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Tao's proof of Szemerédi's theorem

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Introduction Szemerédi's celebrated theorem on arithmetic progressions (Szemerédi 75) is not obviously a part of measure theory (though by a happy coincidence it is one of the elements of a proof in the work described by A.Olevskiĭ at this conference). Remarkably, however, it has led to significant developments in measure theory. The first was Furstenberg's multiple recurrence theorem (Furstenberg 77, Furstenberg 81, Furstenberg & Katznelson 85). In this paper I will give an account of a new approach due to T.Tao (Tao p06) which introduces another phenomenon of great interest from a measure-theoretic point of view.

1 Removable intersections

1A Definition (a) Let (X,Σ,μ) be a probability space, T a subalgebra of Σ (*not* necessarily a σ-subalgebra) and $\langle\Sigma_i\rangle_{i\in I}$ a family of σ-subalgebras of Σ. I will say that $\langle\Sigma_i\rangle_{i\in I}$ has **T-removable intersections** if whenever $J\subseteq I$ is finite and not empty, $E_i\in\Sigma_i$ for $i\in J$, $\mu(\bigcap_{i\in J}E_i)=\emptyset$ and $\epsilon>0$, there is a family $\langle F_i\rangle_{i\in J}$ such that $F_i\in\mathrm{T}\cap\Sigma_i$ and $\mu(E_i\setminus F_i)\le\epsilon$ for each $i\in J$, and $\bigcap_{i\in J}F_i=\emptyset$. (This is a stronger version of what Tao p06 calls the 'uniform intersection property'.)

(b) If X is a set and Σ, Σ' are two σ-algebras of subsets of X, $\Sigma\vee\Sigma'$ will be the σ-algebra generated by $\Sigma\cup\Sigma'$. If $\langle\Sigma_i\rangle_{i\in I}$ is a family of σ-algebras of subsets of X, I will write $\bigvee_{i\in I}\Sigma_i$ for the σ-algebra generated by $\bigcup_{i\in I}\Sigma_i$.

(c) If (X,Σ,μ) is a probability space and $\mathcal{A}\subseteq\mathcal{E}\subseteq\Sigma$, I will say that $\mathcal{A}$ is **metrically dense** in $\mathcal{E}$ if for every $E\in\mathcal{E}$ and $\epsilon>0$ there is an $F\in\mathcal{A}$ such that $\mu(E\triangle F)\le\epsilon$.

1B Lemma Let (X,Σ,μ) be a probability space and T a subalgebra of Σ. Let $\langle\Sigma_i\rangle_{i\in I}$ be a family of σ-subalgebras of Σ.

(a) $\langle\Sigma_i\rangle_{i\in I}$ has T-removable intersections iff $\langle\Sigma_i\rangle_{i\in J}$ has T-removable intersections for every $J\subseteq I$.

(b) Suppose there is a family $\langle\Sigma'_j\rangle_{j\in J}$ of σ-subalgebras of Σ which has T-removable intersections, and a function $f:I\to J$ such that $\Sigma'_{f(i)}=\Sigma_i$ for every $i\in I$. Suppose that $\mathrm{T}\cap\Sigma_i$ is metrically dense in Σ_i for every $i\in I$. Then $\langle\Sigma_i\rangle_{i\in I}$ has T-removable intersections.

(c) Suppose that, for each $i\in I$, we have a σ-subalgebra Σ'_i of Σ_i such that for every $E\in\Sigma_i$ there is an $E'\in\Sigma'_i$ such that $E\triangle E_i$ is negligible. If $\langle\Sigma'_i\rangle_{i\in I}$ has T-removable intersections, so does $\langle\Sigma_i\rangle_{i\in I}$.

proof (a) is trivial.

(b) Suppose that $K\subseteq I$ is finite and not empty, that $\langle E_i\rangle_{i\in K}\in\prod_{i\in K}\Sigma_i$ is such that $\mu(\bigcap_{i\in K}E_i)=0$, and $\epsilon>0$. Set $n=\#(K)$ and $\eta=\frac{\epsilon}{n+2}>0$. Set $E'_j=\bigcap_{i\in K,f(i)=j}E_i\in\Sigma'_j$ for $j\in f[K]$; then $\bigcap_{j\in f[K]}E'_j$ is negligible, so we have $F'_j\in\mathrm{T}\cap\Sigma'_j$, for $j\in f[K]$, such that $\bigcap_{j\in f[K]}F'_j=\emptyset$ and $\mu(E'_j\setminus F'_j)\le\eta$ for every $j\in f[K]$. Now $\mathrm{T}\cap\Sigma_i$ is metrically dense in Σ_i for each i, so we can find $G_i\in\mathrm{T}\cap\Sigma_i$ such that $\mu(E_i\triangle G_i)\le\eta$ for each i. Set $G'_j=\bigcap_{i\in K,f(i)=j}G_i$ for each $j\in f[K]$. Then

$$\mu(G'_j\setminus F'_j)\le\mu(G'_j\setminus E'_j)+\mu(E'_j\setminus F'_j)\le\textstyle\sum_{i\in K,f(i)=j}\mu(G_i\setminus E_i)+\eta\le(n+1)\eta.$$

Note that $G'_j\in\mathrm{T}\cap\Sigma'_j$ for each j. Now set $F_i=G_i\setminus(G'_{f(i)}\setminus F'_{f(i)})$ for $i\in K$. Then $F_i\in\mathrm{T}\cap\Sigma_i$ and

$$\mu(E_i\setminus F_i)\le\mu(E_i\setminus G_i)+\mu(G'_{f(i)}\setminus F'_{f(i)})\le(n+2)\eta=\epsilon.$$

Also

$$\begin{aligned}\bigcap_{i\in K}F_i&=\bigcap_{j\in f[K]}\ \bigcap_{i\in K,f(i)=j}G_i\setminus(G'_j\setminus F'_j)\\&=\bigcap_{j\in f[K]}G'_j\setminus(G'_j\setminus F'_j)\subseteq\bigcap_{j\in f[K]}F'_j=\emptyset.\end{aligned}$$

As $\langle E_i\rangle_{i\in K}$ and ϵ are arbitrary, $\langle\Sigma_i\rangle_{i\in I}$ has T-removable intersections.

(c) If $J\subseteq I$ is finite, $\langle E_j\rangle_{j\in J}\in\prod_{j\in J}\Sigma_j$, $\bigcap_{j\in J}E_j$ is negligible and $\epsilon>0$, then for each $j\in J$ let $E'_j\in\Sigma'_j$ be such that $E'_j\triangle E_j$ is negligible. Then $\bigcap_{j\in J}E'_j$ is negligible, so there are $F_j\in\mathrm{T}\cap\Sigma'_j$, for $j\in J$, such that $\mu(E'_j\setminus F_j)\le\epsilon$ for every $j\in J$ and $\bigcap_{j\in J}F_j$ is empty. Now $\mu(E_j\setminus F_j)\le\epsilon$ for every j. As $\langle E_j\rangle_{j\in J}$ and ϵ are arbitrary, $\langle\Sigma_i\rangle_{i\in I}$ has T-removable intersections.

1C Lemma Let (X,Σ,μ) be a probability space and T a subalgebra of Σ. Let I be a set, A an upwards-directed set, and $\langle\Sigma_{\alpha i}\rangle_{\alpha\in A,i\in I}$ a family of σ-subalgebras of Σ such that, setting $\Sigma_i=\bigvee_{\alpha\in A}\Sigma_{\alpha i}$ for each i,

(i) $\Sigma_{\alpha i}\subseteq\Sigma_{\beta i}$ whenever $i\in I$ and $\alpha\le\beta$ in A,
(ii) $\langle\Sigma_{\alpha i}\rangle_{i\in I}$ has T-removable intersections for every $\alpha\in A$,
(iii) Σ_i and $\bigvee_{j\in I}\Sigma_{\alpha j}$ are relatively independent over $\Sigma_{\alpha i}$ for every $i\in I$ and $\alpha\in A$.

Then $\langle\Sigma_i\rangle_{i\in I}$ has T-removable intersections.

proof Take a non-empty finite set $J\subseteq I$, a family $\langle E_i\rangle_{i\in J}$ such that $E_i\in\Sigma_i$ for every $i\in J$ and $\bigcap_{i\in J}E_i$ is negligible, and $\epsilon>0$. Set $\delta=\frac{1}{\#(J)+1}$, $\eta=\delta\sqrt{\frac{\epsilon}{2}}>0$. For each $i\in J$ there are an $\alpha\in A$ and an $E'_i\in\Sigma_{\alpha i}$ such that $\mu(E_i\triangle E'_i)\le\eta^2$ (because A is upwards-directed); we can suppose that it is the same α for each i. Let $g_i:X\to[0,1]$ be a $\Sigma_{\alpha i}$-measurable function which is a conditional expectation of χE_i on $\Sigma_{\alpha i}$; then

$$\|\chi E_i-g_i\|_2\le\|\chi E_i-\chi E'_i\|_2\le\eta$$

(cf. Fremlin 01, 244Nb). Set $E''_i=\{x:g_i(x)\ge 1-\delta\}\in\Sigma_{\alpha i}$; then

$$\mu(E_i\setminus E''_i)=\mu\{x:\chi E_i(x)-g_i(x)>\delta\}\le\frac{\eta^2}{\delta^2}=\frac{\epsilon}{2}.$$

Set $E=\bigcap_{i\in J}E''_i$. Then $\mu E=0$. **P** Since $\mu(\bigcap_{i\in J}E_i)=0$, $\mu E\le\sum_{i\in J}\mu(E\setminus E_i)$. For any $i\in J$, setting $H_i=X\cap\bigcap_{j\in J\setminus\{i\}}E''_j$,

$$\begin{aligned}\mu(E\setminus E_i)&=\mu((E''_i\setminus E_i)\cap H_i)=\int\chi(E''_i\setminus E_i)\times\chi H_i d\mu\\&=\int h_i\times h'_i d\mu=\int h_i\times\chi H_i d\mu\end{aligned}$$

where $h_i=(1-g_i)\chi E''_i$, h'_i are conditional expectations of $\chi(E''_i\setminus E_i)$, χH_i on $\Sigma_{\alpha i}$, because Σ_i and $\bigvee_{j\in J}\Sigma_{\alpha j}$ are relatively independent over $\Sigma_{\alpha i}$. But $h_i\le\delta\chi E''_i$, so

$$\mu(E\setminus E_i)\le\delta\mu(E''_i\cap H_i)=\delta\mu E.$$

Summing, we have

$$\mu E\le\delta\#(J)\mu E;$$

but $\delta\#(J)<1$, so $\mu E=0$. **Q**

Because $\langle\Sigma_{\alpha i}\rangle_{i\in I}$ has T-removable intersections, there are $F_i\in\mathrm{T}\cap\Sigma_{\alpha i}\subseteq\mathrm{T}\cap\Sigma_i$, for $i\in J$, such that $\bigcap_{i\in I}F_i=\emptyset$ and $\mu(E''_i\setminus F_i)\le\frac{\epsilon}{2}$ for each i; in which case $\mu(E_i\setminus F_i)\le\epsilon$ for each i. As $\langle E_i\rangle_{i\in J}$ and ϵ are arbitrary, $\langle\Sigma_i\rangle_{i\in I}$ has T-removable intersections.

1D Lemma Let (X,Σ,μ) be a probability space, T a subalgebra of Σ, and $\langle\Sigma_i\rangle_{i\in I}$ a finite family of σ-subalgebras of Σ which has T-removable intersections; suppose that $\mathrm{T}\cap\Sigma_i$ is metrically dense in Σ_i for each i. Set $\Sigma^*=\bigvee_{i\in I}\Sigma_i$. Suppose that we have a finite set J, a function $g:J\to I$ and a family $\langle\Lambda_j\rangle_{j\in J}$ of σ-subalgebras of Σ such that

$\langle\Lambda_j\rangle_{j\in J}$ is relatively independent over Σ^*,
for each $j\in J$, Λ_j and Σ^* are relatively independent over $\Sigma_{g(j)}$,
for each $j\in J$, $\mathrm{T}\cap\Lambda_j$ is metrically dense in Λ_j.

