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Alternating functionals

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1 Definition Let P be a partially ordered set such that $p \vee q = \sup\{p, q\}$ is defined for all p, $q \in P$. A function $\gamma : P \to [0, \infty]$ is **alternating of all orders** if, writing P^f for $\{p : \gamma(p) < \infty\}$,

(i) $r \in P^f$ whenever p, $q \in P^f$ and $r \leq p \vee q$,

(ii) $\sum_{J \subseteq I} (-1)^{\#(J)} \gamma(p \vee \sup_{i \in J} p_i) \leq 0$ for every $p \in P^f$ and every non-empty finite family $\langle p_i \rangle_{i \in I}$ in P^f, interpreting $p \vee \sup_{i \in \emptyset} p_i$ as p.

2 Remarks In this case, $\gamma(p) \leq \gamma(q)$ whenever $p \leq q$ (if $\gamma(q) < \infty$, $\gamma(p) - \gamma(p \vee q) \leq 0$). If P is a lattice, then γ is submodular, that is, $\gamma(p \wedge q) + \gamma(p \vee q) \leq \gamma(p) + \gamma(q)$ for all p, $q \in P$ (if $\gamma(p)$ and $\gamma(q)$ are both finite, then

$$\gamma(p \wedge q) - \gamma((p \wedge q) \vee p) - \gamma((p \wedge q) \vee p) + \gamma((p \wedge q) \vee p \vee q) \leq 0).$$

3 Theorem Let P be a partially ordered set, with a least element p_*, such that $p \vee q = \sup\{p, q\}$ is defined for all p, $q \in P$. Let $\gamma : P \to [0, \infty[$ be a function. Then the following are equiveridical.

(i) There are a set X, a ring $\mathcal{E}$ of subsets of X, an additive functional $\nu : \mathcal{E} \to [0, \infty[$ and a function $f : P \to \mathcal{E}$ such that $f(p \vee q) = f(p) \cup f(q)$ and $\gamma(p) = \nu f(p)$ for all p, $q \in P$, and $f(p_*) = \emptyset$.

(ii) Whenever X is a set and $f : P \to \mathcal{P}X$ is a function such that $f(p_*) = \emptyset$ and $f(p \vee q) = f(p) \cup f(q)$ for all p, $q \in P$ and

whenever $Q \subseteq P$ is finite and $q \in Q$, there is an $x \in X$ such that, for $p \in Q$, $x \in f(p) \iff p \not\leq q$,

then there is a non-negative additive functional ν, with domain the ring of subsets of X generated by $\{f(p) : p \in P\}$, such that $\nu f(p) = \gamma(p)$ for every $p \in P$.

(iii) There are a strictly localizable measure space (X, Σ, μ) and a function $f : P \to \Sigma$ such that $f(p \vee q) = f(p) \cup f(q)$ and $\mu f(p) = \gamma(p)$ for all p, $q \in P$, and $f(p_*) = \emptyset$.

(iv) There are a localizable measure algebra $(\mathfrak{A}, \bar{\mu})$ and a function $h : P \to \mathfrak{A}$ such that $h(p \vee q) = h(p) \cup h(q)$ and $\gamma(p) = \bar{\mu} h(p)$ for all p, $q \in P$, and $h(p_*) = 0$.

(v) $\gamma(p_*) = 0$ and γ is alternating of all orders.

proof I start with the easy bits.

(iii)⇒(i) is trivial.

(iv)⇒(iii) Take X the Stone space of $\mathfrak{A}$ with its canonical measure, and $f(p) = \widehat{h(p)}$ for $p \in P$.

