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$\mathfrak{m}$, $\mathfrak{p}$ and $\mathfrak{t}$, following SHELAH 885

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I rewrite the material of SHELAH 885. For basic definitions see FREMLIN 84 or FREMLIN 08?.

1 The easy bits

1A Definitions (a) A family $\langle A_\xi\rangle_{\xi<\mathfrak{p}}$ **exemplifies** $\mathfrak{p}$ if every A_ξ is infinite, $\{A_\xi:\xi<\mathfrak{p}\}$ is downwards-directed, and there is no infinite $C\subseteq\mathbb{N}$ such that $C\setminus A_\xi$ is finite for every $\xi<\mathfrak{p}$.

(b) If $A\subseteq\mathbb{N}$, a **segment** of A will be a set of the form $A\cap n\setminus m$ where m, $n\in\mathbb{N}2$; when $m=0$ I will call it an **initial segment**. For σ, $\tau\in[\mathbb{N}]^{<\omega}$ say that $\sigma\vartriangleleft\tau$ if σ is a proper initial segment of τ.

(c) For a set A and functions f, $g\in\mathbb{N}^A$ I will say that $f\le^* g$ if $\{n:n\in A,\ g(n)<f(n)\}$ is finite.

(d) $\mathfrak{t}$ is the least cardinal such that there is a family $\langle A_\xi\rangle_{\xi<\mathfrak{t}}$ of infinite subsets of $\mathbb{N}$ such that $A_\eta\setminus A_\xi$ is finite whenever $\xi\le\eta<\mathfrak{t}$ but there is no infinite $A\subseteq\mathbb{N}$ such that $A\setminus A_\xi$ is finite for every $\xi<\mathfrak{t}$.

1B Proposition $\mathfrak{b}\ge\mathfrak{t}\ge\mathfrak{p}\ge\mathfrak{m}_{\mathrm{K}}$.

proof $\mathfrak{p}\ge\mathfrak{m}_{\mathrm{K}}$ by FREMLIN 84, 11D, or FREMLIN 08?, 517Ob. $\mathfrak{t}\ge\mathfrak{p}$ is obvious from the definitions. As for $\mathfrak{b}$, we can use an argument of ROTHBERGER 48 (see FREMLIN 84, 14B) as follows. Let $\langle f_\xi\rangle_{\xi<\kappa}$ be a family in $\mathbb{N}^{\mathbb{N}}$, where $\kappa<\mathfrak{t}$. Choose a family $\langle A_\xi\rangle_{\xi\le\kappa}$ in $[\mathbb{N}]^\omega$ inductively, as follows. $A_0=\mathbb{N}$. Given A_ξ, where $\xi<\omega$, let $A_{\xi+1}\subseteq A_\xi$ be an infinite set such that $\#(A_{\xi+1}\cap f_\xi(n))\le\frac{1}{2}n$ for every n. Given that $A_\eta\setminus A_\xi$ is finite whenever $\xi\le\eta<\zeta$, where $\zeta\le\kappa$ is a non-zero limit ordinal, let A_ζ be an infinite set such that $A_\zeta\setminus A_\xi$ is finite for every $\xi<\zeta$.

At the end of the induction, let $f\in\mathbb{N}^{\mathbb{N}}$ be the increasing enumeration of A_κ. Then for every $\xi<\kappa$ we have a finite $m=\#(A_\xi\setminus A_\kappa)$, so that

$$\#(A_{\xi+1}\cap f(n))\ge\#(A_\kappa\cap f(n))-m=n-m-1>\tfrac{1}{2}n$$

whenever $n>2m+2$, and $f_\xi(n)<f(n)$ for every $n>2m+2$. Thus $f_\xi\le^* f$ for every $\xi<\kappa$. As $\langle f_\xi\rangle_{\xi<\kappa}$ is arbitrary, $\mathfrak{b}\ge\mathfrak{t}$.

1C Proposition Suppose that $\mathfrak{p}<\mathfrak{t}$. If $\langle A_\xi\rangle_{\xi<\mathfrak{p}}$ exemplifies $\mathfrak{p}$ then there are a regular uncountable $\kappa<\mathfrak{p}$ and a family $\langle B_\alpha\rangle_{\alpha<\kappa}$ such that $B_\alpha\cap A_\xi$ is infinite whenever $\xi<\mathfrak{p}$ and $\alpha<\kappa$, $B_\alpha\setminus B_\beta$ is finite whenever $\beta\le\alpha<\kappa$, and there is no infinite $C\subseteq\mathbb{N}$ such that $C\setminus B_\alpha$ is finite for every $\alpha<\kappa$ and $C\cap A_\xi$ is infinite for every $\xi<\mathfrak{p}$.

proof (a) Choose $\langle B'_\xi\rangle_{\xi<\zeta}$ such that

$B'_0=\mathbb{N}$,

$B'_{\xi+1}=B'_\xi\cap A_\xi$ for every $\xi<\zeta$,

for a non-zero limit ordinal ξ, if there is an infinite $C\subseteq\mathbb{N}$ such that $C\setminus B'_\eta$ is finite for every $\eta<\xi$ and $C\cap A_\gamma$ is infinite for every $\gamma<\mathfrak{p}$, take such a C for B'_ξ; otherwise set $\zeta=\xi$ and stop.

Because there is no infinite set C such that $C\setminus A_\xi$ is finite for every $\xi<\mathfrak{p}$, we surely have to stop at some $\zeta\le\mathfrak{p}$. **?** If $\zeta=\mathfrak{p}$ then $\langle B'_\xi\rangle_{\xi<\zeta}$ witnesses that $\mathfrak{t}\le\mathfrak{p}$. **X** So $\zeta<\mathfrak{p}$; let $\langle\zeta_\alpha\rangle_{\alpha<\kappa}$ be a strictly increasing family of ordinals with supremum ζ, where $\kappa=\operatorname{cf}\zeta$, and set $B_\alpha=B'_{\zeta_\alpha}$ for $\alpha<\kappa$.