Let A be a finite set and $f:A\to I$, $\phi:A\to\mathcal{P}J$ functions such that $\Sigma_{g(j)}\subseteq\Sigma_{f(\alpha)}$ whenever $\alpha\in A$ and $j\in\phi(\alpha)$. Suppose that

for each $\alpha\in A$, $\bigvee_{j\in\phi(\alpha)}\Lambda_j$ and $\Sigma^*\vee\bigvee_{j\in J\setminus\phi(\alpha)}\Lambda_j$ are relatively independent over $\Sigma_{f(\alpha)}$.

Set $\tilde{\Sigma}_\alpha = \Sigma_{f(\alpha)} \vee \bigvee_{j\in\phi(\alpha)} \Lambda_j$ for $\alpha \in A$. Then $\langle\tilde{\Sigma}_\alpha\rangle_{\alpha\in A}$ has T-removable intersections.

proof Of course we can suppose that A is non-empty, and that $J = \bigcup_{\alpha\in A}\phi(\alpha)$.

(a) To begin with, suppose that every Λ_j is actually a finite subalgebra of T.

(i) Take a non-empty set $B \subseteq A$, $\langle E_\alpha\rangle_{\alpha\in B} \in \prod_{\alpha\in B}\tilde{\Sigma}_\alpha$ such that $\bigcap_{\alpha\in B} E_\alpha$ is negligible, and $\epsilon > 0$. Set $K = \bigcup_{\alpha\in B}\phi(\alpha)$. Let $\mathcal{A}$ be the set of atoms of $\bigvee_{j\in K}\Lambda_j$ and set $\eta = \frac{\epsilon}{\#(\mathcal{A})} > 0$.

(ii) For each $H \in \mathcal{A}$ and $\alpha \in B$, let $C(H,\alpha)$ be the atom of $\bigvee_{j\in\phi(\alpha)}\Lambda_j$ including H. Then there is a family $\langle F_{H\alpha}\rangle_{\alpha\in B}$, with empty intersection, such that $F_{H\alpha} \in \mathrm{T}\cap\Sigma_\alpha$ and $\mu(E_\alpha \cap C(H,\alpha)\setminus F_{H\alpha}) \le \eta$ for each $\alpha \in B$. **P** For each $j \in K$, let H_j be the atom of Λ_j including H, $h_j : X \to [0,1]$ a $\Sigma_{g(j)}$-measurable function which is a conditional expectation of χH_j on $\Sigma_{g(j)}$, and $G_j = \{x : h_j(x) > 0\}$. Because Λ_j and Σ^* are relatively independent over $\Sigma_{g(j)}$, and $\Sigma_{g(j)} \subseteq \Sigma^*$, h_j is a conditional expectation of χH_j on Σ^* for each j. Because $\langle\Lambda_j\rangle_{j\in K}$ is relatively independent over Σ^*, $h = \prod_{j\in K} h_j$ is a conditional expectation of $\chi(X \cap \bigcap_{j\in K} H_j)$ on Σ^*. (For the trivial case in which $K = \emptyset$, take $h = \chi X$.) For each $\alpha \in B$ we have an $E'_\alpha \in \Sigma_{f(\alpha)}$ such that $E_\alpha \cap C(H,\alpha) = E'_\alpha \cap C(H,\alpha)$, because $E_\alpha \in \Sigma_{f(\alpha)} \vee \bigvee_{j\in\phi(\alpha)}\Lambda_j$. Now $\bigcap_{\alpha\in B} E'_\alpha \in \Sigma^*$, so

$$\int_{\bigcap_{\alpha\in B} E'_\alpha} h = \mu(\textstyle\bigcap_{\alpha\in B} E'_\alpha \cap \bigcap_{j\in K} H_j) \le \mu(\bigcap_{\alpha\in B}(E_\alpha \cap C(H,\alpha))) = 0;$$

accordingly $\bigcap_{\alpha\in B} E'_\alpha \cap \bigcap_{j\in K} G_j$ is negligible. Set $E''_\alpha = E'_\alpha \cap \bigcap_{j\in\phi(\alpha)} G_j$ for each α; then $E''_\alpha \in \Sigma_{f(\alpha)}$, because $G_j \in \Sigma_{g(j)} \subseteq \Sigma_{f(\alpha)}$ whenever $j \in \phi(\alpha)$. Also $\bigcap_{\alpha\in B} E''_\alpha$ is negligible.

Because $\langle\Sigma_i\rangle_{i\in I}$ has T-removable intersections and $\mathrm{T}\cap\Sigma_i$ is metrically dense in Σ_i for each i, $\langle\Sigma_{f(\alpha)}\rangle_{\alpha\in B}$ has T-removable intersections (Lemma 1Bb). So we have $F_{H\alpha} \in \mathrm{T}\cap\Sigma_{f(\alpha)}$, for $\alpha \in B$, such that $\bigcap_{\alpha\in A} F_{H\alpha} = \emptyset$ and $\mu(E''_\alpha \setminus F_{H\alpha}) \le \eta$ for every α.

If $\alpha \in B$ and $j \in \phi(\alpha)$,

$$0 = \int_{E'_\alpha\setminus G_j} h_j = \mu(H_j \cap E'_\alpha \setminus G_j)$$

because h_j is a conditional expectation of χH_j on Σ^*. So if $\alpha \in B$,

$$\begin{aligned}\mu(E_\alpha \cap C(H,\alpha)\setminus F_{H\alpha}) &= \mu(E'_\alpha \cap C(H,\alpha)\setminus F_{H\alpha})\\ &\le \mu(E''_\alpha\setminus F_{H\alpha}) + \sum_{j\in\phi(\alpha)}\mu(E'_\alpha\cap H_j\setminus G_j)\end{aligned}$$

(because $C(H,\alpha) = X \cap \bigcap_{j\in\phi(\alpha)} H_j$)

$$\le \eta,$$

as required. **Q**

(iii) For $\alpha \in B$ let $\mathcal{A}_\alpha$ be the set of atoms of $\bigvee_{j\in\phi(\alpha)}\Lambda_j$ and set

$$F_\alpha = \textstyle\bigcup_{G\in\mathcal{A}_\alpha}(G \cap \bigcap_{H\in\mathcal{A},H\subseteq G} F_{H\alpha}).$$

Then $F_\alpha \in \mathrm{T}\cap\tilde{\Sigma}_\alpha$ and

$$\begin{aligned}\mu(E_\alpha\setminus F_\alpha) &= \sum_{G\in\mathcal{A}_\alpha}\mu(E_\alpha\cap G\setminus F_\alpha)\\ &\le \sum_{G\in\mathcal{A}_\alpha}\sum_{H\in\mathcal{A},H\subseteq G}\mu(E_\alpha\cap G\setminus F_{H\alpha})\\ &= \sum_{H\in\mathcal{A}}\mu(E_\alpha\cap C(H,\alpha)\setminus F_{H\alpha}) \le \eta\#(\mathcal{A}) = \epsilon.\end{aligned}$$

If $H \in \mathcal{A}$ then $H \subseteq C(H,\alpha) \in \mathcal{A}_\alpha$ and

$$H \cap F_\alpha \subseteq F_\alpha \cap C(H,\alpha) \subseteq F_{H\alpha},$$

for each α. So $H\cap\bigcap_{\alpha\in A}F_\alpha$ is empty. But $X=\bigcup\mathcal{A}$ so $\bigcap_{\alpha\in A}F_\alpha=\emptyset$. As $\langle E_\alpha\rangle_{\alpha\in B}$ and ϵ are arbitrary, $\langle\tilde\Sigma_\alpha\rangle_{\alpha\in A}$ has T-removable intersections.

(b) Next, suppose that each Λ_j is the σ-algebra generated by $\mathrm{T}\cap\Lambda_j$.

(i) For $L\in[\mathrm{T}]^{<\omega}$, $j\in J$ and $\alpha\in A$ write Λ_{Lj} for the algebra generated by $\Lambda_j\cap L$ and $\Sigma'_{L\alpha}=\Sigma_{f(\alpha)}\vee\bigvee_{j\in\phi(\alpha)}\Lambda_{Lj}$. Then

$$\Sigma'_{K\alpha}\subseteq\Sigma'_{L\alpha}\text{ whenever }\alpha\in A\text{ and }K\subseteq L\in[\mathrm{T}]^{<\omega},$$

$$\bigvee_{L\in[\mathrm{T}]^{<\omega}}\Sigma'_{L\alpha}=\Sigma_{f(\alpha)}\vee\bigvee_{j\in\phi(\alpha)}\bigvee_{L\in[\mathrm{T}]^{<\omega}}\Lambda_{Lj}=\Sigma_{f(\alpha)}\vee\bigvee_{j\in\phi(\alpha)}\Lambda_j=\tilde\Sigma_\alpha$$

because each Λ_j is the σ-algebra generated by $\mathrm{T}\cap\Lambda_j=\bigcup_{L\in[\mathrm{T}]^{<\omega}}\Lambda_{Lj}$. By (a), $\langle\Sigma'_{L\alpha}\rangle_{\alpha\in A}$ has T-removable intersections for every $L\in[\mathrm{T}]^{<\omega}$.

(ii) Suppose that $\alpha\in A$ and $L\in[\mathrm{T}]^{<\omega}$. Then $\bigvee_{j\in\phi(\alpha)}\Lambda_j$ and $\Sigma^*\vee\bigvee_{j\in J\setminus\phi(\alpha)}\Lambda_j$ are relatively independent over $\Sigma_{f(\alpha)}$, by hypothesis. So $\tilde\Sigma_\alpha=\Sigma_{f(\alpha)}\vee\bigvee_{j\in\phi(\alpha)}\Lambda_j$ and $\Sigma^*\vee\bigvee_{j\in J\setminus\phi(\alpha)}\Lambda_j$ are relatively independent over $\Sigma_{f(\alpha)}$. Because $\Sigma_{f(\alpha)}\subseteq\Sigma'_{L\alpha}\subseteq\tilde\Sigma_\alpha$, $\tilde\Sigma_\alpha$ and $\Sigma^*\vee\bigvee_{j\in J\setminus\phi(\alpha)}\Lambda_j$ are relatively independent over $\Sigma'_{L\alpha}$. Because $\bigvee_{j\in\phi(\alpha)}\Lambda_{Lj}\subseteq\Sigma'_{L\alpha}$, $\tilde\Sigma_\alpha$ and $\Sigma^*\vee\bigvee_{j\in J}\Lambda_{Lj}$ are relatively independent over $\Sigma'_{L\alpha}$. So $\tilde\Sigma_\alpha$ and

$$\bigvee_{\beta\in A}\Sigma'_{L\beta}=\bigvee_{\beta\in A}(\Sigma_{f(\beta)}\vee\bigvee_{j\in\phi(\beta)}\Lambda_{Lj})\subseteq\Sigma^*\vee\bigvee_{j\in J}\Lambda_{Lj}$$

are relatively independent over $\Sigma'_{L\alpha}$.

(iii) With (i), this shows that the family $\langle\Sigma'_{L\alpha}\rangle_{L\in[\mathrm{T}]^{<\omega},\alpha\in A}$ satisfies the conditions of Lemma 1C, and $\langle\tilde\Sigma_\alpha\rangle_{\alpha\in A}$ has T-removable intersections.

(c) Finally, for the general case, let Λ'_j be the σ-algebra generated by $\Lambda_j\cap\mathrm{T}$ for $j\in J$, and $\Sigma'_\alpha=\Sigma_{f(\alpha)}\vee\bigvee_{j\in\phi(\alpha)}\Lambda'_j$ for $\alpha\in A$. If $j\in J$ and $F\in\Lambda_j$, there is an $F'\in\Lambda'_j$ such that $F\triangle F'$ is negligible; so if $\alpha\in A$ and $E\in\tilde\Sigma_\alpha$, there is an $E'\in\Sigma'_\alpha$ such that $E\triangle E'$ is negligible. By (b), $\langle\Sigma'_\alpha\rangle_{\alpha\in A}$ has T-removable intersections; by Lemma 1Bc, $\langle\tilde\Sigma_\alpha\rangle_{\alpha\in A}$ has T-removable intersections.

