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# Tube outer measure

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## 1 Foundations

**1A Definitions** Let $r \geq 2$ be an integer.

**(a)** A **tube** in $\mathbb{R}^r$ is a set of the form $T = C + L$ where $L$ is a line. If $T$ is neither empty nor $\mathbb{R}^r$, the expression is unique if we demand that $L$ should be a line through 0 and $C$ a subset of the orthogonal complement $L^\perp$. In this case, write $\lambda_t T$ for the normalized $(r-1)$-dimensional Hausdorff outer measure $\nu^*_{r-1} C$ of $C$. (For completeness, say that $\lambda_t \emptyset = 0$ and $\lambda_t \mathbb{R}^r = \infty$.) Now let $\theta_t$ be the outer measure defined from $\lambda_t$, that is,

$$\theta_t A = \inf\{\sum_{n=0}^\infty \lambda_t T_n : \langle T_n\rangle_{n\in\mathbb{N}} \text{ is a sequence of tubes covering } A\}.$$

I will call this **tube outer measure** on $\mathbb{R}^r$.

**(b)** A **slice** in $\mathbb{R}^r$ is a set of the form $S = \{x : \alpha \leq (x|e) \leq \beta\}$ where $\alpha \leq \beta$ and $e$ is a unit vector; its **width** is then $\lambda_s S = \beta - \alpha$; let $\theta_s$ be the outer measure defined from $\lambda_s$; I will call it **slice outer measure** on $\mathbb{R}^r$.

**1B Lemma** We still get tube outer measure if, in the defining formula for $\theta_t$, we require all the tubes to be 'cylindrical tubes', that is, of the form $C + L$ where $C$ is a ball and $L$ is a line.

**proof** Fremlin 01, 261F.

**1C Corollary** (a) $\theta_s A \leq \theta_t A$ for every $A \subseteq \mathbb{R}^r$.
(b) If $r = 2$ then $\theta_s = \theta_t$.

**1D Lemma** (a) For any $A \subseteq \mathbb{R}^r$, $\nu^*_r A \leq \theta_t A \cdot \operatorname{diam} A$.
(b) Suppose that $T \subseteq \mathbb{R}^r$ is a cylindrical tube, with cross-sectional radius $\alpha > 0$, and that $B$ is a ball, of radius $\beta \geq \alpha$, with centre lying on the axis of $T$. Then $\theta_t(T \cap B) \geq \frac{\sqrt{\beta^2-\alpha^2}}{\beta}\lambda_t T$.
(c) $\lambda_t T = \theta_t T$ for any cylindrical tube $T$.

**proof (a)** If $T$ is any tube, $\nu^*_r(T \cap A) \leq \lambda_t T \cdot \operatorname{diam}(T \cap A)$.

**(b)** The diameter of $T \cap B$ is at most $2\beta$, so

$$\theta_t(T \cap B) \geq \frac{1}{2\beta}\nu_r(T \cap B) \geq \frac{\sqrt{\beta^2-\alpha^2}}{\beta}\lambda_t T.$$

**(c)** Let $\beta \to \infty$ in (b).

**1E Theorem** If $B \subseteq \mathbb{R}^r$ is a ball, $\theta_t B$ is the $(r-1)$-dimensional measure of the projection of $B$ onto any hyperplane.

**proof** It is enough to consider the case in which $B$ is the unit ball. Define $f : \operatorname{int} B \to [1, \infty[$ by setting $f(x) = \frac{1}{\sqrt{1-\|x\|^2}}$. Then the line integral of $f$ along any proper chord of $B$ is $\pi$. **P** If the length of the chord is $2a$, then the integral is $\int_{-a}^a \frac{1}{\sqrt{1-h^2-t^2}}dt$, where $h = \sqrt{1-a^2}$ is the distance from 0 to the midpoint of the chord. Now

$$\int_{-a}^{a}\frac{1}{\sqrt{1-h^2-t^2}}dt=\int_{-a}^{a}\frac{1}{\sqrt{a^2-t^2}}dt=\int_{-1}^{1}\frac{1}{\sqrt{a^2-a^2u^2}}a\,du$$
$$=\int_{-1}^{1}\frac{1}{\sqrt{1-u^2}}du=\int_{-\pi/2}^{\pi/2}\frac{1}{\cos\alpha}\cos\alpha\,d\alpha=\pi.\ \mathbf{Q}$$