(i)⇒(iv) Let $\mathcal{J}$ be the ideal $\{E : E \in \mathcal{E},\ \nu E = 0\}$ and $\mathfrak{B}$ the quotient Boolean ring $\mathcal{E}/\mathcal{J}$; define $\bar{\nu} : \mathfrak{B} \to [0, \infty[$ by setting $\nu E^\bullet = \nu E$ for $E \in \mathcal{E}$. Give $\mathfrak{B}$ the Hausdorff uniformity defined by pseudometrics $(b, c) \mapsto \bar{\nu}(e \cap (b \triangle c))$ for $e \in \mathfrak{B}$. Let $\mathfrak{A}$ be the completion of $\mathfrak{B}$ for this uniformity. The ring operations are uniformly continuous, so $\mathfrak{A}$ has a Boolean ring structure. Any upwards-directed set in $\mathfrak{A}$ gives rise to a Cauchy filter, so has a supremum; thus $\mathfrak{A}$ is a Dedekind complete Boolean algebra. If $e \in \mathfrak{B}$ then $b \mapsto \bar{\nu}(b \cap e)$ is uniformly continuous, so has a unique uniformly continuous extension $\bar{\mu}_e$ on $\mathfrak{A}$; set $\bar{\mu} a = \sup_{e \in \mathfrak{B}} \bar{\mu}_e a$ for $a \in \mathfrak{A}$; check that $(\mathfrak{A}, \bar{\mu})$ is a semi-finite measure algebra. Now define $h : P \to \mathfrak{A}$ by setting $h(p) = f(p)^\bullet$ for $p \in P$.

(ii)⇒(i) We have only to check that there is a pair X, f satisfying the hypotheses of (ii); and if we take $X = P$ and $f(p) = \{q : p \not\leq q\}$, this does the job.

(i)⇒(v) Assume (i). Take $p \in P$ and a non-empty finite family $\langle p_i \rangle_{i \in I}$ in P. For $x \in X \setminus f(p)$ set $K_x = \{i : x \in f(p_i)\}$ and consider

$$\begin{aligned}
\sum_{J\subseteq I}(-1)^{\#(J)}\chi(f(p)\cup\bigcup_{i\in J}f(p_i))(x) &= \sum_{J\subseteq I,J\cap K_x\neq\emptyset}(-1)^{\#(J)}\\
&= \sum_{\emptyset\neq J\subseteq K_x}\ \sum_{J'\subseteq I\setminus K_x}(-1)^{\#(J)}(-1)^{\#(J')}\\
&= 0 \text{ if } I\setminus K_x\neq\emptyset\\
&= 0 \text{ if } K_x=\emptyset\\
&= -1 \text{ if } I=K_x\neq\emptyset.
\end{aligned}$$

So

$$\begin{aligned}
\sum_{J\subseteq I}(-1)^{\#(J)}\gamma(p\vee\sup_{i\in J}p_i) &= \sum_{J\subseteq I}(-1)^{\#(J)}\nu f(p\vee\sup_{i\in J}p_i)\\
&= \sum_{J\subseteq I}(-1)^{\#(J)}\nu(f(p)\cup\bigcup_{i\in J}f(p_i))\\
&= \int\sum_{J\subseteq I}(-1)^{\#(J)}\chi(f(p)\cup\bigcup_{i\in J}f(p_i))d\nu
\end{aligned}$$

(where $\int d\nu$ is the additive functional associated with ν)

$$\begin{aligned}
&= \sum_{J\subseteq I}(-1)^{\#(J)}\nu f(p)+\int_{X\setminus f(p)}\sum_{J\subseteq I}(-1)^{\#(J)}\chi(\bigcup_{i\in J}f(p_i))d\nu\\
&= -\nu\{x:x\in X\setminus f(p),\ K_x=I\}\\
&= -\nu(\bigcap_{i\in I}f(p_i)\setminus f(p))\leq 0.
\end{aligned}$$

So (v) is true.

(v)⇒(ii) Now for the main part of the proof. Assume (v).

(a) For $p\in P$ set $f(p)=\{q:q\in P,\ p\not\leq q\}$; then $f(p\vee q)=f(p)\cup f(q)$ for all p, $q\in P$, and $f(p_*)=\emptyset$.

