\end{aligned}$$

(iii) Generally,

$$\begin{aligned}&\int f d\mu \text{ is defined in } [\infty,\infty]\\&\qquad\iff \int f^+ d\mu \text{ and } \int f^- d\mu \text{ are defined and at most one is infinite}\\&\qquad\iff \sum_{i\in I}\int f^+ d\mu_i \text{ and } \sum_{i\in I}\int f^- d\mu_i \text{ are defined and at most one is infinite}\\&\qquad\iff \int f d\mu_i \text{ is defined for every } i \text{ and at most one of } \sum_{i\in I}\int f^+ d\mu_i,\\&\qquad\qquad\qquad\sum_{i\in I}\int f^- d\mu_i \text{ is infinite,}\end{aligned}$$

and in this case

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu = \sum_{i\in I} \int f^+ d\mu_i - \sum_{i\in I} \int f d\mu_i = \sum_{i\in I} \int f d\mu_i.$$

1D Proposition Let (Y, T, ν) be a strictly localizable measure space, X a set and $\langle\mu_i\rangle_{i\in I}$ a countable family of measures on X such that $\mu = \sum_{i\in I} \mu_i$ is totally finite. Let $\mu_i \times \nu$, $\mu \times \nu$ be the c.l.d. product measures on $X \times Y$. Then $\sum_{i\in I} \mu_i \times \nu$ extends $\mu \times \nu$.

proof (a) Suppose to begin with that ν is totally finite. Set $\Sigma_i = \operatorname{dom}\mu_i$ for each i, and $\Sigma = \bigcap_{i\in I} \Sigma_i = \operatorname{dom}\mu$; write λ for $\mu \times \nu$ and λ_i for $\mu_i \times \nu$ for each i. Then $\Sigma\widehat{\otimes}\mathrm{T} \subseteq \operatorname{dom}\lambda_i$ for every i. If $V \in \Sigma\widehat{\otimes}\mathrm{T}$, then

$$\lambda V = \int \mu V^{-1}[\{y\}]\nu(dy) = \sum_{i\in I} \int \mu_i V^{-1}[\{y\}]\nu(dy) = \sum_{i\in I} \lambda_i V$$

(using 1Cb). If $W \in \operatorname{dom}\lambda$, there are V, $V' \in \Sigma\widehat{\otimes}\mathrm{T}$ such that $V \subseteq W \subseteq V'$ and $\lambda(V' \setminus V) = 0$, so $\lambda_i(V' \setminus V)$ for every i, W is measured by every λ_i, and $(\sum_{i\in I} \lambda_i)W = \sum_{i\in I} \lambda_i V = \lambda W$.

(b) In general, apply (a) to each element of a decomposition of Y.

1E Proposition Let $\langle\mu_i\rangle_{i\in I}$ be a family of measures on a set X, and $f : X \to Y$ a function. Then the image measure $(\Sigma_{i\in I}\mu_i)f^{-1}$ on Y is equal to $\sum_{i\in I}(\mu_i f^{-1})$.

1F Measure algebras Let $\langle\mu_i\rangle_{i\in I}$ be a family of measures on a set X, with sum μ, and for each $i \in I$ let $(\mathfrak{A}_i, \bar\mu_i)$ be the measure algebra of μ_i; let $(\mathfrak{B}, \bar\nu)$ be the simple product of $\langle(\mathfrak{A}_i, \bar\mu_i)\rangle_{i\in I}$, and $(\mathfrak{A}, \bar\mu)$ the measure algebra of μ. For $E \in \operatorname{dom}\mu$ let $E^\bullet$ be its equivalence class in $\mathfrak{A}$ and for each $i \in I$ let E_i° be its equivalence class in $\mathfrak{A}_i$. Then we have a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ defined by saying that $\pi E^\bullet = \langle E_i^\circ\rangle_{i\in I}$ for every $E \in \operatorname{dom}\mu$.

It follows that if I is countable, every μ_i is σ-finite, and μ is semi-finite, then μ is σ-finite, because $\mathfrak{B}$ and $\mathfrak{A}$ are ccc.

1G Topological measures: Proposition Let X be a topological space and $\langle\mu_i\rangle_{i\in I}$ a family of measures on X with sum μ.

(a) If every μ_i is a topological measure, then μ is a topological measure.