So if $T$ is a cylindrical tube, then

$$\int_{T\cap B}f\,d\nu_r=\pi\nu_{r-1}E\le\pi\lambda_tT$$

where $E$ is the projection of $T\cap B$ onto a hyperplane orthogonal to the axis of $T$. Consequently

$$\theta_tB\ge\frac{1}{\pi}\int_Bf\,d\nu_r=\nu_{r-1}K$$

where $K$ is the projection of $B$ onto a hyperplane; but of course $\theta_tB\le\nu_{r-1}K$ since $B\subseteq T$ for the cylindrical tube $T$ with cross-section $K$. So we have equality.

**1F Proposition** (M.Csörnyei, L.Wisewell) If $A\subseteq\mathbb{R}^r$ has $\sigma$-finite $(r-1)$-dimensional Hausdorff measure, $\theta_tA=0$.

**proof (a)** Suppose to begin with that $A$ is $(r-1)$-rectifiable, that is, is the image of a bounded subset $D$ of $\mathbb{R}^{r-1}$ under a Lipschitz function $f$ (Federer 69, 3.2.14).

**(i)** Scaling $D$ up if necessary, we can suppose that $f$ is 1-Lipschitz. By Rademacher's theorem (Fremlin 01, 262Q), there is a $D'\subseteq D$ such that $D\setminus D'$ is $\nu_{r-1}$-negligible and $f\restriction D'$ is differentiable relative to its domain. (Here I am thinking of $\nu_{r-1}$ as Lebesgue measure on $\mathbb{R}^{r-1}$.) Let $\gamma\ge0$ be such that $\|x\|<\gamma$ for every $x\in D$. Let $\beta_{r-1}$, $\beta_{r-2}$ be the measures of the unit balls in $\mathbb{R}^{r-1}$, $\mathbb{R}^{r-2}$ respectively, counting $\beta_0$ as 1 if $r=2$. Let $G\supseteq D$ be a bounded open set, and fix $\epsilon>0$ for the time being.

**(ii)** Let $\mathcal{I}$ be the set of non-trivial closed balls $B\subseteq G$ such that $\theta_t(f[B\cap D'])\le\epsilon\nu_{r-1}B$. Then every point of $D'$ belongs to arbitrarily small members of $\mathcal{I}$. $\mathbf{P}$ Set $\eta=\frac{\beta_{r-1}\epsilon}{2\beta_{r-2}}$. If $x\in D'$, let $T:\mathbb{R}^{r-1}\to\mathbb{R}$ be the derivative of $f$ at $x$ relative to $D'$. Then there is a $\delta_0>0$ such that $B(x,\delta_0)\subseteq G$ and $\|f(y)-f(x)-T(y-x)\|\le\eta\|y-x\|$ whenever $y\in B(x,\delta_0)$. Let $H\subseteq\mathbb{R}^r$ be a hyperplane including $T[\mathbb{R}^{r-1}]$ and $L\subseteq H$ a line through 0; let $P$ be the orthogonal projection onto the hyperplane $L^\perp$. Take any $\delta\in\,]0,\delta_0]$ and consider $P[f[D'\cap B(x,\delta)]]$. We have

$$f[B(x,\delta)]\subseteq f(x)+((H+\eta\delta B(0,1))\cap B(0,\delta)),$$

so

$$P[f[B(x,\delta)]]\subseteq Pf(x)+((H'+B(0,\eta\delta))\cap B(0,\delta)),$$

where $H'=P[H]$ is a subspace of $L^\perp$ of dimension $r-2$. Now, identifying $L^\perp$ with $H'\times L'$, where $L'$ is a line,

$$(H'+B(0,\eta\delta))\cap B(0,\delta)\subseteq(H'\cap B(0,\delta))\times[-\eta\delta,\eta\delta],$$

so that

$$\theta_t(f[B(x,\delta)])\le\nu_{r-1}P[f[B(x,\delta)]]\le\nu_{r-1}((H'+B(0,\eta\delta))\cap B(0,\delta))$$
$$\le2\eta\delta\nu_{r-2}(H'\cap B(0,\delta))=2\eta\delta\beta_{r-2}\delta^{r-2}$$
$$=\epsilon\beta_{r-1}\delta^{r-1}=\epsilon\nu_{r-1}B(x,\delta),$$

and $B(x,\delta)\in\mathcal{I}$. This is true for every $\delta\le\delta_0$, so that $x$ belongs to arbitrarily small members of $\mathcal{I}$. $\mathbf{Q}$