(b) Write $\mathcal{Q}$ for the family of finite subsets of P containing p_* and closed under $\vee$. For $Q\in\mathcal{Q}$ and $L\subseteq Q$ set

$$\begin{aligned}
\delta_Q(L) &= -\sum_{J\subseteq Q\setminus L}(-1)^{\#(J)}\gamma(\sup(L\cup J))\\
&= -\sum_{J\subseteq Q\setminus L}(-1)^{\#(J)}\gamma(\sup L\vee\sup_{p\in J}p)\geq 0
\end{aligned}$$

at least when $L\neq Q$. If $L\subseteq Q$ is such that there is a $p\in Q\setminus L$ such that $p\leq\sup L$, then

$$\begin{aligned}
\delta_Q(L) &= -\sum_{J\subseteq Q\setminus(L\cup\{p\})}(-1)^{\#(J)}\big(\gamma(\sup(L\cup J))-\gamma(\sup(L\cup J\cup\{p\})\big)\\
&= 0.
\end{aligned}$$

For $p\in Q\in\mathcal{Q}$ set $L_{Qp}=\{q:q\in Q,\ q\leq p\}$; then if $L\subseteq Q$ and $\delta_Q(L)\neq 0$, we must have $p\not\leq\sup L$ for every $p\in Q\setminus L$, and $L=L_{Q,\sup L}$. Also, of course, $p\mapsto L_{Qp}:Q\to\mathcal{P}Q$ is injective.

(c) For $Q\in\mathcal{Q}$ set $Q^-=Q\setminus\{\max Q\}$, and let $\mathcal{E}_Q$ be the ring of subsets of X generated by $\{f(p):p\in Q\}$. For each $q\in Q$ let $x_{Qq}\in X$ be such that $\{p:p\in Q,\ x_{Qq}\in f(p)\}=\{p:p\in Q,\ p\not\leq q\}$.

Let μ_Q be the point-supported measure on X defined by saying that

$$\mu_QE=\textstyle\sum_{p\in Q^-,x_{Qp}\in E}\delta_Q(L_{Qp})$$

for $E\subseteq X$. (If $p\in Q^-$ then $L_{Qp}\neq Q$ and $\delta_Q(L_{Qp})\geq 0$.) Then if $p\in Q$ we have

$$\mu_Q f(p) = \sum_{\substack{q\in Q^- \\ x_{Qq}\in f(p)}} \delta_Q(L_{Qq}) = \sum_{\substack{L\subseteq Q \\ p\notin L}} \delta_Q(L)$$

(because, for $q \in Q$, $p \notin L_{Qq}$ iff $p \not\leq q$ iff $q \in Q^-$ and $x_{Qq} \in f(p)$)

$$\begin{aligned}
&= -\sum_{L\subseteq Q\setminus\{p\}} \sum_{J\subseteq Q\setminus L} (-1)^{\#(J)}\gamma(\sup(L\cup J)) \\
&= -\sum_{L\subseteq Q} \sum_{p\in J\subseteq Q\setminus L} (-1)^{\#(J)}\gamma(\sup(L\cup J)) \\
&\qquad - \sum_{L\subseteq Q\setminus\{p\}} \sum_{J\subseteq Q\setminus(L\cup\{p\})} (-1)^{\#(J)}\gamma(\sup(L\cup J)) \\
&= -\sum_{M\subseteq Q} \sum_{p\in J\subseteq M} (-1)^{\#(J)}\gamma(\sup M) \\
&\qquad - \sum_{M\subseteq Q\setminus\{p\}} \sum_{J\subseteq M} (-1)^{\#(J)}\gamma(\sup M) \\
&= \gamma(p) - \gamma(p_*)
\end{aligned}$$

(because $\sum_{p\in J\subseteq M}(-1)^{\#(J)}$ is zero unless $M = \{p\}$, while $\sum_{J\subseteq M}(-1)^{\#(J)}$ is zero unless $M = \emptyset$)

$$= \gamma(p).$$

(d) If Q, $R \in \mathcal{Q}$ and $Q \subseteq R$, then $\mathcal{E}_Q \subseteq \mathcal{E}_R$; also μ_Q agrees with μ_R on $\mathcal{E}_Q$. **P** This is just the Monotone Class Theorem, upside down. Set $Y = f(\max Q)$, so that $\mathcal{E}_Q$ is a finite subalgebra of $\mathcal{P}Y$. Set

$$\mathcal{I} = \{Y \setminus f(p) : p \in Q\}.$$

If $p \in Q$, then

$$\begin{aligned}
\mu_Q(Y \setminus f(p)) &= \mu_Q f(\max Q) - \mu_Q f(p) = \gamma(\max Q) - \gamma(f(p)) \\
&= \mu_R f(\max Q) - \mu_R f(p) = \mu_R(Y \setminus f(p)).
\end{aligned}$$