(b) If every μ_i is τ-additive, then μ is a τ-additive topological measure.

proof (a) Immediate from the definition.

(b) Let $\mathcal{G}$ be an upwards-directed family of open sets measured by μ with union H, also measured by μ. Then $\mu_i H = \sup_{G\in\mathcal{G}} \mu_i G$ for every $i \in I$; as $\mathcal{G}$ is upwards-directed, it follows that $\mu H = \sup_{G\in\mathcal{G}} \mu G$.

1H Indefinite-integral measures: Proposition Let (X, Σ, μ) be a measure space and $\langle f_i\rangle_{i\in I}$ a family of non-negative μ-virtually measurable real-valued functions with conegligible domains such that $f = \sum_{i\in I} f_i$ is μ-virtually measurable and finite almost everywhere. Let $\langle\nu_i\rangle_{i\in I}$, ν be the associated indefinite-integral measures (Fremlin 01, §234); set $\nu' = \sum_{i\in I} \nu_i$.

(a) If F is measured by ν then it is measured by ν' and $\nu' F \le \nu F$.

(b) If I is countable then $\nu = \nu'$.

proof Set $G = \{x : f(x) > 0\}$ and $G_i = \{x : f_i(x) > 0\}$ for $i \in I$,

(a) $G_i \cap \operatorname{dom} f \subseteq G$ for every i, so $G_i \setminus G$ is always negligible. Now, writing $\hat\Sigma$ for the domain of the completion of μ,

$$\operatorname{dom}\nu = \{E : E \subseteq X,\, E \cap G \in \hat\Sigma\} \subseteq \{E : E \subseteq X,\, E \cap G_i \in \hat\Sigma\} = \operatorname{dom}\nu_i$$

(Fremlin 01, 234Da). This is true for every $i \in I$ so $\operatorname{dom}\nu \subseteq \operatorname{dom}\nu'$. Next, if $F \in \operatorname{dom}\nu$ and $J \subseteq I$ is finite,

$$\sum_{i\in J} \nu_i F = \sum_{i\in J} \int_F f_i = \int_F \sum_{i\in J} f_i \le \int_F f = \nu F.$$

Taking the supremum over $J \in [I]^{<\omega}$, $\nu' F \le \nu F$.

(b) If I is countable then $G\triangle\bigcup_{i\in I}G_i$ is negligible, so $\operatorname{dom}\nu=\bigcap_{i\in I}\operatorname{dom}\nu_i=\operatorname{dom}\nu'$. If $\nu'F$ is finite, then B.Levi's theorem tells us that $\int_F f=\sum_{i\in I}\int_F f_i$ (see 123Xa), that is, that $\nu F=\nu'F$; so $\nu=\nu'$.

1I Examples (a) Consider $X=\{0,1\}^2$, Σ_1 the algebra of sets determined by the first coordinate, Σ_2 the algebra of sets determined by the second coordinate; suppose that $\mu_1X=\mu_1\{(1,0),(1,1)\}=1$, $\mu_2X=\mu_2\{(0,1),(1,1)\}=1$. Then $\mu_1+\mu_2$ has domain $\{\emptyset,X\}$ while $\hat\mu_1+\hat\mu_2$ measures $\{(0,0)\}$, so is not equal to $\widehat{\mu_1+\mu_2}$.

(b) Let $X=[0,1]$; let μ_1 be counting measure and μ_2 Lebesgue measure. Then $\mu_1+\mu_2$ is the restriction of counting measure to the domain of μ_2, so is not localizable or locally determined, even though both μ_1 and μ_2 are strictly localizable, and both are inner regular with respect to $\Sigma_1\cap\Sigma_2$.

(c) Consider $X=\mathbb{N}^2$, Σ_1 the algebra of sets determined by the first coordinate, Σ_2 the algebra of sets determined by the second coordinate; suppose that $\mu_1(\{n\}\times\mathbb{N})=\mu_2(\mathbb{N}\times\{n\})=1$ for every n. Then $\mu_1+\mu_2$ has domain $\{\emptyset,X\}$ so is purely infinite, while μ_1 and μ_2 are both complete and σ-finite.