**(iii)** By Vitali's theorem (Fremlin 02, 261B), there is a countable disjoint $\mathcal{I}_0\subseteq\mathcal{I}$ such that $\nu_{r-1}(D'\setminus\bigcup\mathcal{I}_0)=0$. Now

$$\theta_t(f[D'\cap\bigcup\mathcal{I}_0])\le\sum_{B\in\mathcal{I}_0}\theta_tf[D'\cap B]\le\sum_{B\in\mathcal{I}_0}\epsilon\nu_{r-1}B\le\epsilon\nu_{r-1}G.$$

On the other hand, $\nu_{r-1}[D\setminus\bigcup\mathcal{I}_0]=0$. Taking $L\subseteq\mathbb{R}^r$ to be any line through 0 and $P$ the orthogonal projection onto $L^\perp$,

$$\theta_t(f[D\setminus\bigcup\mathcal{I}_0])\le\nu_{r-1}(P[f[D\setminus\bigcup\mathcal{I}_0]])=0$$

because $Pf:D\to L^\perp$ is Lipschitz. So $\theta_t(A)=\theta_t(f[D])\le\epsilon\nu_{r-1}G$. As $\epsilon$ is arbitrary, $\theta_t(A)=0$, as claimed.

**(b)** Now suppose that $A$ has finite $(r-1)$-dimensional measure. Then there is an $(r-1)$-rectifiable set $C\subseteq A$ such that almost every projection of $A\setminus C$ onto a hyperplane is $\nu_{r-1}$-negligible (). Accordingly $\theta_tA\le\theta_tC+\theta_t(A\setminus C)=0$. It follows at once that a countable union of such sets is $\theta_t$-negligible.

**1G Bang's Theorem** (Bang 51) If $K\subseteq\mathbb{R}^r$ is a compact convex set, then $\theta_sK=\operatorname{mw}(K)$ where $\operatorname{mw}(K)=\min\{\lambda_sS:S$ is a slice including $K\}$ is the minimal width of $K$.

**proof** Of course $\theta_sK$ is less than or equal to the minimal width of $K$, so this proof is devoted to the reverse inequality.

**(a)** If $K\subseteq\mathbb{R}^r$ is a non-empty compact convex set and $e\in\mathbb{R}^r$ is a unit vector, then there are $a\in K$ and $\gamma\ge\operatorname{mw}(K)$ such that $a+\gamma e\in K$. **P** Set $\gamma=\sup\{\alpha:\alpha e\in K-K\}$; because $K-K$ is compact, $\gamma e\in K-K$ and there is an $a\in K$ such that $a+\gamma e\in K$. Take any $\gamma'>\gamma$. Then $\gamma'e$ does not belong to the compact convex set $K-K$, so there is a unit vector $e'\in\mathbb{R}^r$ such that

$$\gamma'\ge(\gamma'e|e')>\sup_{x,y\in K}(y-x|e)\ge\operatorname{mw}(K).$$

As $\gamma'$ is arbitrary, $\gamma\ge\operatorname{mw}(K)$, as required. **Q**

**(b)** If $K\subseteq\mathbb{R}^r$ is a compact convex set and $v\in\mathbb{R}^r$ then $\operatorname{mw}((K+v)\cap(K-v))\ge\operatorname{mw}(K)-2\|v\|$. **P** We need consider only the case $\operatorname{mw}(K)>2\|v\|$. Express $v$ as $\delta e$ where $e$ is a unit vector. By (a), there are $a\in K$ and $\gamma\ge\operatorname{mw}(K)$ such that $a+\gamma e\in K$. Set $z=a+\frac{\gamma}{2}e$ and consider $K'=z+\frac{\gamma-2\delta}{\gamma}(K-z)$. Then

$$\operatorname{mw}(K')=\frac{\gamma-2\delta}{\gamma}\operatorname{mw}(K)\ge\operatorname{mw}(K)-2\delta.$$