Since $\mathcal{I}$ is closed under $\cap$ and contains Y, this is enough to show that μ_Q and μ_R agree on the σ-algebra of subsets of Y generated by $\mathcal{I}$, which is $\mathcal{E}_Q$. **Q**

(e) Since any finite subset of P generates a member of $\mathcal{Q}$, $\mathcal{Q}$ is upwards-directed and has union P; so $\langle\mathcal{E}_Q\rangle_{Q\in\mathcal{Q}}$ is upwards-directed and has union the ring $\mathcal{E}$ generated by $\{f(p) : p \in P\}$. By (d), we have an additive functional $\nu : \mathcal{E} \to [0,\infty[$ such that ν agrees with μ_Q on $\mathcal{E}_Q$ for every $Q \in \mathcal{Q}$, in which case $\nu f(p)$ must be $\gamma(p)$ for every $p \in P$. Thus (ii) is true.

This completes the proof.

4 Definition Following Fremlin 03, I say that a **capacity** on a topological space X is a function $\gamma : \mathcal{P}X \to [0,\infty]$ such that

$\gamma(A) \leq \gamma(B)$ whenever $A \subseteq B \subseteq X$;

$\lim_{n\to\infty} \gamma(A_n) = \gamma(A)$ whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of subsets of X with union A;

$\gamma(K) = \inf\{\gamma(G) : G \supseteq K \text{ is open}\}$ for every compact set $K \subseteq X$.

I will say that γ is **biregular** if moreover

$\gamma(G) = \sup\{\gamma(K) : K \subseteq G \text{ is compact}\}$ for every open $G \subseteq X$,

$\gamma(A) = \inf\{\gamma(G) : G \supseteq A \text{ is open}\}$ for every $A \subseteq X$.

5 Theorem Let X be a locally compact Hausdorff space and $\gamma : \mathcal{P}X \to [0,\infty]$ a biregular capacity on X which is alternating of all orders, zero on $\emptyset$ and finite on compact sets. Let $\mathcal{C}^+$ be the family of non-empty closed subsets of X with the Fell topology. Then there is a unique Radon measure μ on $\mathcal{C}^+$ such that $\gamma(A) = \mu^*\{C : C \in \mathcal{C}^+,\ A \cap C \neq \emptyset\}$ whenever $A \subseteq X$ is covered by a sequence of compact sets.

proof (a) Recall that the set $\mathcal{C}$ of all closed subsets of X is compact and Hausdorff in the Fell topology (FREMLIN 03, 4A2T), so that $\mathcal{C}^+$ is locally compact. Define $f : \mathcal{P}X \to \mathcal{P}\mathcal{C}^+$ by setting $f(A) = \{C : A \cap C \neq \emptyset\}$; let $\mathcal{I}$ be the ideal of subsets of X of finite capacity. Because $\mathcal{C}^+$ contains all non-empty finite sets, $f{\restriction}\mathcal{I}$ satisfies the hypothesis in condition (ii) of Theorem 3, and there is a non-negative additive functional ν defined on the ring generated by $\{f(A) : A \in \mathcal{I}\}$ such that $\gamma(A) = \nu f(A)$ for every $A \in \mathcal{I}$.

(b) Let T be the ring of subsets of $\mathcal{C}^+$ generated by $\{f(A) : A \in \mathcal{I},\ A \text{ is either compact or open}\}$. Then $\nu{\restriction}\mathrm{T}$ is inner regular with respect to the compact sets, in the sense that $\nu E = \sup\{\nu F : F \in \mathrm{T} \text{ is compact}, F \subseteq E\}$ for every $E \in \mathrm{T}$.

P Note first that the set

$$\mathcal{E} = \{E : E \in \mathrm{T},\ \nu E = \sup\{\nu F : F \in \mathrm{T} \text{ is compact}, F \subseteq E\}\}$$

is a sublattice of T.