(d) On $[0,1]^2$ let ν_1 be the restriction of Lebesgue measure to sets determined by the first coordinate, and ν_2 the restriction of Lebesgue measure to sets determined by the second coordinate. Let μ be Lebesgue measure on $[0,1]$, and $\nu=\hat\nu_1+\hat\nu_2$. Every member of $\mathcal{N}(\nu)=\mathcal{N}(\nu_1)\cap\mathcal{N}(\nu_2)$ is included in a set of the form $E\times E$ where $\mu E=0$, and if $F\in\operatorname{dom}\nu$, then either F or its complement belongs to $\mathcal{N}(\nu)$. So if $W\in\operatorname{dom}(\mu\times\nu)$, there is an $F\in\mathcal{N}(\nu)$ such that for μ-almost every $x\in[0,1]$, either $W[\{x\}]$ or its complement is included in F.

Consequently $\{(x,x,x):x\in[0,1]\}$ is not measured by $\mu\times\nu$; but it is negligible for $(\mu\times\nu_1)+(\mu\times\nu_2)$.

2 Locally coherent sums

2A Definition A family $\langle\mu_i\rangle_{i\in I}$ of measures on a set X is **locally coherent** if every μ_i is inner regular with respect to $\{E:\mu E$ is defined and finite$\}$, where $\mu=\sum_{i\in I}\mu_i$.

Note that if I is finite, then $\langle\mu_i\rangle_{i\in I}$ is locally coherent iff every μ_i is inner regular with respect to $\{E:\mu_jE$ is defined and finite for every $j\in I\}$.

2B Proposition Let $\langle\mu_i\rangle_{i\in I}$ be a locally coherent family of measures on a set X, with sum μ.

(a) If $\mathcal{K}$ is a family of sets such that every μ_i is inner regular with respect to $\mathcal{K}$, $\mathcal{K}$ is closed under finite unions and the intersection of a non-increasing sequence in $\mathcal{K}$ always belongs to $\mathcal{K}$, then μ is inner regular with respect to $\mathcal{K}$.

(b) μ is semi-finite.

(c) For each i, let $\tilde\mu_i$ be the c.l.d. version of μ_i. Then $\langle\tilde\mu_i\rangle_{i\in I}$ is locally coherent and $\sum_{i\in I}\tilde\mu_i$ is the c.l.d. version of μ.

(d) If every μ_i is complete and locally determined, so is μ.

(e) Let (Y,T,ν) be another measure space. Write $\lambda=\mu\times\nu$, $\lambda_i=\mu_i\times\nu$ for the product measures on $X\times Y$. Then $\langle\lambda_i\rangle_{i\in I}$ is locally coherent and $\lambda=\sum_{i\in I}\lambda_i$.

proof Set $\Sigma_i=\operatorname{dom}\mu_i$ for each $i\in I$, and $\Sigma=\operatorname{dom}\mu=\bigcap_{i\in I}\Sigma_i$; write Σ^f for $\{E:E\in\Sigma,\ \mu E<\infty\}$.

(a) By Fremlin 03, 412Ac, every μ_i is inner regular with respect to $\mathcal{K}\cap\Sigma^f$. If $E\in\operatorname{dom}\mu$ and $0\le\gamma<\mu E$ there is a finite set $J\subseteq I$ such that $\sum_{i\in J}\mu_iE>\gamma$. Now we can find a family $\langle K_i\rangle_{i\in J}$ in $\mathcal{K}\cap\Sigma^f$ such that $K_i\subseteq E$ for every i and $\sum_{i\in J}\mu_iK_i\ge\gamma$. Set $K=\bigcup_{i\in J}K_i$; then $K\in\mathcal{K}\cap\Sigma^f$, $K\subseteq E$ and $\mu K\ge\gamma$. As E and γ are arbitrary, μ is inner regular with respect to $\mathcal{K}$.

(b) Apply (a) with $\mathcal{K}=\Sigma^f$.

(c)(i) Every μ_i is inner regular with respect to Σ^f, so every $\tilde\mu_i$ is inner regular with respect to Σ^f (Fremlin 03, 412Ha); *a fortiori*, every $\tilde\mu_i$ is inner regular with respect to $\{E:\sum_{i\in I}\tilde\mu_iE$ is defined and finite$\}$.