If $x\in K$ then

$$\begin{aligned}z+\frac{\gamma-2\delta}{\gamma}(x-z)+v&=\frac{\gamma-2\delta}{\gamma}x+\frac{2\delta}{\gamma}z+\delta e=\frac{\gamma-2\delta}{\gamma}x+\frac{2\delta}{\gamma}(z+\frac{\gamma}{2}e)\\&=\frac{\gamma-2\delta}{\gamma}x+\frac{2\delta}{\gamma}(a+\gamma e)\in K,\\z+\frac{\gamma-2\delta}{\gamma}(x-z)-v&=\frac{\gamma-2\delta}{\gamma}x+\frac{2\delta}{\gamma}z-\delta e=\frac{\gamma-2\delta}{\gamma}x+\frac{2\delta}{\gamma}(z-\frac{\gamma}{2}e)\\&=\frac{\gamma-2\delta}{\gamma}x+\frac{2\delta}{\gamma}a\in K;\end{aligned}$$

thus $y+v$ and $y-v$ belong to $K$ for every $y\in K'$, and $K'\subseteq(K+v)\cap(K-v)$. But this means that $\operatorname{mw}((K+v)\cap(K-v))$ must be at least $\operatorname{mw}(K')\ge\operatorname{mw}(K)-2\|v\|$. **Q**

**(c)** Let $K\subseteq\mathbb{R}^r$ be a compact convex set, and $\langle v_i\rangle_{i\in I}$ a finite family in $\mathbb{R}^r$. For $\boldsymbol{\epsilon}=\langle\epsilon_i\rangle_{i\in I}\in\{-1,1\}^I$ write $w_{\boldsymbol{\epsilon}}=\sum_{i\in I}\epsilon_iv_i$. Then

$$\operatorname{mw}(\bigcap_{\boldsymbol{\epsilon}\in\{-1,1\}^I}K+w_{\boldsymbol{\epsilon}})\ge\operatorname{mw}(K)-2\sum_{i\in I}\|v_i\|.$$

**P** The case $\#(I)=1$ is (b) above; now induce on $\#(I)$. **Q**

**(d) ?** Suppose, if possible, that $K\subseteq\mathbb{R}^r$ is a compact convex set such that $\theta_sK<\operatorname{mw}(K)$. Then we have a sequence $\langle S_n\rangle_{n\in\mathbb{N}}$ of slices, covering $K$, such that $\sum_{n=0}^\infty\lambda_sS_n<\operatorname{mw}(K)$. Widening each $S_n$ slightly if necessary, we can suppose that $K\subseteq\bigcup_{n\in\mathbb{N}}\operatorname{int}S_n$. Then $K\subseteq\bigcup_{i<n}\operatorname{int}S_i$ for some $n$. Express each $S_i$ as $\{x:|(x|e_i)-\gamma_i|\le\delta_i\}$ where $e_i$ is a unit vector and $\delta_i\ge0$. Set $v_i=\delta_ie_i$ for each $i$, and for $\boldsymbol{\epsilon}\in\{0,1\}^n$ define $w_{\boldsymbol{\epsilon}}$ as in (c).

Because $\operatorname{mw}(K)>\sum_{i=0}^{n-1}\lambda_sS_i=2\sum_{i=0}^n\|v_i\|$, (c) tells us that there is an $x\in\bigcap_{\boldsymbol{\epsilon}\in\{-1,1\}^n}K+w_{\boldsymbol{\epsilon}}$. Take $\boldsymbol{\epsilon}\in\{-1,1\}^n$ maximising the expression

$$2(x|w_{\boldsymbol{\epsilon}})-2\sum_{i=0}^{n-1}\epsilon_i\gamma_i\delta_i+\|w_{\boldsymbol{\epsilon}}\|^2.$$

Then $x+w_{\boldsymbol{\epsilon}}\in K$ so there is a $j<n$ such that $x+w_{\boldsymbol{\epsilon}}\in\operatorname{int}S_j$. Define $\boldsymbol{\epsilon}'\in\{-1,1\}^n$ by saying that

$$\begin{aligned}\epsilon'_i &= -\epsilon_i \text{ if } i = j\\ &= \epsilon_i \text{ otherwise.}\end{aligned}$$