We need to examine sets of the form $E = f(A) \setminus f(B)$ where A, $B \in \mathcal{I}$ are either compact or open. If $\epsilon > 0$, we have a compact $K \subseteq A$ and an open $G \supseteq B$ such that $\gamma(A) - \gamma(K) \leq \epsilon$ and $\gamma(G) - \gamma(B) \leq \epsilon$, because ν is biregular. Now $f(K)$ is a closed subset of $\mathcal{C}$ included in $\mathcal{C}^+$, so is compact, while $f(G)$ is open; thus $F = f(K) \setminus f(G)$ is compact; of course $F \in \mathrm{T}$ and $F \subseteq E$. Now

$$\begin{aligned}\nu(f(A) \setminus f(B)) &\leq \nu F + \nu(f(A) \setminus f(K)) + \nu(f(G) \setminus f(B))\\ &= \nu F + \nu f(A) - \nu f(K) + \nu f(G) - \nu f(B)\\ &= \nu F + \gamma(A) - \gamma(K) + \gamma(G) - \gamma(B) \leq \nu F + 2\epsilon.\end{aligned}$$

As ϵ is arbitrary, $E \in \mathcal{E}$.

Next, any member of T is expressible as a finite union of finite intersections of sets of this kind, and therefore belongs to $\mathcal{E}$, as required. **Q**

(c) Let $\mathcal{K}$ be the family of compact subsets of $\mathcal{C}^+$. If $F \in \mathcal{K}$, then it is a closed subset of $\mathcal{C}$ not containing $\emptyset$, so there must be a compact set $K \subseteq X$ such that $F \subseteq f(K)$. Thus every member of $\mathcal{K}$ is covered by a member of T, and we have a functional $\phi : \mathcal{K} \to [0, \infty[$ defined by saying that $\phi F = \inf\{\nu E : F \subseteq E \in \mathrm{T}\}$ for $F \in \mathcal{K}$.

I seek to apply Theorem 413I of FREMLIN 03. Of course $\emptyset \in \mathcal{K}$ and $\mathcal{K}$ is closed under finite disjoint unions and countable intersections; moreover, if $\langle F_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection, one of the F_n must be empty, so $\inf_{n \in \mathbb{N}} \phi F_n = 0$. Now turn to condition (α) of the theorem:

$$\phi F_1 = \phi F_0 + \sup\{\phi F : F \in \mathcal{K},\ F \subseteq F_1 \setminus F_0\} \text{ whenever } F_0,\ F_1 \in \mathcal{K} \text{ and } F_0 \subseteq F_1.$$

(i) If F_0, $F \in \mathcal{K}$ are disjoint, then $\phi(F_0 \cup F) \geq \phi F_0 + \phi F$. **P** The topology $\mathfrak{S}$ of $\mathcal{C}^+$ is generated by sets of the form $f(G)$, where $G \subseteq X$ is open, and $\mathcal{C}^+ \setminus f(K)$, where $K \subseteq X$ is compact. Since every point of X is included in an open set of finite capacity (relatively compact sets have finite capacity), $\mathfrak{S}$ is generated by sets of the form $f(G) \setminus f(K)$ where G is an open set of finite capacity and K is compact. But these all belong to T. So $\mathfrak{S} \cap \mathrm{T}$ is a base for $\mathfrak{S}$ and disjoint compact sets in $\mathcal{C}^+$ can be separated by members of T; let $E_0 \in \mathrm{T}$ be such that $F_0 \subseteq E_0 \subseteq \mathcal{C}^+ \setminus F$. Now if $E \in \mathrm{T}$ and $E \supseteq F_0 \cup F$,

$$\nu E = \nu(E \cap E_0) + \nu(E \setminus E_0) \geq \phi F_0 + \phi F;$$

as E is arbitrary, $\phi(F_0 \cup F) \geq \phi F_0 + \phi F$. **Q**

(ii) If F_0, $F_1 \in \mathcal{K}$, $F_0 \subseteq F_1$ and $\epsilon > 0$, there is an $F \in \mathcal{K}$ such that $F \subseteq F_1 \setminus F_0$ and $\phi F_1 \leq \phi F_0 + \phi F + 3\epsilon$. **P** Let E_0, $E_1 \in \mathrm{T}$ be such that $F_0 \subseteq E_0$, $F_1 \subseteq E_1$ and $\nu E_0 \leq \phi F_0 + \epsilon$. By (b), there is an $F' \in \mathcal{K} \cap \mathrm{T}$ such that $F' \subseteq E_1 \setminus E_0$ and $\nu F' \geq \nu(E_1 \setminus E_0) - \epsilon$. Set $F = F' \cap F_1$. Then $F \in \mathcal{K}$ and $F \subseteq F_1 \setminus F_0$. Let $E \in \mathrm{T}$ be such that $F \subseteq E$ and $\nu E \leq \phi F + \epsilon$. Then $F_1 \subseteq E_0 \cup E \cup ((E_1 \setminus E_0) \setminus F')$, so