(ii) Let $\tilde\mu$ be the c.l.d. version of μ, and suppose that $\tilde\mu$ measures E. Then for any $F\in\Sigma^f$ we have E_0, $E_1\in\Sigma^f$ such that $E_0\subseteq E\cap F\subseteq E_1\subseteq F$ and $\mu(E_1\setminus E_0)=0$. In this case, $\mu_i(E_1\setminus E_0)=0$ and $\tilde\mu_i(E\cap F)$ is defined and equal to μ_iE_0, for every $i\in I$. Adding, $\mu'(E\cap F)$ is defined and equal to $\mu E_0=\tilde\mu(E\cap F)$.

Generally, for $E \in \operatorname{dom}\tilde{\mu}$ and $i \in I$, $E \cap F$ is measured by $\tilde{\mu}_i$ for every $F \in \Sigma^f$; as $\tilde{\mu}_i$ is inner regular with respect to Σ^f, $\tilde{\mu}_i$ measures E (Fremlin 03, 412Ja). So $\mu' E$ is defined and

$$\mu' E = \sum_{i \in I} \sup_{F \in \Sigma^f} \tilde{\mu}_i(E \cap F) = \sup_{F \in \Sigma^f} \sum_{i \in I} \tilde{\mu}_i(E \cap F)$$
$$= \sup_{F \in \Sigma^f} \tilde{\mu}(E \cap F) = \mu E.$$

Thus μ' extends $\tilde{\mu}$.

(iii) In the other direction, if $E \in \operatorname{dom}\mu'$ and $F \in \Sigma^f$, then $J = \{i : i \in I,\ \mu_i F > 0\}$ is countable. For each $i \in J$, we have G_i, $H_i \in \Sigma^f$ such that $G_i \subseteq E \cap F \subseteq H_i \subseteq F$ and $\mu_i(G_i \setminus H_i) = 0$. (The point here is that $\Sigma^f \cap \mathcal{P}F$ is a σ-algebra of subsets of F.) Set $G = \bigcup_{i \in J} G_i$ and $H = F \cap \bigcap_{i \in J} H_i$; then $G \subseteq E \cap F \subseteq H$ and $\mu(H \setminus G) = 0$, so the completion of μ measures $E \cap F$. As F is arbitrary, $\tilde{\mu}$ measures E.

Thus $\mu' = \tilde{\mu}$.

(d) follows at once from (c).

(e)(i) For each $i \in I$, let μ_i' be the restriction of μ_i to Σ. Then $\mu = \sum_{i \in I} \mu_i'$; moreover, because the c.l.d. versions of μ_i' and μ_i agree on, and are inner regular with respect to, $\operatorname{dom}\mu_i'$, they are equal (Fremlin 03, 412L). So $\mu_i' \times \nu = \mu_i \times \nu$ (Fremlin 01, 251S). We may therefore suppose that $\Sigma_i = \Sigma$ for every i. Write $\tilde{\lambda}$ for $\sum_{i \in I} \lambda_i$ and $\tilde{\Lambda}$ for its domain.

(ii) Let Λ_0^f be the family of sets belonging to $\Sigma \widehat{\otimes} \mathrm{T}$ which are included in a set of the form $E \times F$ where $E \in \Sigma^f$ and $F \in \mathrm{T}^f$, that is, $F \in \mathrm{T}$ and $\nu F < \infty$. Then every λ_i is inner regular with respect to Λ_0^f. **P** If $\lambda_i W > \gamma$ there are $E \in \Sigma_i$, $F \in \mathrm{T}^f$ such that $\lambda_i E < \infty$ and $\lambda_i(W \cap (E \times F)) > \gamma$; now there is an $E' \in \Sigma^f$ such that $E' \subseteq E$ and $\mu_i(E \setminus E) \cdot \nu F < \lambda_i(W \cap (E \times F)) - \gamma$, in which case $\lambda_i(W \cap (E' \times F)) > \gamma$. Next, there is a $V \in \Sigma \widehat{\otimes} \mathrm{T}$ such that $V \subseteq W \cap (E' \times F)$ and $\lambda_i V \geq \gamma$. In this case, $V \in \Lambda_0^f$; as W is arbitrary, λ_i is inner regular with respect to Λ_0^f. **Q**

If $E \in \Sigma^f$ and $F \in \mathrm{T}^f$ then

$$\tilde{\lambda}(E \times F) = \textstyle\sum_{i \in I} \lambda_i(E \times F) = \sum_{i \in I} \mu_i E \cdot \nu F = \mu E \cdot \nu F$$

is finite. So Λ_0^f is included in $\tilde{\Lambda}^f = \{W : W \in \tilde{\Lambda},\ \tilde{\lambda} W < \infty\}$, and every λ_i is inner regular with respect to $\tilde{\Lambda}^f$. Accordingly $\langle \lambda_i \rangle_{i \in I}$ is locally coherent.