1E Theorem (Tao p06, Theorem 4.2) Let (X,Σ,μ) be a probability space, and T a subalgebra of Σ. Let Γ be a partially ordered set such that $\gamma\wedge\delta=\inf\{\gamma,\delta\}$ is defined in Γ for all γ, $\delta\in\Gamma$, and $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma}$ a family of σ-subalgebras of Σ such that

(i) $\mathrm{T}\cap\Sigma_\gamma$ is metrically dense in Σ_γ for every $\gamma\in\Gamma$,

(ii) if γ, $\delta\in\Gamma$ and $\gamma\le\delta$ then $\Sigma_\gamma\subseteq\Sigma_\delta$,

(iii) if $\gamma\in\Gamma$ and Δ, Δ' are finite subsets of Γ such that $\delta\wedge\gamma\in\Delta'$ for every $\delta\in\Delta$, then Σ_γ and $\bigvee_{\delta\in\Delta}\Sigma_\delta$ are relatively independent over $\bigvee_{\delta\in\Delta'}\Sigma_\delta$.

Then $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma}$ has T-removable intersections.

proof (a) To begin with (down to the end of (d) below) suppose that Γ is finite. In this case, we have a rank function $r:\Gamma\to\mathbb{N}$ such that $r(\gamma)=\min\{n:n\in\mathbb{N},\ r(\delta)<n\text{ for every }\delta<\gamma\}$ for each $\gamma\in\Gamma$. For $a\subseteq\Gamma$ set $\tilde\Sigma_a=\bigvee_{\gamma\in a}\Sigma_\gamma$; note that $\mathrm{T}\cap\tilde\Sigma_a$ is always metrically dense in $\tilde\Sigma_a$.

Let A be the family of those sets $a\subseteq\Gamma$ such that $\gamma\in a$ whenever $\gamma\le\delta\in a$. For $n\in\mathbb{N}$ set $\Gamma_n=\{\gamma:r(\gamma)=n\}$ and $A_n=\{a:a\in A,\ r(\gamma)<n\text{ for every }\gamma\in a\}$

(b) Suppose that a, b, c are subsets of Γ and that $\gamma\wedge\delta\in c$ whenever $\gamma\in a$ and $\delta\in b\cup(a\setminus\{\gamma\})$. Then

(i) $\langle\Sigma_\gamma\rangle_{\gamma\in a}$ is relatively independent over $\tilde\Sigma_c$,

(ii) $\tilde\Sigma_a$ and $\tilde\Sigma_b$ are relatively independent over $\tilde\Sigma_c$.

P Induce on $\#(a)$. If $a=\emptyset$ then $\tilde\Sigma_a=\{\emptyset,X\}$ and the result is trivial. For the inductive step, take $\gamma_0\in a$ and set $a'=a\setminus\{\gamma_0\}$. Then the inductive hypothesis tells us that $\langle\Sigma_\gamma\rangle_{\gamma\in a'}$ is relatively independent over $\tilde\Sigma_c$ and that $\tilde\Sigma_{a'}$ and $\tilde\Sigma_b$ are relatively independent over $\tilde\Sigma_c$. We also see that $\gamma_0\wedge\delta\in c$ whenever $\delta\in a'$, so that Σ_{γ_0} and $\tilde\Sigma_{a'}$ are relatively independent over $\tilde\Sigma_c$. But this means that $\langle\Sigma_\gamma\rangle_{\gamma\in a}$ is relatively independent over $\tilde\Sigma_c$. Similarly, because in fact $\gamma_0\wedge\delta\in c$ for every $\delta\in a'\cup b$, Σ_{γ_0} and $\tilde\Sigma_{a'}\vee\tilde\Sigma_b$ are relatively independent over $\tilde\Sigma_c$; so the triple Σ_{γ_0}, $\tilde\Sigma_{a'}$ and $\tilde\Sigma_b$ are relatively independent over $\tilde\Sigma_c$, and $\tilde\Sigma_a = \Sigma_{\gamma_0} \vee \tilde\Sigma_{a'}$ and $\tilde\Sigma_b$ are relatively independent over $\tilde\Sigma_c$. Thus the induction continues. **Q**

(c) For each $n \in \mathbb{N}$, $\langle\tilde\Sigma_a\rangle_{a\in A_n}$ has T-removable intersections. **P** Induce on n. If $n = 0$ then $A_n = \{\emptyset\}$ and the result is trivial. For the inductive step to $n + 1 \ge 1$, apply Lemma 1D, as follows. The inductive hypothesis tells us that $\langle\tilde\Sigma_a\rangle_{a\in A_n}$ has T-removable intersections, and we know that $\mathrm{T}\cap\tilde\Sigma_a$ is always metrically dense in $\tilde\Sigma_a$. Set

$$\Sigma^* = \bigvee_{a\in A_n} \tilde\Sigma_a = \tilde\Sigma_d$$

where $d = \bigcup_{m<n} \Gamma_m$ is the largest member of A_n. Define $g : \Gamma_n \to A_n$ by setting $g(\gamma) = \{\delta : \delta < \gamma\}$. Then $\gamma \wedge \delta \in d$ for all distinct γ, $\delta \in \Gamma_n$, so $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma_n}$ is relatively independent over Σ^*, by (b-i) just above. If $\gamma \in \Gamma_n$ and $\delta \in d$, $\gamma \wedge \delta \in g(\gamma)$, so Σ_γ and Σ^* are relatively independent over $\tilde\Sigma_{g(\gamma)}$, by (b-ii). Of course $\mathrm{T} \cap \Sigma_\gamma$ is metrically dense in Σ_γ for every $\gamma \in \Gamma_n$.

For $a \in A_{n+1}$, set $\phi(a) = a \cap \Gamma_n$ and

$$f(a) = a \setminus \phi(a) = a \cap \bigcup_{m<n} \Gamma_m \in A_n.$$

If $\gamma \in \phi(a)$ then $g(\gamma) \subseteq a$, by the definition of A, so $g(\gamma) \subseteq f(a)$ and $\tilde\Sigma_{g(\gamma)} \subseteq \tilde\Sigma_{f(a)}$. Finally, by (b-ii), $\bigvee_{\gamma\in\phi(a)} \Sigma_\gamma$ and $\Sigma^* \vee \bigvee_{\gamma\in\Gamma_n\setminus\phi(a)} \Sigma_\gamma$ are relatively independent over $\tilde\Sigma_{f(a)}$, because if $\gamma \in \phi(a)$ and $\delta \in d \cup (\Gamma_n \setminus \phi(a))$ then $\gamma \wedge \delta \in g(\gamma) \subseteq f(a)$.

So all the hypotheses of Lemma 1D are satisfied, and

$$\langle\tilde\Sigma_{f(a)} \vee \bigvee_{\gamma\in\phi(a)} \Sigma_\gamma\rangle_{a\in A_{n+1}} = \langle\tilde\Sigma_a\rangle_{a\in A_{n+1}}$$

has T-removable intersections. Thus the induction proceeds. **Q**

(d) Because Γ is finite, there is some n such that $A = A_n$. Now, for each $\gamma \in \Gamma$, set $e_\gamma = \{\delta : \delta \le \gamma\}$; then $e_\gamma \in A$ and $\Sigma_\gamma = \tilde\Sigma_{e_\gamma}$. By Lemma 1Bb, or otherwise, $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma}$ has T-removable intersections, as required.

(e) Thus the theorem is true when Γ is finite. For the general case, take any finite $\Gamma_0 \subseteq \Gamma$ and set $\Gamma' = \{\inf a : a \subseteq \Gamma_0 \text{ is non-empty}\}$. Then Γ' is finite and closed under $\wedge$, and $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma'}$ satisfies the conditions of the theorem. So $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma'}$ and $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma_0}$ have T-removable intersections. As Γ_0 is arbitrary, $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma}$ has T-removable intersections, and the proof is complete.

1F Corollary Let (X, Σ, μ) be a probability space, $\langle\Sigma_i\rangle_{i\in I}$ an independent family of σ-subalgebras of Σ, and T a subalgebra of Σ such that $\mathrm{T} \cap \Sigma_i$ is metrically dense in Σ_i for every $i \in I$. For $J \subseteq I$ set $\tilde\Sigma_J = \bigvee_{i\in J} \Sigma_i$. Then $\langle\tilde\Sigma_J\rangle_{J\subseteq I}$ has T-removable intersections.

proof Of course T is metrically dense in $\tilde\Sigma_J$ for every J, and $\tilde\Sigma_J \subseteq \tilde\Sigma_K$ whenever $J \subseteq K \subseteq I$. If $J \subseteq I$ and $\mathcal{K}$, $\mathcal{K}'$ are families of subsets of I such that $J \cap K \in \mathcal{K}'$ for every $K \in \mathcal{K}$, set $L = \bigcup\mathcal{K}$ and $L' = \bigcup\mathcal{K}'$; then $\bigvee_{K\in\mathcal{K}} \tilde\Sigma_K = \tilde\Sigma_L$ and $\bigvee_{K\in\mathcal{K}'} \tilde\Sigma_K = \tilde\Sigma_{L'}$. Also $J \cap L \subseteq L'$. So $\tilde\Sigma_{J\setminus L'}$, $\tilde\Sigma_{L\setminus L'}$ and $\tilde\Sigma_{L'}$ are independent; consequently $\tilde\Sigma_{J\setminus L'}$ and $\tilde\Sigma_{L\setminus L'}$ are relatively independent over $\tilde\Sigma_{L'}$, $\tilde\Sigma_{J\cup L'} = \tilde\Sigma_J \vee \tilde\Sigma_{L'}$ and $\tilde\Sigma_{L\cup L'} = \tilde\Sigma_L \vee \tilde\Sigma_{L'}$ are relatively independent over $\tilde\Sigma_{L'}$, and $\tilde\Sigma_J$ and $\tilde\Sigma_L$ are relatively independent over $\tilde\Sigma_{L'}$. But this is just what is required to make Theorem 1E applicable.

2 Permutation-invariant measures on $\mathcal{P}([I]^{<\omega})$

2A Definitions and basic properties Let I be a set. Then $\mathcal{P}([I]^{<\omega})$ is a compact Hausdorff space, if we give it its usual topology, generated by sets of the form $\{R : a \in R,\ b \notin R\}$ for finite sets a, $b \subseteq I$. Let G_I be the set of permutations of I, and for $\phi \in G_I$, $R \subseteq [I]^{<\omega}$ set

$$\phi\bullet R = \{\phi[a] : a \in R\},$$

so that $\bullet$ is an action of G_I on $\mathcal{P}([I]^{<\omega})$, and $R \mapsto \phi\bullet R$ is a homeomorphism for every $\phi \in G_I$. (It is in fact easy to check that if I is given its discrete topology and G_I is given its topology of pointwise convergence as a subset of I^I, then the action is continuous.) Let P_I be the set of Radon probability measures on $\mathcal{P}([I]^{<\omega})$. Then we have an action of G_I on P_I defined by saying that

$$\phi\bullet E = \{\phi\bullet R : R \in E\}$$

for $\phi \in G_I$ and $E \subseteq \mathcal{P}([I]^{<\omega})$, and

$$(\phi\bullet\mu)(E) = \mu(\phi^{-1}\bullet E)$$

for $\phi \in G_I$, $\mu \in P_I$ and Borel sets $E \subseteq \mathcal{P}([I]^{<\omega})$. Because $R \mapsto \phi\bullet R$ is a homeomorphism, the map $\mu \mapsto \phi\bullet\mu$ is a homeomorphism when P_I is given its narrow topology corresponding to the weak* topology on $C(\mathcal{P}([I]^{<\omega}))^*$ (Fremlin 03, §437).

If $\mu \in P_I$ and G is a subgroup of G_I, I will say that μ is G-**invariant** if $\mu = \phi\bullet\mu$ for every $\phi \in G$. Note that $\{\mu : \mu \in P_I$ is G-invariant$\}$ is always a closed subset of P_I.