$$\phi F_1 \leq \nu E_0 + \nu E + \nu(E_1 \setminus E_0) - \nu F' \leq \phi F_0 + \phi F + 3\epsilon,$$

as required. **Q**

Putting this together with (i), the final condition of the theorem is satisfied.

(d) We therefore have a complete locally determined measure μ on $\mathcal{C}^+$ extending ϕ and inner regular with respect to $\mathcal{K}$. For $E \subseteq \mathcal{C}^+$, μ measures E iff μ measures $E \cap F$ for every $F \in \mathcal{K}$ (FREMLIN 03, 412J);

so μ measures all closed subsets of $\mathcal{C}^+$, therefore all Borel sets. Of course μ is inner regular with respect to the compact sets. If $C \in \mathcal{C}^+$, there is a relatively compact open subset G of X meeting C (because X is locally compact), and now $f(G)$ is an open set containing C, while $f(\overline{G})$ is compact and

$$\mu f(G) \le \mu f(\overline{G}) = \phi f(\overline{G}) = \nu f(\overline{G}) = \gamma(\overline{G}) < \infty.$$

Thus μ is locally finite and is a Radon measure.

(e) I noted in (d) that

$$\mu f(K) = \phi f(K) = \nu f(K) = \gamma(K)$$

for compact $K \subseteq X$. If $G \subseteq X$ is open, we also have $\mu f(G) = \gamma(G)$. **P** If $\gamma(G)$ is finite, then

$$\begin{aligned}\mu f(G) &= \sup\{\mu F : F \subseteq f(G) \text{ is compact}\}\\ &= \sup\{\phi F : F \subseteq f(G) \text{ is compact}\} \le \nu f(G) = \gamma(G)\\ &= \sup\{\gamma(K) : K \subseteq G \text{ is compact}\}\\ &= \sup\{\mu f(K) : K \subseteq G \text{ is compact}\} \le \mu f(G).\end{aligned}$$

If $\gamma(G)$ is infinite, then

$$\begin{aligned}\mu f(G) \le \gamma(G) &= \sup\{\gamma(K) : K \subseteq G \text{ is compact}\}\\ &= \sup\{\mu f(K) : K \subseteq G \text{ is compact}\} \le \mu f(G)\end{aligned}$$

and again we have equality. **Q**

(f) We need an elementary remark about Fell topologies on locally compact spaces. If $F \subseteq \mathcal{C}$ is closed, then $\bigcup F$ is a closed subset of X. **P** Take $x \in X \setminus \bigcup F$, and consider $\{\mathcal{C} \setminus f(\overline{G}) : G$ is a relatively compact neighbourhood of $x\}$. This is an open cover of F, and is upwards-directed, so there is a neighbourhood G of X such that $F \cap f(\overline{G}) = \emptyset$. Now $\bigcup F$ does not meet G, so $x \notin \overline{\bigcup F}$. **Q**

(g) Now take any relatively compact $A \subseteq X$. We have

$$\mu^* f(A) \le \inf\{\mu f(G) : G \supseteq A \text{ is open}\} = \inf\{\gamma G : G \supseteq A \text{ is open}\} = \gamma A$$

by the other half of the biregularity condition. **?** Suppose, if possible, that $\mu^* f(A) < \gamma(A)$. Let $G \supseteq A$ be a relatively compact open set. Then there is a compact $F \subseteq f(G) \setminus f(A)$ such that $\mu F > \mu f(G) - \gamma(A)$. Set $L = \bigcup_{C \in F} C$; then L is closed and disjoint from A, $G \setminus L$ is open, $A \subseteq G \setminus L$ and $f(G \setminus L)$ is disjoint from F. But this means that

$$\gamma(A) \le \gamma(G \setminus L) = \mu f(G \setminus L) \le \mu f(G) - \mu F < \gamma(A),$$

which is absurd. **X** So $\mu^* f(A) = \gamma(A)$.