(iii) $\tilde{\lambda}$ and λ are both complete locally determined measures, inner regular with respect to Λ_0^f. They agree on sets of the form $E \times F$ where $E \in \Sigma^f$ and $F \in \mathrm{T}^f$; by the Monotone Class Theorem (Fremlin 00, 136C), they agree on Λ_0^f; so they are equal (Fremlin 03, 412L).

2C (Quasi-)Radon measures: Proposition (a) Let X be a topological space and $\langle \mu_i \rangle_{i \in I}$ a family of quasi-Radon measures on X. If $\mu = \sum_{i \in I} \mu_i$ is effectively locally finite, it is a quasi-Radon measure.

(b) Let X be a Hausdorff space and $\langle \mu_i \rangle_{i \in I}$ a family of Radon measures on X. If $\mu = \sum_{i \in I} \mu_i$ is locally finite, it is a Radon measure.

proof (a) μ is a τ-additive topological measure by 1Ca and 1G. Let $\mathcal{G}$ be the family of open sets of finite measure for μ, and $\mathcal{K}$ the family of closed sets included in members of $\mathcal{G}$. Then μ gives finite measure to every member of $\mathcal{K}$. Also every μ_i is inner regular with respect to $\mathcal{K}$. **P** Suppose that $\mu_i E > \gamma \geq 0$. Since $\mu_i(E \setminus \bigcup \mathcal{G}) = \mu(X \setminus \bigcup \mathcal{G}) = 0$, and G is upwards-directed, $\sup_{G \in \mathcal{G}} \mu_i(E \cap G) = \mu(E \cap \bigcup \mathcal{G})$ (Fremlin 03, 414Ea) and there is a $G \in \mathcal{G}$ such that $\mu_i(E \cap G) > \gamma$. Now there is a closed set $K \subseteq E \cap G$ such that $\mu_i K \geq \gamma$, and $K \in \mathcal{K}$. **Q**

By 2B, μ is complete and locally determined and inner regular with respect to $\mathcal{K}$, therefore inner regular with respect to the closed sets, and is a quasi-Radon measure.

(b) Because μ is a locally finite topological measure (1Ga), every compact set has finite μ-measure; by 2Ba, μ is inner regular with respect to the compact sets, therefore effectively locally finite. By (a), it is a locally finite quasi-Radon measure, so is a Radon measure.

2D (Quasi-)Radon product measures: Proposition (a) Let X be a topological space and $\langle \mu_i \rangle_{i \in I}$ a family of quasi-Radon measures on X with sum μ; suppose that μ is effectively locally finite, therefore a quasi-Radon measure. Let $(Y,\mathfrak{S},\mathrm{T},\nu)$ be a quasi-Radon measure space and write $\tilde\lambda$, $\tilde\lambda_i$ for the quasi-Radon product measures of μ and ν, μ_i and ν respectively (Fremlin 03, 417G). Then $\tilde\lambda=\sum_{i\in I}\tilde\lambda_i$.

(b) Let X be a Hausdorff space and $\langle\mu_i\rangle_{i\in I}$ a family of Radon measures on X with sum μ; suppose that μ is locally finite, therefore a Radon measure. Let $(Y,\mathfrak{S},\mathrm{T},\nu)$ be a Radon measure space and write $\tilde\lambda$, $\tilde\lambda_i$ for the Radon product measures of μ and ν, μ_i and ν respectively. Then $\tilde\lambda=\sum_{i\in I}\tilde\lambda_i$.