By the choice of $\boldsymbol{\epsilon}$,

$$2(x|w_{\boldsymbol{\epsilon}'}) - 2\textstyle\sum_{i=0}^{n-1}\epsilon'_i\gamma_i\delta_i + \|w_{\boldsymbol{\epsilon}'}\|^2 \le 4(x|w_{\boldsymbol{\epsilon}}) - 2\textstyle\sum_{i=0}^{n-1}\epsilon_i\gamma_i\delta_i + \|w_{\boldsymbol{\epsilon}}\|^2,$$

that is,

$$2(x|w_{\boldsymbol{\epsilon}} - 2\epsilon_j v_j) - 2\sum_{i=0}^{n-1}\epsilon_i\gamma_i\delta_i + 4\epsilon_j\gamma_j\delta_j + \|w_{\boldsymbol{\epsilon}} - 2\epsilon_j v_j\|^2$$

$$\le 4(x|w_{\boldsymbol{\epsilon}}) - 2\sum_{i=0}^{n-1}\epsilon_i\gamma_i\delta_i + \|w_{\boldsymbol{\epsilon}}\|^2,$$

$$-4\epsilon_j(x|v_j) + 4\epsilon_j\gamma_j\delta_j + \|w_{\boldsymbol{\epsilon}}\|^2 - 4\epsilon_j(w_{\boldsymbol{\epsilon}}|v_j) + 4\|v_j\|^2 \le \|w_{\boldsymbol{\epsilon}}\|^2,$$

$$\begin{aligned}\delta_j^2 = \|v_j\|^2 &\le \epsilon_j(x|v_j) - \epsilon_j\gamma_j\delta_j + \epsilon_j(w_{\boldsymbol{\epsilon}}|v_j)\\ &\le |(x + w_{\boldsymbol{\epsilon}}|v_j) - \gamma_j\delta_j| = \delta_j|(x + w_{\boldsymbol{\epsilon}}|e_j)\gamma_j|\end{aligned}$$

and either $\delta_j = 0$ or $|(x + w_{\boldsymbol{\epsilon}}|e_j) - \gamma_j| \ge \delta_j$, so that $x + w_{\boldsymbol{\epsilon}} \notin \operatorname{int} S_j$. **X**

This contradiction shows that the theorem is true.

## 2 Outer measure properties

**2A Proposition** (M.Csörnyei, L.Wisewell) The measure $\mu$ on $\mathbb{R}^r$ defined by Carathéodory's method from $\theta_t$ has trivial measure algebra.

**proof (a)** Suppose that $A$ is a non-negligible set. Then for any $\epsilon > 0$ there is a $\delta > 0$ such that for every $\delta' \in [0, \delta]$ there is a cylindrical tube $T$ of cross-sectional radius $\delta'$ such that $(1 - \epsilon)\lambda_t T \le \theta_t(T \cap A)$.

**P** Let $C$ be the unit ball in $\mathbb{R}^{r-1}$. There is a $\gamma > 0$ such that for every $\delta \in ]0, 1]$ there is a disjoint family $\mathcal{D}$ of balls of radius $\delta$ included in $C$ and with $\sum_{D\in\mathcal{D}}\nu_{r-1}D \ge \gamma\nu_{r-1}C$. (If you like, start by packing $C$ with cubes of side $2\delta$.) Next, we can find a bounded non-negligible set $A' \subseteq A$. There is a sequence $\langle T_n\rangle_{n\in\mathbb{N}}$ of cylindrical tubes, covering $A'$, such that $\sum_{n=0}^{\infty}\lambda_t T_n \le (1 + \gamma\epsilon)\theta_t A'$. So there must be an $n$ such that $0 < \lambda_t T_n \le (1 + \gamma\epsilon)\theta_t(T_n \cap A')$. Express $T_n$ as $C_0 + L$ where $L$ is a line through and $C_0$ is a ball in $L^\perp$; let $\delta$ be the radius of $C_0$. If $\delta' \in ]0, \delta]$, let $\mathcal{D}$ be a disjoint family of $(r-1)$-balls of radius $\delta'$, included in $C_0$, of total $(r-1)$-dimensional measure at least $\gamma\nu_{r-1}C_0$. Set $H = C_0 \setminus \bigcup\mathcal{D}$. Then we have