If $A \subseteq X$ is covered by a sequence of compact sets, then it is expressible as the union of a non-decreasing sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ of relatively compact sets; in this case $\langle f(A_n) \rangle_{n \in \mathbb{N}}$ is non-decreasing and has union $f(A)$, so that

$$\mu^* f(A) = \sup_{n \in \mathbb{N}} \mu^* f(A_n) = \sup_{n \in \mathbb{N}} \gamma(A_n) = \gamma(A).$$

Thus μ has all the properties declared.

(h) To see that μ is unique, consider the base $\mathcal{V}$ for the topology of $\mathcal{C}^+$ consisting of sets of the form $\bigcap_{i \in I} f(G_i) \setminus f(L)$ where $\langle G_i \rangle_{i \in I}$ is a non-empty finite family of relatively compact open subsets of X and L is a compact subset of X. The condition that μ must satisfy determines its values on any set of the form $f(G) \cup f(L) = f(G \cup L)$ where G is open and relatively compact and L is compact, and therefore determines its values on $\mathcal{V}$. By Fremlin 03, 415H, μ is fixed by these.

6 Proposition Let X be a regular Hausdorff space and $\mathcal{K}$ the family of compact subsets of X. Let γ be a capacity on X such that every compact set has finite capacity and $\gamma \restriction \mathcal{K}$ is alternating of all orders. Then there is a biregular capacity γ' on X, alternating of all orders, and agreeing with γ on $\mathcal{K}$.

proof (a) For open sets $G \subseteq X$ set $\gamma_1(G) = \sup\{\gamma(K) : K \in \mathcal{K},\ K \subseteq G\}$. Then γ_1 is alternating of all orders. **P** The basic fact is that if $G_0, \dots, G_n$ are open and $K \subseteq \bigcup_{i \le n} G_i$ is compact, then K is expressible as $\bigcup_{i\le n} K_i$ where K_i is a compact subset of G_i for each i; this is where we need to know that X is regular. Because γ is subadditive (see §2), it follows that γ_1 is subadditive and $\mathcal{G}=\{G:\gamma_1(G)<\infty\}$ is closed under $\cup$. Take $G\in\mathcal{G}$, a non-empty finite family $\langle G_i\rangle_{i\in I}$ in $\mathcal{G}$ and $\epsilon>0$. For each $J\subseteq I$ let $K_J\subseteq G\cup\bigcup_{i\in J}G_i$ be a compact set such that $\gamma(K_J)\ge\gamma_1(G\cup\bigcup_{i\in J}G_i)-\epsilon$; let K_{Ji}, for $i\in J$, be compact subsets of G_i, and K'_J a compact subset of G, such that $K_J=K'_J\cup\bigcup_{i\in J}K_{Ji}$. Set $\tilde K=\bigcup_{J\subseteq I}K'_J$, so that $\tilde K$ is a compact subset of G, and $\tilde K_i=\bigcup_{i\in J\subseteq I}K_{Ji}$, so that $\tilde K_i$ is a compact subset of G_i for each i; observe that if $J\subseteq I$ then

$$K_J=K'_J\cup\bigcup\nolimits_{i\in J}K_{Ji}\subseteq\tilde K\cup\bigcup\nolimits_{i\in J}\tilde K_i\subseteq G\cup\bigcup\nolimits_{i\in J}G_i,$$

so that

$$\gamma(\tilde K\cup\bigcup\nolimits_{i\in J}\tilde K_i)\le\gamma_1(G\cup\bigcup\nolimits_{i\in J}G_i\le\gamma(\tilde K\cup\bigcup\nolimits_{i\in J}\tilde K_i)+\epsilon.$$

Now

$$\sum\nolimits_{J\subseteq I}(-1)^{\#(J)}\gamma_1(G\cup\bigcup\nolimits_{i\in J}G_i)\le\sum\nolimits_{J\subseteq I}(-1)^{\#(J)}\gamma(\tilde K\cup\bigcup\nolimits_{i\in J}\tilde K_i)+2^{\#(I)}\epsilon\le 2^{\#(I)}\epsilon$$

because $\gamma\restriction\mathcal{K}$ is alternating of all orders. As G, $\langle G_i\rangle_{i\in I}$ and ϵ are arbitrary, γ_1 is alternating of all orders. **Q**