proof (a) Set $\tilde\lambda'=\sum_{i\in I}\tilde\lambda_i$. Then $\tilde\lambda'$ is effectively locally finite. **P** Suppose that $\tilde\lambda'W>0$. Then there is a $j\in I$ such that $\tilde\lambda_jW>0$. Let $E\subseteq X$, $F\subseteq Y$ be such that μ_jE and νF are finite and $\tilde\lambda_j(W\cap(E\times F))>0$. Now μ_j is inner regular with respect to the sets included in open sets of finite μ-measure (see the proof of 2Ca), so there is an open set $G\subseteq X$ such that $\mu G<\infty$ and $\mu_j(E\setminus G)\cdot\nu F<\tilde\lambda_j(W\cap(E\times F))$. Accordingly $\tilde\lambda_j(W\cap(G\times F))>0$. similarly, there is an open set $H\subseteq Y$ such that $\nu H<\infty$ and $\tilde\lambda_j(W\cap(G\times H))>0$. Now

$$\tilde\lambda'(G\times H)=\textstyle\sum_{i\in I}\tilde\lambda_i(G\times H)=\sum_{i\in I}\mu_iG\cdot\nu H=\mu G\cdot\nu H$$

is finite, while $\tilde\lambda'(W\cap(G\times H))\ge\tilde\lambda_j(W\cap(G\times H))>0$. **Q**

So $\tilde\lambda'$ is a quasi-Radon measure, like $\tilde\lambda$. Since they agree on products of open sets, they must be equal (Fremlin 03, 415H(v)).

(b) follows at once, since of course μ is effectively locally finite.

2E Subspaces Let $\langle\mu_i\rangle_{i\in I}$ be a locally coherent family of complete locally determined measures on a set X, with sum μ.

(a) Let μ_i^*, μ^* be the outer measures defined from μ_i, μ. Then $\mu^*A=\sum_{i\in I}\mu_i^*A$ for every $A\subseteq X$.

(b) Let A be a subset of X, and $(\mu_i)_A$, μ_A the subspace measures on A defined from μ_i, μ. Then $\sum_{i\in I}(\mu_i)_A$ extends μ_A.

proof (a)(i) There is an $E\in\operatorname{dom}\mu$ such that $A\subseteq E$ and

$$\mu^*A=\mu E=\textstyle\sum_{i\in I}\mu_iE\ge\sum_{i\in I}\mu_i^*A.$$

(ii) Set $\Sigma^f=\{E:\mu E\text{ is defined and finite}\}$. Then $\mu^*(A\cap E)\le\sum_{i\in I}\mu_i^*A$. **P** Set $J=\{i:\mu_iE>0\}$, so that J is countable. For each $i\in J$ let F_i be a μ_i-measurable envelope of $A\cap E$, and $E_i\in\Sigma^f$ a subset of $E\setminus F_i$ such that $\mu_iE_i=\mu_i(E\setminus F_i)$; set $F=E\setminus\bigcup_{i\in J}E_i$. Then $A\cap E\subseteq F$ and

$$\mu^*(A\cap A)\le\mu F=\textstyle\sum_{i\in J}\mu_i(E\setminus E_i)=\sum_{i\in J}\mu_i^*(A\cap E)\le\sum_{i\in I}\mu_i^*A.\ \mathbf{Q}$$

By 2Bd and Fremlin 03, 412Jc, $\mu^*A\le\sum_{i\in I}\mu_i^*A$ and we have equality.

(b) Set $\mu'_A=\sum_{i\in I}(\mu_i)_A$. If $B\subseteq A$ is measured by μ_A, it is of the form $A\cap E$ where E is measured by μ, in which case $A\cap E$ is measured by $(\mu_i)_A$ for every i, and therefore by μ'_A. Now

$$\mu_AB=\mu^*B=\textstyle\sum_{i\in I}\mu_i^*B=\sum_{i\in I}(\mu_i)_AB=\mu'_AB.$$

2F Example In Fremlin 01, 216D, we have an example of a locally coherent family of complete locally determined probability measures on $X=\omega_2\times\omega_2$ such that the subspace measure on $A=\omega_1\times\omega_1$ of the sum is not the sum of the subspace measures.

References

Fremlin D.H. [00] *Measure Theory, Vol. 1: The Irreducible Minimum.* Torres Fremlin, 2000.
Fremlin D.H. [01] *Measure Theory, Vol. 2: Broad Foundations.* Torres Fremlin, 2001.
Fremlin D.H. [03] *Measure Theory, Vol. 4: Topological Measure Spaces.* Torres Fremlin, 2003.