$$\begin{aligned}\lambda_t(H + L) + \sum_{D\in\mathcal{D}}\lambda_t(D + L) &= \nu_{r-1}H + \sum_{D\in\mathcal{D}}\nu_{r-1}D = \nu_{r-1}C_0\\ &= \lambda_t T_n < (1 + \gamma\epsilon)\theta_t A'\\ &\le (1 + \gamma\epsilon)\big(\lambda_t(H + L) + \sum_{D\in\mathcal{D}}\theta_t(A \cap (D + L))\big),\end{aligned}$$

so

$$\begin{aligned}\sum_{D\in\mathcal{D}}\lambda_t(D + L) - \theta_t(A \cap (D + L)) &\le \gamma\epsilon\big(\lambda_t(H + L) + \sum_{D\in\mathcal{D}}\theta_t(A \cap (D + L))\big)\\ &\le \gamma\epsilon\nu_{r-1}C_0 \le \epsilon\sum_{D\in\mathcal{D}}\nu_{r-1}D = \epsilon\sum_{D\in\mathcal{D}}\lambda_t(D + L)\end{aligned}$$

and there must be a $D \in \mathcal{D}$ such that

$$(1 - \epsilon)\lambda_t T \le \theta_t(T \cap A') \le \theta_t(T \cap A)$$

where $T = D + L$ is a cylindrical tube with cross-sectional radius $\delta'$. **Q**

**(b) ?** Suppose, if possible, that there is a measurable set $E \subseteq \mathbb{R}^r$ such that neither $E$ nor its complement is negligible. By (a) we can find cylindrical tubes $T_1$ and $T_2$, of the same non-zero cross-sectional radius $\beta$, such that $\frac{2}{3}\lambda_t T_1 \le \theta_t(T_1 \cap E)$ and $\frac{2}{3}\lambda_t T_2 \le \theta_t(T_2 \setminus E)$. Since $E$ is supposed to be measurable,

$$\theta_t(T_1 \setminus E) = \theta_t T_1 - \theta_t(T_1 \cap E) = \lambda_t T_1 - \theta_t(T_1 \cap E) \le \frac{1}{3}\lambda_t T_1,$$

and similarly $\theta_t(T_2 \cap E) \le \frac{1}{3}\lambda_t T_2$.

Let $B_1 \subseteq T_1$ and $B_2 \subseteq T_2$ be balls of radius $\beta$, and let $T$ be the cylindrical tube, with cross-sectional radius $\beta$, and with axis running through the centres of $B_1$ and $B_2$. Then

$$\begin{aligned}\theta_t(T \cap E) &\ge \theta_t(T \cap B_1 \cap E) = \theta_t(T \cap B_1) - \theta_t(T \cap B_1 \setminus E)\\ &\ge \theta_t T_1 - \theta_t(T_1 \setminus E) \ge \frac{2}{3}\theta_t T_1,\end{aligned}$$

using 1Bc and 1E to see that

$$\theta_t(T_1 \cap B_1) = \theta_t B_1 = \lambda_t T_1 = \theta_t T_1.$$

Similarly, $\theta_t(T \setminus E) > \frac{2}{3}\theta_t T$; but this is impossible if $E$ is measurable. **X**

**2B** Recall that an outer measure $\phi$ is **submodular** or **strongly subadditive** if $\phi(A \cup B) + \phi(A \cap B) \le \phi A + \phi B$ for all $A$ and $B$.

**Proposition** $\theta_t$ is not submodular.

**proof** Let $B_1$, $B_2$, $B_3$ be three balls of radius 1 with centres in an equilateral triangle so large that no line meets more than two of the balls. Set $A_1 = B_2 \cup B_3$, $A_2 = B_1 \cup B_3$, $A_3 = B_1 \cup B_2$ and $A = B_1 \cup B_2 \cup B_3$.