(b) Now set $\gamma'(A)=\inf\{\gamma_1(G):G\supseteq A\text{ is open}\}$ for $A\subseteq X$. Because γ_1 is subadditive, so is γ'. Because γ is a capacity, and $\gamma(K)\le\gamma_1(G)\le\gamma(G)$ whenever K is compact and $G\supseteq K$ is open, $\gamma'\restriction\mathcal{K}=\gamma\restriction\mathcal{K}$.

If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of sets with union A, $\gamma'(A_n)$ is finite for every n, and $\epsilon>0$, then for each $n\in\mathbb{N}$ choose $G_n\supseteq A_n$ such that $\gamma_1(G_n)\le\gamma'(A_n)+2^{-n}\epsilon$. Set $H_n=\bigcup_{i\le n}G_i$ for each n. Then $\gamma_1(H_n)\le\gamma'(A_n)+2\epsilon-2^{-n}\epsilon$ for every n. **P** Induce on n. For $n=0$ this is trivial. For the inductive step, note that $G_{n+1}\supseteq A_n$, so that

$$\gamma_1H_{n+1}\le\gamma_1H_n+\gamma_1G_{n+1}-\gamma_1(H_n\cap G_{n+1})$$

(because γ_1 is submodular)

$$\begin{aligned}&\le\gamma'(A_n)+2\epsilon-2^{-n}\epsilon+\gamma'(A_{n+1})+2^{-n-1}\epsilon-\gamma'(A_n)\\&=\gamma'(A_{n+1})+2\epsilon-2^{-n-1}\epsilon\end{aligned}$$

and the induction continues. **Q** So

$$\begin{aligned}\gamma'(\bigcup_{n\in\mathbb{N}}A_n)&\le\gamma_1(\bigcup_{n\in\mathbb{N}}G_n)=\gamma_1(\bigcup_{n\in\mathbb{N}}H_n)\\&=\lim_{n\to\infty}\gamma_1(H_n)\le\lim_{n\to\infty}\gamma'(A_n)+2\epsilon.\end{aligned}$$

Consequently γ' is a capacity.

(c) To see that γ' is alternating of all orders, we can copy the ideas of (a). First, it is subadditive, so $\mathcal{A}=\{A:\gamma'(A)<\infty\}$ is an ideal of subsets of X. Next, given $A\in\mathcal{A}$, a finite family $\langle A_i\rangle_{i\in I}$ in $\mathcal{A}$, and $\epsilon>0$, then for each $J\subseteq I$ let G_J be an open set including $A\cup\bigcup_{i\in J}A_i$ and such that $\gamma_1(G_J)\le\gamma'(A\cup\bigcup_{i\in J}A_i)+\epsilon$. Set $H=\bigcap_{J\subseteq I}G_J$, $H_i=\bigcap_{i\in J\subseteq I}G_J$ for $i\in I$; then

$$A\cup\bigcup\nolimits_{i\in J}A_i\subseteq H\cup\bigcup\nolimits_{i\in J}H_i\subseteq G_J,$$

$$\gamma'(A\cup\bigcup\nolimits_{i\in J}A_i)\le\gamma_1(H\cup\bigcup\nolimits_{i\in J}H_i)\le\gamma'(A\cup\bigcup\nolimits_{i\in J}A_i)+\epsilon$$

for every $J\subseteq I$. So

$$\begin{aligned}\sum_{J\subseteq I}(-1)^{\#(J)}\gamma'(A\cup\bigcup_{i\in J}A_i)&\le\sum_{J\subseteq I}(-1)^{\#(J)}\gamma_1(H\cup\bigcup_{i\in J}H_i)+2^{\#(I)}\epsilon\\&\le 2^{\#(I)}\epsilon.\end{aligned}$$

As A, $\langle A_i\rangle_{i\in I}$ and ϵ are arbitrary, γ' is alternating of all orders.

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References

Fremlin D.H. [03] *Measure Theory, Vol. 4: Topological Measure Spaces.* Torres Fremlin, 2003.