For $R \subseteq [I]^{<\omega}$ and $J \subseteq I$ I write $R\lceil J$ for the trace $R \cap \mathcal{P}J \subseteq [J]^{<\omega}$ of R on J. If we take $\mathcal{V}$ to be the family of sets of the form $\{R : R \subseteq [I]^{<\omega},\ R\lceil J = S\}$ where $J \subseteq I$ is finite and $S \subseteq \mathcal{P}J$, then $\mathcal{V}$ is a base for the topology of $\mathcal{P}([I]^{<\omega})$ and every open-and-closed subset of $\mathcal{P}([I]^{<\omega})$ is expressible as a finite disjoint union of members of $\mathcal{V}$. Consequently two Radon measures on $\mathcal{P}([I]^{<\omega})$ which agree on $\mathcal{V}$ will be the same.

If I, J are sets and $f : I \to J$ is a function, I define $\tilde{f} : \mathcal{P}([J]^{<\omega}) \to \mathcal{P}([I]^{<\omega})$ by setting $\tilde{f}(R) = \{a : a \in [I]^{<\omega},\ f[a] \in R\}$ for $R \subseteq [J]^{<\omega}$. Note that $\tilde{f}$ is continuous, since $\{R : a \in \tilde{f}(R)\} = \{R : f[a] \in R\}$ is a basic open-and-closed set in $\mathcal{P}([J]^{<\omega})$ for every $a \in [I]^{<\omega}$. If $I \subseteq J$ and f is the identity function, then $\tilde{f}(R) = R\lceil I$ for every $R \subseteq [J]^{<\omega}$. Observe that when $\phi \in G_I$ and $R \subseteq [I]^{<\omega}$ then $\tilde{\phi}(R) = \phi^{-1}\bullet R$.

2B Lemma Let I be a set. For $J \subseteq I$ write Σ_J for the σ-algebra of subsets of $\mathcal{P}([I]^{<\omega})$ generated by sets of the form $E_a = \{R : a \in R \subseteq [I]^{<\omega}\}$ where $a \in [J]^{<\omega}$. Let $\mathbb{T}$ be the set of σ-algebras of subsets of $\mathcal{P}([I]^{<\omega})$.

(a) We have an action of G_I on $\mathcal{P}(\mathcal{P}([I]^{<\omega}))$ defined by saying that

$$\phi\bullet E = \{\phi\bullet R : R \in E\} = \{R : R \subseteq [I]^{<\omega},\ \phi^{-1}\bullet R \in E\}$$

for $\phi \in G_I$ and $E \subseteq \mathcal{P}([I]^{<\omega})$. If $a \in [I]^{<\omega}$ and $\phi \in G_I$, $\phi\bullet E_a = E_{\phi[a]}$.

(b) We have an action of G_I on $\mathbb{R}^{\mathcal{P}([I]^{<\omega})}$ defined by saying that

$$(\phi\bullet f)(R) = f(\phi^{-1}\bullet R)$$

for $\phi \in G_I$, $f : \mathcal{P}([I]^{<\omega}) \to \mathbb{R}$ and $R \subseteq [I]^{<\omega}$.

(c)(i) We have an action of G_I on $\mathbb{T}$ defined by saying that

$$\phi\bullet\mathrm{T} = \{\phi\bullet E : E \in \mathrm{T}\}$$

for $\phi \in G_I$ and $\mathrm{T} \in \mathbb{T}$.

(ii) If $\langle \mathrm{T}_\gamma\rangle_{\gamma\in\Gamma}$ is a family in $\mathbb{T}$, then $\phi\bullet(\bigvee_{\gamma\in\Gamma} \mathrm{T}_\gamma) = \bigvee_{\gamma\in\Gamma} \phi\bullet\mathrm{T}_\gamma$ for every $\phi \in G_I$.

(iii) If $\phi \in G_I$ and $J \subseteq I$ then $\phi\bullet\Sigma_J = \Sigma_{\phi[J]}$.

(iv) If $\mathrm{T} \in \mathbb{T}$ and $f : \mathcal{P}([I]^{<\omega}) \to \mathbb{R}$ is T-measurable, then $\phi\bullet f$ is $\phi\bullet\mathrm{T}$-measurable for every $\phi \in G_I$.

proof (a) It is elementary to check that the formulae for $\phi\bullet E$ agree and define an action of G_I on $\mathcal{P}(\mathcal{P}([I]^{<\omega}))$, just because we are starting from an action of G_I on $\mathcal{P}([I]^{<\omega})$.

If $a \in [I]^{<\omega}$ then

$$\phi\bullet E_a = \{\phi\bullet R : a \in R\} = \{\phi\bullet R : \phi[a] \in \phi\bullet R\} = E_{\phi[a]}.$$

(b) Again because we are starting from an action of G_I on $\mathcal{P}([I]^{<\omega})$, the formula defines an action of G_I on $\mathbb{R}^{\mathcal{P}([I]^{<\omega})}$.

(c) These are equally elementary. For any $\phi \in G_I$, the map $E \mapsto \phi\bullet E : \mathcal{P}(\mathcal{P}([I]^{<\omega})) \to \mathcal{P}(\mathcal{P}([I]^{<\omega}))$ is a Boolean automorphism, so takes σ-algebras to σ-algebras and preserves joins of σ-algebras. If $J \subseteq G_I$ then $\phi\bullet\Sigma_J = \{\phi\bullet E : E \in \Sigma_J\}$ is the σ-algebra generated by

$$\{\phi\bullet E_a : a \in [J]^{<\omega}\} = \{E_{\phi[a]} : a \in [J]^{<\omega}\} = \{E_a : a \in [\phi[J]\,]^{<\omega}\},$$

which is $E_{\phi[J]}$. If $T \in \mathbb{T}$ and $f : \mathcal{P}([I]^{<\omega}) \to \mathbb{R}$ is measurable and $\alpha \in \mathbb{R}$, then

$$\{R : (\phi\bullet f)(R) > \alpha\} = \{R : f(\phi^{-1}\bullet R) > \alpha\} = \{R : \phi^{-1}\bullet R \in E\}$$

(where $E = \{S : f(S) > \alpha\} \in T$)

$$= \phi\bullet E \in \phi\bullet T,$$

so $\phi\bullet f$ is $(\phi\bullet T)$-measurable.

2C Lemma Let I be a set, G a subgroup of G_I and μ a G-invariant measure in P_I. Write $\mathbb{T}_\mu$ for the set of σ-subalgebras of $\operatorname{dom}\mu$. For $\mathrm{T} \in \mathbb{T}_\mu$, write V_{T} for the norm-closed Riesz subspace of $L^1(\mu)$ consisting of equivalence classes of T-measurable functions, and $A_{\mathrm{T}} : L^1(\mu) \to V_{\mathrm{T}}$ for the conditional expectation operator (Fremlin 01, 242J).

(a)(i) We have an action of G on $L^0(\mu)$ defined by saying that

$$\phi\bullet f^\bullet = (\phi\bullet f)^\bullet$$

whenever $\phi \in G$ and $f : \mathcal{P}([I]^{<\omega}) \to \mathbb{R}$ is $(\operatorname{dom}\mu)$-measurable. For any $p \in [1,\infty]$, $u \in L^p(\mu)$ and $\phi \in G$, $\phi\bullet u \in L^p(\mu)$ and $\|\phi\bullet u\|_p = \|u\|_p$.

(ii) If $\phi \in G$, $J \subseteq I$ and $\phi(i) = i$ for $i \in J$, then $\phi\bullet u = u$ for every $u \in V_{\Sigma_J}$.

(b)(i) If $\mathrm{T} \in \mathbb{T}_\mu$ and $\phi \in G$, then $\phi\bullet\mathrm{T}$, as defined in 2B, belongs to $\mathbb{T}_\mu$; so we have an action of G on $\mathbb{T}_\mu$.

(ii) $\phi\bullet u \in V_{\phi\bullet\mathrm{T}}$ whenever $\mathrm{T} \in \mathbb{T}_\mu$, $u \in V_{\mathrm{T}}$ and $\phi \in G$.

(iii) If $\phi \in G$, $\mathrm{T} \in \mathbb{T}_\mu$ and $u \in L^1(\mu)$, then $\phi\bullet A_{\mathrm{T}}u = A_{\phi\bullet\mathrm{T}}(\phi\bullet u)$.

proof (a)(i) As in 2B, this is just because we are starting from an action of G on $\mathcal{P}([I]^{<\omega})$ and μ is G-invariant (Fremlin 03, 441J-441K).

(ii) If $a \in [J]^{<\omega}$ then $\phi\bullet E_a = E_{\phi[a]} = E_a$, as noted in 2Ba. Now $\{E : E \subseteq \mathcal{P}([I]^{<\omega}),\ \phi\bullet E = E\}$ is a σ-algebra of sets, so includes Σ_J. From the formulae in the proof of 2Bc we see that $\phi\bullet f = f$ for every Σ_J-measurable f, so that $\phi\bullet u = u$ for every $u \in V_{\Sigma_J}$.

(b)(i) Because μ is G-invariant, $\phi\bullet(\operatorname{dom}\mu) = \operatorname{dom}\mu$, so $\phi\bullet\mathrm{T} \in \mathbb{T}_\mu$ for every $\mathrm{T} \in \mathbb{T}_\mu$.

(ii) This is immediate from 2B(c-iv) and (a-i) here.

(iii) Express u as $f^\bullet$ where $f : \mathcal{P}([I]^{<\omega}) \to \mathbb{R}$ is μ-integrable. Take any $E \in \phi\bullet\mathrm{T}$. Then

$$\int_E \phi\bullet u = \int_E \phi\bullet f = \int (\phi\bullet f) \times \chi E = \int (\phi\bullet f) \times (\phi\bullet\phi^{-1}\bullet\chi E)$$
$$= \int \phi\bullet(f \times (\phi^{-1}\bullet\chi E)) = \int f \times (\phi^{-1}\bullet\chi E)$$

(because μ is ϕ-invariant)

$$= \int_{\phi^{-1}\bullet E} u = \int_{\phi^{-1}\bullet E} A_{\mathrm{T}}u$$

because $\phi^{-1}\bullet E \in \mathrm{T}$. Applying this to $A_{\mathrm{T}}u$ in place of u, we see that

$$\int_E \phi\bullet(A_{\mathrm{T}}u) = \int_{\phi^{-1}\bullet E} A_{\mathrm{T}}(A_{\mathrm{T}}u) = \int_{\phi^{-1}\bullet E} A_{\mathrm{T}}u = \int_E \phi\bullet u,$$

and this is true for every $E \in \phi\bullet\mathrm{T}$. Since (ii) tells us that $\phi\bullet(A_{\mathrm{T}}u) \in V_{\phi\bullet\mathrm{T}}$, $\phi\bullet(A_{\mathrm{T}}u)$ satisfies the defining characterization of $A_{\phi\bullet\mathrm{T}}(\phi\bullet u)$, and they are equal.

2D I continue with the notation from 2B and 2C.

Lemma Let I be a set. Let $\langle J_\gamma\rangle_{\gamma\in\Gamma}$ be a non-empty finite family of subsets of I, and set $\mathrm{T} = \bigvee_{\gamma\in\Gamma} \Sigma_{J_\gamma}$. Let $K \subseteq I$ be finite, and $S \subseteq \mathcal{P}K$; for each $\gamma \in \Gamma$ let $K_\gamma \subseteq I$ be finite and $S_\gamma \subseteq \mathcal{P}K_\gamma$. Set $E = \{R : R \subseteq [I]^{<\omega},\ R\lceil K = S\}$ and $F = \bigcap_{\gamma\in\Gamma}\{R : R \subseteq [I]^{<\omega},\ R\lceil K_\gamma = S_\gamma\}$.