Set $\gamma = \theta_t A$ and take $\epsilon > 0$. Let $\delta > 0$ be such that no tube of cross-sectional diameter at most $\delta$ can meet all three of $B_1$, $B_2$ and $B_3$. Then we can cover $A$ by a sequence $\langle T_n\rangle_{n\in\mathbb{N}}$ of tubes with $\sum_{n=0}^{\infty} \lambda_t T_n \le \gamma + \epsilon$; moreover, this can be done with tubes of cross-sectional diameter at most $\delta$. If we look at the tubes missing each $B_i$ in turn, we see that there is some $j$ such that

$$\theta_t A_j \le \frac{2}{3}\sum_{n=0}^{\infty} \lambda_t T_n \le \frac{2}{3}(\gamma + \epsilon).$$

So in fact $\theta_t A_j \le \frac{2}{3}(\gamma + \epsilon)$ for every $j$; as $\epsilon$ is arbitrary, $\theta_t A_j \le \frac{2}{3}\gamma$ for every $j$. At the same time, we know from Theorem 1D that

$$\theta_t B_1 = \theta_t B_2 = \theta_t B_3 = \theta_t A_1 = \theta_t A_2 = \lambda_t T$$

for any cylindrical tube $T$ with radius 1; so $\gamma \le 2\theta_t B$.

Putting these together,

$$\theta_t(A_1 \cup A_2) + \theta_t(A_1 \cap A_2) = \gamma + \theta_t B \ge \frac{3}{2}\gamma > \frac{4}{3}\gamma \ge \theta_t A_1 + \theta_t A_2,$$

and $\theta_t$ is not submodular.

**2C** Because every tube can be covered by an open tube with almost the same cross-sectional volume, $\theta_t$ is outer regular with respect to the open sets, that is, $\theta_t A = \inf\{\theta_t G : G \supseteq A \text{ is open}\}$ for every $A \subseteq \mathbb{R}^r$.

**2D (a)** Let $D \subseteq \mathbb{R}^r$ be any non-trivial bounded set, and let $\mu$ be the subspace measure on $D$ induced by $\nu_r$. Then $\frac{1}{\operatorname{diam} D}\mu A \le \theta_t A$ for every $A$ measured by $\mu$.

This shows that there are many measures dominated by $\theta_t$.

**(b)** Let $\Delta \subseteq \mathbb{R}^2$ be an equilateral triangle of side 1, together with its interior. By Bang's theorem (1G), $\theta_t\Delta = \frac{\sqrt{3}}{2}$. Consider the standard cover of $\Delta$ by four equilateral triangles of side $\frac{1}{2}$; let $E_1$, $E_2$, $E_3$ be the three trapeziums obtained by deleting corner triangles. Then $\theta_t E_i = \frac{\sqrt{3}}{4}$ for each $i$. **?** If $\mu$ is a measure on $\Delta$ such that $\mu\Delta = \theta_t\Delta$ and $\mu K \le \theta_t K$ for all compact $K \subseteq \Delta$, then we have

$$\frac{\sqrt{3}}{2} = \mu\Delta \le \tfrac{1}{2}\textstyle\sum_{i=1}^{3}\mu E_i \le \tfrac{3}{2}\cdot\tfrac{\sqrt{3}}{4}.\ \mathbf{X}$$

Thus $\theta_t$ is not the supremum of the measures it dominates, even on compact convex sets.

**2E Example** Take $r = 2$. Then there is an open set $G \subseteq \mathbb{R}^2$ such that $\theta_t G = \infty$ but $\theta_t(G \cap B) \le 1$ for every bounded set $B$.

**proof** I aim to build $G$ as the union of a non-decreasing sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of bounded open sets such that $\theta_t\overline{G}_n < 1$ for every $n$. At each stage, I shall have a family $\langle T_{ni}\rangle_{i<2^n}$ of cylindrical tubes (= slices) of width $2^{-n}$, all in different directions, such that $\overline{G}_n$ is covered by their interiors.

Start with $G_0 = \emptyset$ and $T_{00}$ any slice of width 1. Given $G_{n-1}$ and $\langle T_{n-1,i}\rangle_{i<2^{n-1}}$, start by choosing $\langle T_{ni}\rangle_{i<2^n}$ to be slices of width $2^{-n}$, all in different directions, with their interiors covering $\overline{G}_{n-1}$. (Replace each $T_{n-1,i}$ by two slices, first splitting $T_{n-1,i}$ in half lengthways, then moving the two parts together a little, then varying the direction of one of them slightly.) Let $\delta_n > 0$ be the least angle between the axes of $T_{ni}$ and $T_{nj}$ for distinct $i$, $j < n$, and let $\beta_n \ge 1$ be so large that

$$2n < \tfrac{1}{8}\beta_n \sin\tfrac{1}{2}\delta_n.$$

Now $G_n$ will be $G_{n-1} \cup \bigcup_{i<2^n} H_{ni}$ where each $H_{ni}$ is an open rectangle of width $2^{-n-1}$ and length $\beta_n$, disjoint from $\bigcup_{j\ne i} T_{nj}$ and the disc of radius $n$ centred at 0, lying centered along the axis of $T_{ni}$. Continue.