Let G be a subgroup of G_I such that for every finite $L \subseteq I$ there is a $\phi \in G$ such that

$$\phi(i) = i \text{ for every } i \in K, \quad \phi[K_\gamma] \cup \phi^n[L \cap J_\gamma] \subseteq J_\gamma \text{ for every } n \in \mathbb{Z} \text{ and } \gamma \in \Gamma.$$

Let μ be a G-invariant measure in P_I. Then E and F are relatively independent over T with respect to μ.

Remark For the basic theory of relative independence, see Fremlin 03, §458.

proof (a) For $L \subseteq I$ set $\mathrm{T}_L = \bigvee_{\gamma\in\Gamma} \Sigma_{J_\gamma\cap L} \subseteq \mathrm{T}$. For any $u \in L^2(\mu)$ and $\epsilon > 0$ there is a finite $L \subseteq I$ such that $\|A_{\mathrm{T}'}u - A_{\mathrm{T}}u\|_2 \le \epsilon$ whenever T' is a σ-subalgebra of T including T_L. **P** Because T is the σ-algebra generated by

$$\bigcup_{L\subseteq I \text{ is finite}}\bigcup_{\gamma\in\Gamma}\Sigma_{J_\gamma\cap L},$$

$\bigcup_{L\subseteq I \text{ is finite}} L^2(\mu)\cap V_{\mathrm{T}_L}$ is norm-dense in $L^2(\mu)\cap V_{\mathrm{T}}$, and there are a finite $L\subseteq I$ and a $v\in L^2(\mu)\cap V_{\mathrm{T}_L}$ such that $\|v-A_{\mathrm{T}}u\|_2\le\epsilon$. If now $\mathrm{T}_L\subseteq\mathrm{T}'\subseteq\mathrm{T}$, $v\in L^2(\mu)\cap V_{\mathrm{T}'}$, while $A_{\mathrm{T}'}\restriction L^2(\mu)$ is the orthogonal projection onto $L^2(\mu)\cap V_{\mathrm{T}'}$ (Fremlin 01, 244Nb), so

$$\|A_{\mathrm{T}'}u-A_{\mathrm{T}}u\|_2=\|A_{\mathrm{T}'}A_{\mathrm{T}}u-A_{\mathrm{T}}u\|_2\le\|v-A_{\mathrm{T}}u\|_2\le\epsilon.\ \mathbf{Q}$$

(b) Let g, $h:\mathcal{P}([I]^{<\omega})\to[0,1]$ be T-measurable functions which are conditional expectations of χE, χF respectively on T. Set $u=\chi E^\bullet$ and $v=\chi F^\bullet$, so that $g^\bullet=A_{\mathrm{T}}u$ and $h^\bullet=A_{\mathrm{T}}v$. Let $\epsilon>0$; by (a), there is an $L\in[I]^{<\omega}$ such that

$$\|A_{\mathrm{T}'}u-A_{\mathrm{T}}u\|_2\le\epsilon,\quad\|A_{\mathrm{T}'}v-A_{\mathrm{T}}v\|_2\le\epsilon,\quad\|A_{\mathrm{T}'}(u\times v)-A_{\mathrm{T}}(u\times v)\|_2\le\epsilon$$

whenever T' is a σ-subalgebra of T including T_L. We can suppose that $L\supseteq\bigcup_{\gamma\in\Gamma}K_\gamma$. By the hypothesis on G, there is a $\phi\in G$ such that $\phi(i)=i$ for every $i\in K$ and $\phi[K_\gamma]\cup\phi^n[L\cap J_\gamma]\subseteq J_\gamma$ for every $n\in\mathbb{Z}$ and $\gamma\in\Gamma$; set $M_\gamma=\bigcup_{n\in\mathbb{Z}}\phi^n[L\cap J_\gamma]$, so that

$$L\cap J_\gamma\subseteq M_\gamma=\phi[M_\gamma]\subseteq J_\gamma$$

for each $\gamma\in\Gamma$.

Now

$$\phi\bullet u=u$$

by 2C(a-ii); also

$$\|\phi\bullet(A_{\mathrm{T}_L}u)-A_{\mathrm{T}}u\|_2\le3\epsilon.$$

$\mathbf{P}$ Set

$$\mathrm{T}'=\bigvee_{\gamma\in\Gamma,n\in\mathbb{Z}}\Sigma_{\phi^n[J_\gamma\cap L]}.$$

Then $\phi\bullet\mathrm{T}'=\mathrm{T}'$, by 2B(c-ii) and 2B(c-iii), so $\phi\bullet(A_{\mathrm{T}'}u)=A_{\mathrm{T}'}(\phi\bullet u)=A_{\mathrm{T}'}u$, by 2C(b-iii). Because $\phi^n[J_\gamma\cap L]\subseteq J_\gamma$ for every γ and n, $\mathrm{T}'\subseteq\mathrm{T}$, and of course $\mathrm{T}_L\subseteq\mathrm{T}'$. But now

$$\begin{aligned}\|\phi\bullet(A_{\mathrm{T}_L}u)-A_{\mathrm{T}}u\|_2&\le\|\phi\bullet(A_{\mathrm{T}_L}u)-\phi\bullet(A_{\mathrm{T}'}u)\|_2+\|A_{\mathrm{T}'}u-A_{\mathrm{T}}u\|_2\\&=\|A_{\mathrm{T}'}u-A_{\mathrm{T}_L}u\|_2+\|A_{\mathrm{T}'}u-A_{\mathrm{T}}u\|_2\\&\le2\|A_{\mathrm{T}'}u-A_{\mathrm{T}}u\|_2+\|A_{\mathrm{T}}u-A_{\mathrm{T}_L}u\|_2\le3\epsilon.\ \mathbf{Q}\end{aligned}$$

(c) Set

$$\mathrm{T}^*=\bigvee_{\gamma\in\Gamma}\Sigma_{K_\gamma\cup M_\gamma}.$$

Then T^* and

$$\phi\bullet\mathrm{T}^*=\bigvee_{\gamma\in\Gamma}\Sigma_{\phi[K_\gamma]\cup M_\gamma}$$

include T_L, while $\phi\bullet\mathrm{T}^*\subseteq\mathrm{T}$ because $\phi[K_\gamma]\cup M_\gamma\subseteq J_\gamma$ for every γ. Also $F\in\mathrm{T}^*$, because $\{R:R\subseteq[I]^{<\omega}$, $R\restriction K_\gamma=S_\gamma\}$ belongs to $\Sigma_{K_\gamma}\subseteq\mathrm{T}^*$ for every γ. Now

$$\begin{aligned}&\|A_{\mathrm{T}}u\times A_{\mathrm{T}}v-A_{\mathrm{T}_L}u\times A_{\mathrm{T}_L}v\|_2\\&\qquad\le\|A_{\mathrm{T}}u\times(A_{\mathrm{T}}v-A_{\mathrm{T}_L}v)\|_2+\|(A_{\mathrm{T}}u-A_{\mathrm{T}_L}v)\times A_{\mathrm{T}_L}v\|_2\\&\qquad\le\|A_{\mathrm{T}}v-A_{\mathrm{T}_L}v\|_2+\|A_{\mathrm{T}}u-A_{\mathrm{T}_L}v\|_2\end{aligned}$$

(because $\|A_{\mathrm{T}}u\|_\infty$ and $\|A_{\mathrm{T}_L}v\|_\infty$ are both at most 1)

$$\le2\epsilon,$$

and

$$\begin{aligned}\|A_{\mathrm{T}_L}(u\times v)-A_{\mathrm{T}_L}u\times A_{\mathrm{T}_L}v\|_2&=\|A_{\mathrm{T}_L}A_{\mathrm{T}^*}(u\times v)-A_{\mathrm{T}_L}u\times A_{\mathrm{T}_L}v\|_2\\&=\|A_{\mathrm{T}_L}(v\times A_{\mathrm{T}^*}u)-A_{\mathrm{T}_L}(v\times A_{\mathrm{T}_L}u)\|_2\end{aligned}$$

(because $v\in V_{\mathrm{T}^*}$ and $A_{T_L}u\in V_{T_L}$, see Fremlin 01, 242L)

$$\leq \|v \times A_{\mathrm{T}^*}u - v \times A_{\mathrm{T}_L}u\|_2 \leq \|A_{\mathrm{T}^*}u - A_{\mathrm{T}_L}u\|_2$$

(because $\|v\|_\infty \leq 1$)

$$= \|\phi\bullet(A_{\mathrm{T}^*}u) - \phi\bullet(A_{\mathrm{T}_L}u)\|_2$$
$$= \|A_{\phi\bullet\mathrm{T}^*}(\phi\bullet u) - \phi\bullet(A_{\mathrm{T}_L}u)\|_2$$
$$\leq \|A_{\phi\bullet\mathrm{T}^*}u - A_{\mathrm{T}}u\|_2 + \|A_{\mathrm{T}}u - \phi\bullet(A_{\mathrm{T}_L}u)\|_2 \leq 4\epsilon.$$

Putting these together,

$$\begin{aligned}\|A_{\mathrm{T}}(u \times v) - A_{\mathrm{T}}u \times A_{\mathrm{T}}v\|_2 &\leq \|A_{\mathrm{T}}(u \times v) - A_{\mathrm{T}_L}(u \times v)\|_2 \\ &\qquad + \|A_{\mathrm{T}_L}(u \times v) - A_{\mathrm{T}_L}u \times A_{\mathrm{T}_L}v\|_2 \\ &\qquad + \|A_{\mathrm{T}}u \times A_{\mathrm{T}}v - A_{\mathrm{T}_L}u \times A_{\mathrm{T}_L}v\|_2 \\ &\leq 7\epsilon.\end{aligned}$$

As ϵ is arbitrary, $A_{\mathrm{T}}(u \times v) = A_{\mathrm{T}}u \times A_{\mathrm{T}}v$, that is, $g \times h$ is a conditional expectation of $\chi(E \cap F)$ on T, and E and F are relatively independent over T.

2E Still in the same notation, we come to a theorem of T.Tao.

Theorem (Tao p06) Let I be an infinite set and $\mathcal{J}$ a filter on I not containing any finite set. Then whenever $J_0 \in \mathcal{J}$ and $\mathcal{K}, \mathcal{K}' \subseteq \mathcal{J}$ are such that $J \cap J_0 \in \mathcal{K}'$ for every $J \in \mathcal{K}$, Σ_{J_0} and $\bigvee_{J\in\mathcal{K}} \Sigma_J$ are relatively independent over $\bigvee_{J\in\mathcal{K}'} \Sigma_J$ with respect to any G_I-invariant measure $\mu \in P_I$.

proof Write T for $\bigvee_{J\in\mathcal{K}'} \Sigma_J$.

(a) Suppose that $\mathcal{K}_0 \subseteq \mathcal{K}$ is finite and not empty, $K \in [J_0]^{<\omega}$, $K_J \in [J]^{<\omega}$ for every $J \in \mathcal{K}_0$, $S \subseteq \mathcal{P}K$, $E = \{R : R \subseteq \mathcal{P}([I]^{<\omega}),\ R\lceil K = S\}$, $S_J \subseteq \mathcal{P}K_J$ for $J \in \mathcal{K}_0$, and $F = \{R : R \subseteq [I]^{<\omega},\ R\lceil K_J = S_J$ for every $J \in \mathcal{K}_0\}$. Then E and F are relatively independent over T. **P** Set $u = \chi E^\bullet$ and $v = \chi F^\bullet$ in $L^2(\mu)$, and take $\epsilon > 0$. Then there is a finite set $\mathcal{K}_1 \subseteq \mathcal{K}'$, including $\{J \cap J_0 : J \in \mathcal{K}_0\}$, such that, setting $\mathrm{T}' = \bigvee_{J\in\mathcal{K}_1} \Sigma_J$,

$$\|A_{\mathrm{T}'}u - A_{\mathrm{T}}u\| \leq \epsilon, \quad \|A_{\mathrm{T}'}v - A_{\mathrm{T}}v\| \leq \epsilon, \quad \|A_{\mathrm{T}'}(u \times v) - A_{\mathrm{T}}(u \times v)\| \leq \epsilon.$$

Set $\Gamma = (\mathcal{K}_0 \times \{0\}) \cup (\mathcal{K}_1 \times \{1\})$, and fix $J_1 \in \mathcal{K}_0$. Define $\langle J'_\gamma\rangle_{\gamma\in\Gamma}$, $\langle K'_\gamma\rangle_{\gamma\in\Gamma}$ and $\langle S'_\gamma\rangle_{\gamma\in\Gamma}$ by saying

— if $\gamma = (J, 0)$ where $J \in \mathcal{K}_0$, then $J'_\gamma = J \cap J_0$, $K'_\gamma = K_J$, $S'_\gamma = S_J$,

— if $\gamma = (J, 1)$ where $J \in \mathcal{K}_1$ then $J'_\gamma = J$, $K'_\gamma = \emptyset$ and $S'_\gamma = S_{J_1}\lceil\emptyset$.

Then

$$K'_\gamma \cap K \subseteq J'_\gamma \in \mathcal{J}$$

for every $\gamma \in \Gamma$; $F = \{R : R\lceil K'_\gamma = S'_\gamma$ for every $\gamma \in \Gamma\}$; and $\mathrm{T}' = \bigvee_{\gamma\in\Gamma} \Sigma_{J'_\gamma}$. Moreover, if $L \subseteq I$ is finite, set $L' = L \cup \bigcup_{\gamma\in\Gamma} K'_\gamma$. Let $M \subseteq \bigcap_{\gamma\in\Gamma} J'_\gamma \setminus (L' \cup K)$ be a set of size $\#(L' \setminus K)$, and let $\phi \in G_I$ be such that $\phi[L' \setminus K] = M$, ϕ^2 is the identity and $\phi(i) = i$ for $i \notin M \cup (L' \setminus K)$. Then $\phi[L' \cap J'_\gamma] \subseteq J'_\gamma$ for every γ; consequently

$$\phi(i) = i \text{ for every } i \in K, \quad \phi[K'_\gamma] \cup \phi^n[L \cap J'_\gamma] \subseteq J'_\gamma \text{ for every } n \in \mathbb{Z} \text{ and } \gamma \in \Gamma.$$

As L is arbitrary, G_I satisfies the condition of Lemma 2D, and we see that $A_{\mathrm{T}'}(u \times v) = A_{\mathrm{T}'}u \times A_{\mathrm{T}'}v$. Consequently $\|A_{\mathrm{T}}(u \times v) - A_{\mathrm{T}}u \times A_{\mathrm{T}}v\|_2 \leq 3\epsilon$. As ϵ is arbitrary, $A_{\mathrm{T}}(u \times v) = A_{\mathrm{T}}u \times A_{\mathrm{T}}v$ and E, F are relatively independent over T. **Q**

(b) Suppose that $\mathcal{K}_0 \subseteq \mathcal{K}$ is finite and not empty, $K \in [J_0]^{<\omega}$, $K_J \in [J]^{<\omega}$ for every $J \in \mathcal{K}_0$, $E \in \Sigma_K$ and $F \in \bigvee_{J\in\mathcal{K}_0} \Sigma_{K_J}$. Then E and F are relatively independent over T. **P** E and F can be expressed as $\bigcup_{l\in L} E_l$ and $\bigcup_{m\in M} F_m$ where $\langle E_l\rangle_{l\in L}$ is a finite partition of E, $\langle F_m\rangle_{m\in M}$ is a finite partition of F, and E_l, F_m satisfy the conditions of (a) for suitable S, S_J. In this case E_l and F_m are relatively independent over T for all l and m, so E and F also are. **Q**

(c) Suppose that $E \in \Sigma_{J_0}$ and $F \in \bigvee_{J\in\mathcal{K}} \Sigma_J$. Then E and F are relatively independent over T. **P** If $\mathcal{K}$ is empty, this is trivial. Otherwise, for any $\epsilon > 0$ there are E_1, F_1 such that $\mu(E \triangle E_1) \leq \epsilon^2$ and $\mu(F \triangle F_1) \leq \epsilon^2$ and $E_1 \in \Sigma_K$ for some $K \in [J_0]^{<\omega}$, while $F_1 \in \bigvee_{J\in\mathcal{K}_0} \Sigma_{K_J}$ for some non-empty finite $\mathcal{K}_0 \subseteq \mathcal{K}$ and suitable finite sets $K_J \subseteq J$. In this case, setting $u = \chi E^\bullet$, $v = \chi F^\bullet$, $u_1 = \chi E_1^\bullet$ and $v_1 = \chi F_1^\bullet$,

$$\|u - u_1\|_2 \le \epsilon, \quad \|v - v_1\|_2 \le \epsilon, \quad \|u \times v - u_1 \times v_1\|_2 \le 2\epsilon,$$

$$\|A_{\mathrm{T}}u - A_{\mathrm{T}}u_1\|_2 \le \epsilon, \quad \|A_{\mathrm{T}}v - A_{\mathrm{T}}v_1\|_2 \le \epsilon,$$

$$\|A_{\mathrm{T}}(u \times v) - A_{\mathrm{T}}(u_1 \times v_1)\|_2 \le 2\epsilon,$$

$$\|(A_{\mathrm{T}}u \times A_{\mathrm{T}}v) - (A_{\mathrm{T}}u_1 \times A_{\mathrm{T}}v_1)\|_2 \le 2\epsilon,$$

while

$$A_{\mathrm{T}}(u_1 \times v_1) = A_{\mathrm{T}}u_1 \times A_{\mathrm{T}}v_1$$

by (b); consequently

$$\|A_{\mathrm{T}}(u \times v) - A_{\mathrm{T}}u \times A_{\mathrm{T}}v\|_2 \le 4\epsilon.$$

As ϵ is arbitrary, $A_{\mathrm{T}}(u \times v) = A_{\mathrm{T}}u \times A_{\mathrm{T}}v$ and E, F are relatively independent over T. **Q**

But this is just the definition of relative independence for the σ-algebras Σ_{J_0} and $\bigvee_{J \in \mathcal{K}} \Sigma_J$.

2F The following is now easy.

Theorem (Tao p06) Let I be a set and $\mathcal{J}$ a filter on I not containing any finite set. Let T be the algebra of open-and-closed subsets of $\mathcal{P}([I]^{<\omega})$, and μ a G_I-invariant measure in P_I. Then, defining Σ_J as in 2B, $\langle \Sigma_J \rangle_{J \in \mathcal{J}}$ has T-removable intersections with respect to μ.

proof If $J \subseteq I$ and $a \in [J]^{<\omega}$ then $\{R : a \in R\}$ belongs to $\mathrm{T} \cap \Sigma_J$; accordingly Σ_J is the σ-algebra generated by $\mathrm{T} \cap \Sigma_J$ and $\mathrm{T} \cap \Sigma_J$ is metrically dense in Σ_J. Ordering $\mathcal{J}$ by $\subseteq$, condition (ii) of 1E is obviously satisfied. And condition (iii) is just Theorem 2E. So Theorem 1E gives the result.

3 The hypergraph removal lemma and Szemerédi's theorem

3A Construction Suppose we are given a sequence $\langle (m_n, T_n) \rangle_{n \in \mathbb{N}}$ and a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ such that

(α) $\langle m_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\mathbb{N} \setminus \{0\}$ and $\lim_{n \to \mathcal{F}} m_n = \infty$,

(β) $T_n \subseteq \mathcal{P} m_n$ for each n.

Then for any set I there is a G_I-invariant $\mu \in P_I$ such that

$$\mu\{R : R\lceil K = S\} = \lim_{n \to \mathcal{F}} \tfrac{1}{m_n^{\#(K)}} \#(\{z : z \in m_n^K, \tilde{z}(T_n) = S\})$$

whenever $K \subseteq I$ is finite and $S \subseteq \mathcal{P}K$.

proof (a) For each $n \in \mathbb{N}$ let ν_n be the usual measure on m_n^I, the product of I copies of the uniform probability measure on the finite set m_n. Then $w \mapsto \tilde{w}(T_n) : m_n^I \to \mathcal{P}([I]^{<\omega})$ is continuous, since for any $a \in [I]^{<\omega}$ the set $\{w : a \in \tilde{w}(T_n)\} = \{w : w(a) \in T_n\}$ is determined by coordinates in the finite set a. So we have a corresponding Radon probability measure μ_n on $\mathcal{P}([I]^{<\omega})$ defined by saying that $\mu_n E = \nu_n\{w : \tilde{z}(T_n) \in E\}$ for every Borel set $E \subseteq \mathcal{P}([I]^{<\omega})$ (Fremlin 03, 418I). If $K \subseteq I$ is finite and $w \in m_n^I$, then

$$\tilde{w}(T_n)\lceil K = \{a : a \subseteq K, w[a] \in T_n\} = \{a : a \subseteq K, (w\restriction K)[a] \in T_n\} = (w\restriction K)^{\sim}(T_n).$$

So if $S \subseteq \mathcal{P}K$, then

$$\begin{aligned} \mu_n\{R : R\lceil K = S\} &= \nu_n\{w : w \in m_n^I, \tilde{w}(T_n)\lceil K = S\} \\ &= \nu_n\{w : w \in m_n^I, (w\restriction K)^{\sim}(T_n) = S\} \\ &= \tfrac{1}{m_n^{\#(K)}} \#(\{z : z \in m_n^K, \tilde{z}(T_n) = S\}). \end{aligned}$$

Let μ be the limit $\lim_{n \to \mathcal{F}} \mu_n$ in the narrow topology on P_I; then

$$\begin{aligned} \mu\{R : R\lceil K = S\} &= \lim_{n \to \mathcal{F}} \mu_n\{R : R\lceil K = S\} \\ &= \lim_{n \to \mathcal{F}} \tfrac{1}{m_n^{\#(K)}} \#(\{z : z \in m_n^K, \tilde{z}(T_n) = S\}) \end{aligned}$$

whenever $K \subseteq I$ is finite and $S \subseteq \mathcal{P}K$.

(b) Now let $\phi : I \to I$ be any bijection and $\tilde{\phi} : \mathcal{P}([I]^{<\omega}) \to \mathcal{P}([I]^{<\omega})$ the corresponding bijection. Then for any finite $K \subseteq I$,

$$\begin{aligned}\tilde{\phi}(R)\lceil K &= \{a : a \subseteq K,\, a \in \tilde{\phi}(R)\}\\ &= \{a : a \subseteq K,\, \phi[a] \in R\} = \{a : a \in [I]^{<\omega},\, \phi[a] \in R\lceil\phi[K]\}.\end{aligned}$$

Fix $n \in \mathbb{N}$ for the moment. If $K \subseteq I$ is finite, and $S \subseteq \mathcal{P}K$, then

$$\begin{aligned}\mu_n\tilde{\phi}^{-1}\{R : R\lceil K = S\} &= \mu_n\{R : \tilde{\phi}(R)\lceil K = S\}\\ &= \mu_n\{R : S = \{a : a \in [I]^{<\omega},\, \phi[a] \in R\lceil\phi[K]\}\}\\ &= \mu_n\{R : R\lceil\phi[K] = \{\phi[a] : a \in S\}\}\\ &= \nu_n\{w : \tilde{w}(T_n)\lceil\phi[K] = \{\phi[a] : a \in S\}\}\\ &= \nu_n\{w : \{a : a \subseteq \phi[K],\, w[a] \in T_n\} = \{\phi[a] : a \in S\}\}\\ &= \nu_n\{w : \{\phi[a] : a \subseteq K,\, w[\phi[a]] \in T_n\} = \{\phi[a] : a \in S\}\}\\ &= \nu_n\{w : \{a : a \subseteq K,\, (w\phi)[a] \in T_n\} = S\}\\ &= \nu_n\{w : \{a : a \subseteq K,\, w[a] \in T_n\} = S\}\end{aligned}$$

(because $w \mapsto w\phi : m_n^I \to m_n^I$ is an automorphism for the measure ν_n)

$$= \nu_n\{w : \tilde{w}(T_n)\lceil K = S\} = \mu_n\{R : R\lceil K = S\}.$$

So μ_n and $\mu_n\tilde{\phi}^{-1}$ agree on the family $\mathcal{V}$ of basic open-and-closed sets. As this is true for every n, we also have $\mu V = \mu\tilde{\phi}^{-1}[V]$ for every $V \in \mathcal{V}$, and $\mu = \mu\tilde{\phi}^{-1}$. As ϕ is arbitrary, μ is G_I-invariant.