The construction ensures that the intersection of $G$ with the disc of radius $n$ is included in $G_{n-1}$ for every $n \ge 1$, so always has outer measure at most $\sum_{i<2^n}\lambda_t T_{ni} = 1$. **?** Suppose, if possible, that $G$ can be covered by a sequence $\langle S_k\rangle_{k\in\mathbb{N}}$ of slices with $\sum_{k=0}^{\infty}\lambda_t S_k < \infty$. Let $h_k$ be the width of $S_k$ for each $k$. Let $m$ be such that $\sum_{k=m}^{\infty}\lambda_t S_k \le \frac{1}{8}$, and take $n$ such that $m \le 2^{n-2}$ and $\sum_{k=0}^{\infty} h_k \le n$.

For $i < 2^n$ let $K_i$ be the set of those $k \in \mathbb{N}$ such that the angle between the axes of $T_{ni}$ and $S_k$ is less than $\frac{1}{2}\delta_n$; then the $K_i$ are disjoint. If $i < 2^n$ and $k \in \mathbb{N}\setminus K_i$, $S_k \cap H_i$ is a parallelogram with width $2^{-n-1}$ in one direction, width $h_k$ in the other and angle at least $\frac{1}{2}\delta_n$, so its area is at most $\frac{2^{-n-1}h_k}{\sin(\delta_n/2)}$. Accordingly

$$\sum_{i<2^n}\sum_{k\in\mathbb{N}\setminus K_i}\nu_2(S_k\cap H_i) \le \frac{2}{\sin\frac{1}{2}\delta_n}\sum_{k=0}^{\infty} h_k \le \frac{2n}{\sin\frac{1}{2}\delta_n}.$$

Set $J = \{i : K_i \cap m \ne \emptyset\}$ and $I = 2^n \setminus J$; then $\#(J) \le m \le 2^{n-2}$ so $\#(I) \ge \frac{3}{4}\cdot 2^n$. Now, because the $K_i$ are disjoint,

$$\begin{aligned}\sum_{i\in I,k\in K_i}\nu_2(S_k\cap H_i) &\le \sum_{k=m}^{\infty}\max\{\nu_2(S_k\cap H_i) : i<2^n\}\\ &\le \sum_{k=m}^{\infty} h_k\cdot\max\{\operatorname{diam} H_i : i<2^n\} \le \tfrac{1}{8}(\beta_n+1) \le \tfrac{1}{4}\beta_n.\end{aligned}$$

Putting these together, and remembering that the $S_k$ are supposed to cover the $H_{ni}$,

$$\begin{aligned}\tfrac{3}{8}\beta_n \le 2^{-n-1}\#(I)\beta_n &= \sum_{i\in I}\nu_2 H_{ni}\\ &\le \sum_{i\in I,k\in\mathbb{N}}\nu_2(S_k\cap H_{ni}) \le \frac{2n}{\sin\frac{1}{2}\delta_n} + \tfrac{1}{4}\beta_n,\end{aligned}$$

which is impossible, by the choice of $\beta_n$. $\mathbf{X}$

So $\theta_t G = \infty$, as claimed.

**3 Problems**

**(a)** With $r = 2$, find $\theta_t S$, where $S$ is the Koch snowflake curve; or anyway determine whether $\theta_t S = 0$.

**(b)** Suppose that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of subsets of $\mathbb{R}^r$ such that $A = \bigcup_{n\in\mathbb{N}} A_n$ is bounded. Must $\theta_t A$ be $\sup_{n\in\mathbb{N}}\theta_t A_n$?

If so we shall be able to speak of the **tube capacity** of a set $A$ as $\sup\{\theta_t(A\cap B) : B\subseteq\mathbb{R}^r \text{ is bounded}\}$.

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