Remark In fact it is the case that the measures in P_I obtainable by this construction are precisely the extreme points of the set of G_I-invariant measures in P_I; see Fremlin n06.

3B Definition If I, J are sets, $R \subseteq \mathcal{P}I$ and $S \subseteq \mathcal{P}J$, I will say for the purposes of this section that an **embedding** of (I, R) in (J, S) is an injective function $f : I \to J$ such that $f[a] \in S$ for every $a \in R$, that is (when $S \subseteq [J]^{<\omega}$), $R \subseteq \tilde{f}(S)$.

3C Theorem (Nagle Rödl & Schacht 06, Theorem 3) Let L be a finite set with r members, and $T \subseteq \mathcal{P}L$. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever I is a finite set, $R \subseteq \mathcal{P}I$ and the number of embeddings of (L, T) in (I, R) is at most $\delta\#(I)^r$, there is an $S \subseteq \mathcal{P}I$ such that $\#(R \cap [I]^k \setminus S) \le \epsilon\#(I)^k$ for every k and there is no embedding of (L, T) in (I, S).

proof (Tao p06) **?** Suppose, if possible, otherwise.

(a) We have a sequence $\langle(m_n, T_n)\rangle_{n\in\mathbb{N}}$ such that

$m_n \in \mathbb{N} \setminus \{0\}$, $T_n \subseteq \mathcal{P}m_n$; the number of embeddings of (L, T) in (m_n, T_n) is at most $2^{-n}m_n^r$;
if $S \subseteq \mathcal{P}m_n$ and $\#(T_n \cap [m_n]^k \setminus S) \le \epsilon m_n^k$ for every k then there is an embedding of (L, T) in (m_n, S)

for every $n \in \mathbb{N}$. Of course (L, T) always has at least one embedding in (m_n, T_n) so $\lim_{n\to\infty} m_n = \infty$. Let I be an infinite set including L and $\mathcal{F}$ a non-principal ultrafilter on $\mathbb{N}$. Let $\mu \in P_I$ be the G_I-invariant measure defined from $\langle(m_n, T_n)\rangle_{n\in\mathbb{N}}$ and $\mathcal{F}$ by the process of 3A.

(b) Let $\mathcal{J}$ be $\{J : I \setminus L \subseteq J \subseteq I,\, J \cap L \in T\}$. For $J \in \mathcal{J}$ let Σ_J be the σ-algebra of subsets of $\mathcal{P}([I]^{<\omega})$ generated by sets of the form $E_a = \{R : a \in R \subseteq [I]^{<\omega}\}$ where $a \in [J]^{<\omega}$. By Theorem 2F, applied to the filter on I generated by $\{I \setminus L\}$, $\langle\Sigma_J\rangle_{J\in\mathcal{J}}$ has T-removable intersections with respect to μ, where T is the open-and-closed algebra of $\mathcal{P}([I]^{<\omega})$.

(c) Set $E_J = \{R : R \in [I]^{<\omega},\, J \cap L \in R\}$ for $J \in \mathcal{J}$. Then $E_J \in \Sigma_J$ for every $J \in \mathcal{J}$, and

$$\begin{aligned}
\mu(\bigcap_{J\in\mathcal{J}} E_J) = \mu\{R : T \subseteq R\} &= \sum_{T\subseteq T'\subseteq \mathcal{P}L} \mu\{R : R\lceil L = T'\}\\
&= \sum_{T\subseteq T'\subseteq \mathcal{P}L} \lim_{n\to\mathcal{F}} \frac{1}{m_n^r}\#(\{z : z\in m_n^L,\ \tilde{z}(T_n) = T'\})\\
&= \lim_{n\to\mathcal{F}} \frac{1}{m_n^r} \sum_{T\subseteq T'\subseteq \mathcal{P}L} \#(\{z : z\in m_n^L,\ \tilde{z}(T_n) = T'\})\\
&= \lim_{n\to\mathcal{F}} \frac{1}{m_n^r}\#(\{z : z\in m_n^L,\ T\subseteq \tilde{z}(T_n)\})\\
&= \lim_{n\to\mathcal{F}} \frac{1}{m_n^r}\#(\{z : z\in m_n^L \text{ is injective},\ T\subseteq \tilde{z}(T_n)\})\\
&= \lim_{n\to\mathcal{F}} \frac{1}{m_n^r}\#(\{z : z \text{ is an embedding of } (L,T) \text{ in } (m_n,T_n)\}) = 0.
\end{aligned}$$

(d) Take $\eta > 0$ such that $2\eta\#(\mathcal{J}) < \epsilon$, and $\langle F_J\rangle_{J\in\mathcal{J}}$ such that $\bigcap_{J\in\mathcal{J}} F_J = \emptyset$ and $F_J \in \mathrm{T}\cap\Sigma_J$ and $\mu(E_J\setminus F_J) \le \eta$ for every $J\in\mathcal{J}$. Every F_J is open-and-closed, so there is an $M\in[I]^{<\omega}$ such that $L\subseteq M$ and every F_J is determined by coordinates in M, that is,

$$F_J = \{R : R\in[I]^{<\omega},\ R\lceil M \in H_J\}$$

for some $H_J\subseteq\mathcal{P}M$. Because $\bigcap_{J\in\mathcal{J}} F_J$ is empty, so is $\bigcap_{J\in\mathcal{J}} H_J$.

For $J\in\mathcal{J}$ set $E'_J = \{S : J\cap L\in S\subseteq\mathcal{P}M\}$, so that $E_J = \{R : R\subseteq[I]^{<\omega},\ R\lceil M\in E'_J\}$. Note that E'_J, H_J and therefore $E'_J\setminus H_J$ are determined by coordinates in $(J\cap L)\cup(M\setminus L)$.

(e) Let $n\ge r$ be such that

$$\begin{aligned}
\frac{1}{m_n^{\#(M)}}\#(\{z : z\in m_n^M,\ \tilde{z}(T_n)\in E'_J\setminus H_J\}) &\le \eta + \mu\{R : R\lceil M\in E'_J\setminus H_J\}\\
&= \eta + \mu\{R : R\in E_J\setminus F_J\} \le 2\eta
\end{aligned}$$

for every $J\in\mathcal{J}$. For $w\in m_n^{M\setminus L}$ and $J\in\mathcal{J}$ set

$$Q_{wJ} = \{z\restriction J\cap L : w\subseteq z\in m_n^M,\ z\restriction L \text{ is injective},\ \tilde{z}(T_n)\in E'_J\setminus H_J\}.$$

Then

$$\begin{aligned}
\sum_{J\in\mathcal{J},w\in m_n^{M\setminus L}} m_n^{\#(L\setminus J)}\#(Q_{wJ}) &\le \sum_{J\in\mathcal{J},w\in m_n^{M\setminus L}} \#(\{z\restriction L : w\subseteq z\in m_n^M,\ \tilde{z}(T_n)\in E'_J\setminus H_J\})\\
&= \sum_{J\in\mathcal{J}} \#(\{z : z\in m_n^M,\ \tilde{z}(T_n)\in E'_J\setminus H_J\})\\
&\le 2\eta\#(\mathcal{J})m_n^{\#(M)},
\end{aligned}$$

so there must be a $w\in m_n^{M\setminus L}$ such that

$$\textstyle\sum_{J\in\mathcal{J}} m_n^{\#(L\setminus J)}\#(Q_{wJ}) \le 2\eta\#(\mathcal{J})m_n^r < \epsilon m_n^r.$$

(f) Consider

$$\textstyle S = T_n\setminus\bigcup_{k\in\mathbb{N}}\bigcup_{J\in\mathcal{J},\#(J\cap L)=k}\{z[J\cap L] : z\in Q_{wJ}\}.$$

Then

$$\#(T_n\cap[m_n]^k\setminus S) = \#(T_n\cap[m_n]^k\cap \bigcup_{\substack{J\in\mathcal{J}\\ \#(J\cap L)=k}}\{z[J\cap L] : z\in Q_{wJ}\})$$

(because $z\restriction L$ is injective whenever $z\in\bigcup_{J\in\mathcal{J}} Q_{wJ}$)

$$\leq \sum_{J\in\mathcal{J},\#(J\cap L)=k} \#(Q_{wJ})$$
$$= \frac{1}{m_n^{r-k}} \sum_{J\in\mathcal{J},\#(J\cap L)=k} m_n^{\#(L\setminus J)}\#(Q_{wJ})$$
$$\leq \frac{1}{m_n^{r-k}} \sum_{J\in\mathcal{J}} m_n^{\#(L\setminus J)}\#(Q_{wJ}) < \epsilon m_n^k$$

for every k. So by the choice of (m_n, T_n) there is an embedding v of (L, T) in (m_n, S); take $z = v \cup w$, so that $w \subseteq z \in m_n^M$ and $z\restriction L = v$ is injective and $z[a] \in S$ for every $a \in T$. There is some $J \in \mathcal{J}$ such that $\tilde{z}(T_n) \notin H_J$. On the other hand,

$$z[J \cap L] \in S \subseteq T_n,$$

so $J \cap L \in \tilde{z}(T_n)$ and $\tilde{z}(T_n) \in E'_J$. But now $z \in Q_{wJ}$ and $z[J \cap L]$ cannot belong to S. **X**

This contradiction proves the theorem.

3D Corollary: the Hypergraph Removal Lemma For every $\epsilon > 0$ and $r \geq 2$ there is a $\delta > 0$ such that whenever I is a finite set, $R \subseteq [I]^r$ and $\#(\{J : J \in [I]^{r+1},\ [J]^r \subseteq R\}) \leq \delta\#(I)^{r+1}$, there is an $S \subseteq R$ such that $\#(R \setminus S) \leq \epsilon\#(I)^r$ and there is no $J \in [I]^{r+1}$ such that $[J]^r \subseteq S$.

proof Take $\#(L) = r + 1$, $T = [L]^r$ in Theorem 3C.

3E Corollary: Szemerédi's Theorem (Szemerédi 75) For every $\epsilon > 0$ and $r \geq 2$ there is an $n_0 \in \mathbb{N}$ such that whenever $n \geq n_0$, $A \subseteq n$ and $\#(A) \geq \epsilon n$ there is an arithmetic progression of length $r + 1$ in A.

proof (Frankl & Rödl 02) Set $\eta = \frac{1}{r!}(\frac{\epsilon}{2r!})^r$. Take $\delta > 0$ such that whenever I is a finite set, $R \subseteq [I]^r$ and $\#(\{J : J \in [I]^{r+1},\ [J]^r \subseteq R\}) \leq \delta\#(I)^{r+1}$, there is an $S \subseteq R$ such that $\#(R \setminus S) \leq \frac{\eta}{2(r+1)^r}\#(I)^r$ and there is no $J \in [I]^{r+1}$ such that $[J]^r \subseteq S$. Let n_0 be such that $\epsilon n \geq 2r \cdot r!$ and $n(r + 1)^{r+1}\delta \geq 1$ whenever $n \geq n_0$. Take $n \geq n_0$ and $A \subseteq n$ such that $\#(A) \geq \epsilon n$.

Let $C \subseteq n^r$ be the set

$$\{(i_0, i_1, \dots, i_{r-1}) : \textstyle\sum_{j=0}^{r-1}(j + 1)i_j \in A\}.$$

Then $\#(C) \geq \eta n^r$. **P** For $m < r!$ set $A_m = \{i : i \in A,\ i \equiv m \mod r!\}$. Then there is an m such that $\#(A_m) \geq \frac{\epsilon n}{r!}$. Now we have an injection $\phi : [A_m]^r \to C$ given by saying that if $l_0 < \ldots < l_{r-1}$ in A_m then

$$\phi(\{l_0, \dots, l_{r-1}\})(j) = l_0 \text{ if } j = 0$$
$$= \tfrac{1}{j+1}(l_j - l_{j-1}) \text{ if } 0 < j < r.$$

So

$$\#(C) \geq \#([A_m]^r) \geq \frac{1}{r!}(\frac{\epsilon n}{r!} - r)^r \geq \frac{1}{r!}(\frac{\epsilon n}{2r!})^r = \eta n^r. \text{ } \mathbf{Q}$$

Let I be $n \times (r + 1)$ and for $c = (i_0, \dots, i_{r-1}) \in C$ set

$$J_c = \{(i_j, j) : j < r\} \cup \{(\textstyle\sum_{j=0}^{r-1} i_j, r)\} \in [I]^{r+1}.$$

Observe that if c, $c' \in C$ are distinct, then $[J_c]^r \cap [J_{c'}]^r = \emptyset$, since given any face of the r-simplex J_c we can read off all but at most one of the coordinates of c and calculate the last. Set $R = \bigcup_{c\in C}[J_c]^r \subseteq [I]^r$.

? Suppose, if possible, that the only r-simplices $J \in [I]^{r+1}$ such that $[J]^r \subseteq R$ are of the form J_c for some $c \in C$. Then there are at most

$$\#(C) \leq n^r \leq \delta\#(I)^{r+1}$$

such simplices; by the choice of δ, there is an $S \subseteq R$ such that $R \setminus S$ covers no r-simplices and

$$\#(S) \leq \frac{\eta}{2(r+1)^r}\#(I)^r = \frac{\eta}{2}n^r < \#(C).$$

But every J_c must have a face in S, and no two J_c share a face, so this is impossible. **X**

So we have an r-simplex $J \in [I]^{r+1}$, which is not of the form J_c where $c \in C$, such that $[J]^r \subseteq R$. Now since the only faces put into R come from the J_c, and therefore meet each of the $r+1$ levels $n \times \{k\}$ in at most one point, J must be of the form $\{(i_j, j) : j < r\} \cup \{(l, r)\}$. Since $\{(i_j, j) : j < r\}$ is a face of some J_c, $c = (i_0, \dots, i_{r-1}) \in C$. Set $l' = i_0 + \dots + i_{r-1}$; then $l' \neq l$ because $J \neq J_c$. For each $k < r$, $J \setminus \{(i_k, k)\}$ is a face of J and therefore of $J_{c'}$ for some $c' \in C$; now $J_{c'}$ must be

$$(J \setminus \{(i_k, k)\}) \cup \{(l - \textstyle\sum_{j<r, j\neq k} i_j, k)\} = (J \setminus \{(i_k, k)\}) \cup \{(i_k + l - l', k)\}$$

and

$$\textstyle\sum_{j=0}^{r} (j+1)i_j + (k+1)(l - l')$$

belongs to A. Since this is true for every $k < r$, and we also have $\sum_{j=0}^{r}(j+1)i_j \in A$ because $c \in C$, we have an arithmetic progression in A of length $r+1$, as required.

4 The multiple recurrence theorem

The first 'measure-theoretic' proof of Szemerédi's theorem was due to FURSTENBURG 77, and relied on a deep analysis of the structure of measure-preserving transformations. While the methods described here do not seem to give us any information on this structure, it is apparently a folklore result that the hypergraph removal lemma provides a quick proof of the basic theorem used in Furstenburg's approach.

4A Lemma (cf. SOLYMOSI 03) Suppose that $r \geq 1$ and $n \in \mathbb{N}$. For $0 \leq j$, $k < r$ set $e_j(k) = 1$ if $k = j$, 0 otherwise. For $z \in n^r$ and $A \subseteq n^r$ write

$$\Delta(z, A) = \{k : k \in \mathbb{Z},\ z + ke_i \in A \text{ for every } i < r\}, \quad q(z, A) = \#(\Delta(z, A)).$$

Then for every $\epsilon > 0$ there is a $\delta > 0$ such that $\#(\{z : z \in n^r,\ q(z, A) \geq \delta n\}) \geq \delta n^r$ whenever $n \in \mathbb{N}$, $A \subseteq n^r$ and $\#(A) \geq \epsilon n^r$.

proof We can use the same approach as in 3E. Let $\epsilon' > 0$ be such that $2^r r^r \epsilon' < \epsilon$. Let $\delta > 0$ be such that whenever I is finite, $R \subseteq [I]^r$ and $\#(\{J : J \in [I]^{r+1},\ [J]^r \subseteq R\}) \leq \delta \#(I)^{r+1}$ there is an $S \subseteq R$ such that $\#(S) \leq \epsilon' \#(I)^r$ and there is no $J \in [I]^{r+1}$ such that $[J]^r \subseteq R \setminus S$ (3D).

Take $n \in \mathbb{N}$ and $A \subseteq n^r$ such that $\#(A) \geq \epsilon n^r$. Set $I = (r \times n) \cup (\{r\} \times nr)$, so that $\#(I) = 2nr$.

For $z \in A$ set

$$J_z = \{(i, z(i)) : i < r\} \cup \{(r, \textstyle\sum_{i=0}^{r-1} z(i))\} \in [I]^{r+1};$$

set $R = \bigcup_{z\in A} [J_z]^r$. Observe that if z, $w \in A$ are distinct then $[J_z]^r$ and $[J_w]^r$ are disjoint. If $S \subseteq R$ and $\#(S) \leq \epsilon' \#(I)^r$, then $\#(S) < \epsilon n^r$ and there must be a $z \in A$ such that $[J_z]^r \cap S = \emptyset$ and $[J_z]^r \subseteq R \setminus S$. Consequently

$$\mathcal{K} = \{K : K \in [I]^{r+1},\ [K]^r \subseteq R\}$$

must have at least $\delta \#(I)^{r+1} \geq 2\delta n^{r+1}$ members.

Next, $\#(\mathcal{K}) = \sum_{z\in n^r} q(z, A)$. **P** Set $B = \{(z, k) : z \in n^r,\ k \in \mathbb{Z},\ z + ke_i \in A \text{ for every } i < r\}$; then $\#(B) = \sum_{z\in n^r} q(z, A)$. For any $K \in \mathcal{K}$, there must be a $z_K \in A$ such that $(i, z_K(i)) \in K$ for every $i < r$ while $(r, k_K + \sum_{i=0}^{r-1} z_K(i)) \in K$ for some k_K; in this case, $z_K + k_K e_i \in A$ for every $i < r$ and $(z_K, k_K) \in B$. Conversely, starting from $(z, k) \in B$, $\{(i, z(i)) : i < r\} \cup \{(r, k + \sum_{i=0}^{r-1} z(i))\}$ belongs to K. So $K \mapsto (z_K, k_K)$ is a bijection from $\mathcal{K}$ to B and $\#(\mathcal{K}) = \#(B)$. **Q**

Thus $\sum_{z\in n^r} q(z, A) \geq 2\delta n^{r+1}$. Of course $q(z, A) \leq n$ for every $z \in A$. So setting $C = \{z : z \in n^r,\ q(z, A) \geq \delta n\}$, we have

$$2\delta n^{r+1} \leq n\#(C) + \delta n \cdot n^r \leq n\#(C) + \delta n^{r+1}$$

and $\#(C) \geq \delta n^r$, as claimed.

4B Theorem (FURSTENBURG 81, 7.13) Let (X, Σ, μ) be a probability space, $r \geq 1$ and $\langle T_i \rangle_{i<r}$ a commuting family of automorphisms of (X, Σ, μ). If $E \in \Sigma$ and $\mu E > 0$, then there is an $\eta > 0$ such that

$$\textstyle\sum_{k=0}^{n-1} \mu\{x : T_i^k x \in E \text{ for every } i < r\} \geq \eta n$$

for every $n \in \mathbb{N}$.

proof Doubling r if necessary, we can suppose that for every $i < r$ there is a $j < r$ such that $T_i^{-1} = T_j$. In Lemma 4A, set $\epsilon = \frac{1}{2}\mu E$ and take a corresponding $\delta > 0$; set $\eta = \frac{1}{2}\delta^2\epsilon$. Now, given $n \in \mathbb{N}$, then for $z \in n^r$ set $\tilde{T}_z = T_0^{z(0)}T_1^{z(1)}\dots T_{r-1}^{z(r-1)}$. Consider the set $W = \{(x, z) : z \in n^r,\ \tilde{T}_z(x) \in E\}$. Then $W^{-1}[\{z\}]$ has measure μE for every z, so if we set $F = \{x : x \in E,\ \#(W[\{x\}]) \geq \epsilon n^r\}$ we have

$$n^r\mu E \leq n^r\mu F + \epsilon n^r, \quad \mu F \geq \epsilon.$$

In the notation of 4A, set

$$V = \{(x, z) : (x, z) \in W,\ q(z, W[\{x\}]) \geq \delta n\};$$

then for any $x \in F$ we have $\#(V[\{x\}]) \geq \delta n^r$, by the choice of δ. There must therefore be a $z \in n^r$ such that $\mu V^{-1}[\{z\}] \geq \delta\mu F \geq \delta\epsilon$. Take any $x \in V^{-1}[\{z\}]$ and $k \in \Delta(z, W[\{x\}])$. Then $z + ke_i \in W[\{x\}]$, that is, $T_i^k\tilde{T}_z(x) \in E$, for every $i < r$. (This is where we need to know that the T_i commute.) Also $|k| < n$. Set $G = \tilde{T}_z[V^{-1}[\{z\}]]$, so that $\mu G \geq \delta\epsilon$ and for every $y \in G$ we have $\#(\{k : |k| < n,\ T_i^k(y) \in E$ for every $i < r\}) \geq \delta n$. But this means that

$$\sum_{k=0}^{n-1} \mu\{x : T_i^k(x) \in E \text{ for every } i < r\}$$

$$\geq \frac{1}{2}\sum_{|k|<n} \mu\{x : T_i^k(x) \in E \text{ for every } i < r\}$$

(because if $T_i^k(x) \in E$ for every $i < r$ then $T_i^{|k|}(x) \in E$ for every $i < r$)

$$= \frac{1}{2}\int \#(\{k : |k| < n,\ T_i^k(x) \in E \text{ for every } i < r\})\mu(dx)$$

$$\geq \frac{1}{2}\delta n\mu G \geq \frac{1}{2}\delta^2\epsilon n = \eta n,$$

as required.

Concluding remarks Regarded as a proof of Szemerédi's theorem, the argument of §§1-3 above has the virtues of reasonable brevity and (I hope) of completeness and correctness. It depends, of course, on non-trivial ideas from measure theory, which for anyone except a measure theorist will compromise the claim of 'brevity'. There are further, more significant, defects. The outstanding problem associated with Szemerédi's theorem is the estimation of n_0 as a function of r and ϵ; and while in a theoretical sense it must be possible to trace through the arguments above to establish rigorous bounds, the methods are not well adapted to such an exercise, and one would not expect the bounds obtained to be good. There is also the point that I have made uninhibited use of the axiom of choice. The ultrafilters in §3 can easily be replaced by appropriate sequences, but all standard treatments of measure theory assume at least the countable axiom of choice; and Szemerédi's theorem is clearly true in significantly weaker theories than ordinary ZF. Of course an analysis giving bounds for the parameters here would presumably do so by reducing all the arguments to purely combinatorial ones, so answering this criticism.

The value of the work here, therefore, lies less in its applications to the hypergraph removal lemma and Szemerédi's theorem, than in the idea of 'removable intersections', where Theorems 1E and 2E give us two remarkable results. It is also an instructive example of a more general phenomenon. Given a sequence of finite objects with quantititative aspects, it is often profitable to seek a measure μ reflecting the asymptotic behaviour of this sequence; this is the idea of the construction in 3A. The 'quantitative aspects' here, as developed in 3C, are the proportion of functions from L to m_n which are embeddings of (L, T) in (m_n, T_n), and the proportion of simplices in $[m_n]^k$ which must be removed from T_n in order to destroy all these embeddings. The measure μ is set up to describe the limits of these proportions as measures of appropriate sets.

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