(b) ? If $\kappa=\omega$, set $\tilde B_n=\bigcap_{i\le n}B_i\setminus n$ for each n, so that $\langle\tilde B_n\rangle_{n\in\mathbb{N}}$ is non-increasing and has empty intersection and $B_n\triangle\tilde B_n$ is finite for every n. Define $f_\xi\in\mathbb{N}^{\mathbb{N}}$, for $\xi<\mathfrak{p}$, by saying that $f_\xi(n)=\min(A_\xi\cap\tilde B_n)$. Because $\mathfrak{p}<\mathfrak{t}\le\mathfrak{b}$, there is an $f\in\mathbb{N}^{\mathbb{N}}$ such that $f_\xi\le^* f$ for every $\xi<\mathfrak{p}$. Consider $C=\bigcup_{n\in\mathbb{N}}(f(n)+1)\cap\tilde B_n$. For any $\xi<\mathfrak{p}$, $f_\xi(n)\in C$ for all but finitely many n so $A_\xi\cap C$ is infinite. For any $n\in\mathbb{N}$, $C\setminus\tilde B_n\subseteq f(n)+1$ is finite; it follows that $C\setminus B'_\xi$ is finite for every $\xi<\zeta$, which is impossible. **X**

1D Proposition Suppose that $\zeta < \mathfrak{t}$ is an ordinal and $\langle A_\xi \rangle_{\xi<\zeta}$, $\langle B_n \rangle_{n\in\mathbb{N}}$ are families in $\mathcal{P}\mathbb{N}$ such that

$B_n \setminus B_m$ is finite whenever $m \le n$,

$A_\xi \cap B_n$ is infinite for all $\xi < \zeta$ and $n \in \mathbb{N}$,

whenever $\eta \le \xi < \zeta$ there is an $n \in \mathbb{N}$ such that $B_n \cap A_\xi \setminus A_\eta$ is finite.

Then there is an infinite $A \subseteq \mathbb{N}$ such that $A \setminus A_\xi$ and $A \setminus B_n$ are finite for all $\xi < \zeta$ and $n \in \mathbb{N}$.

proof Replacing B_n with $\bigcap_{m\le n} B_m \setminus n$ if necessary, we can suppose that $B_{n+1} \subseteq B_n$ for every n and that $\bigcap_{n\in\mathbb{N}} B_n$ is empty. For each $\xi < \zeta$ let $f_\xi \in \mathbb{N}^\mathbb{N}$ be a strictly increasing sequence such that $f_\xi(n) \in B_n \cap A_\xi$ for every n. Because $\zeta < \mathfrak{t} \le \mathfrak{b}$ there is a strictly increasing $f \in \mathbb{N}^\mathbb{N}$ such that $\{n : f_\xi(n) \ge f(n)\}$ is finite for every $\xi < \zeta$. Set

$$B^* = \bigcup_{n\in\mathbb{N}} B_n \cap f(n).$$

Then $A_\xi \cap B^*$ is infinite for every $\xi < \zeta$. **P** $f_\xi(n) \in A_\xi \cap B^*$ whenever $f_\xi(n) < f(n)$. **Q** Next, $B^* \setminus B_n \subseteq \bigcup_{i<n} B_i \cap f(i)$ is finite for every n; consequently, $B^* \cap A_\xi \setminus A_\eta$ is finite whenever $\eta \le \xi < \zeta$. As $\zeta < \mathfrak{t}$ there is an infinite $A \subseteq B^*$ such that $A \setminus A_\xi$ is finite for every $\xi < \zeta$; this works.

2 Peculiar cuts

2A Definition If κ, λ are cardinals and P is a partially ordered set, a **peculiar** (λ, κ^*)**-cut** in P is a pair $(\langle p_\xi \rangle_{\xi<\lambda}, \langle q_\alpha \rangle_{\alpha<\kappa})$ of families in P such that

$p_\xi < p_\eta$ whenever $\xi < \eta < \lambda$,

$q_\beta < q_\alpha$ whenever $\alpha < \beta < \kappa$,

$p_\xi \le q_\alpha$ whenever $\xi < \lambda$ and $\alpha < \kappa$,

whenever $p \in P$ is such that $p \le q_\alpha$ for every $\alpha < \kappa$, there is a $\xi < \lambda$ such that $p \le p_\xi$,

whenever $q \in P$ is such that $p_\xi \le q$ for every $\xi < \lambda$, there is an $\alpha < \kappa$ such that $q_\alpha \le q$.

2B Definition Let S^* be $[\mathbb{N}]^{<\omega} \setminus \emptyset$ and $\mathcal{S}$ the family of infinite subsets S of S^* such that $\min \sigma \ne \min \tau$ for all distinct σ, $\tau \in S$. For S, $S' \in \mathcal{S}$ say that $S \prec S'$ if for all but finitely many $\tau \in S'$ there is a $\sigma \in S$ such that $\sigma \triangleleft \tau$.

2C Proposition Let $\mathcal{S}$, $\prec$ be as in 2B.

(a) $\prec$ is transitive.

(b) If $\zeta < \mathfrak{t}$ is an infinite ordinal and $\langle S_\xi \rangle_{\xi<\zeta}$ is a family in $\mathcal{S}$ such that $S_\eta \prec S_\xi$ whenever $\eta < \xi$ in ζ then there is an $S \in \mathcal{S}$ such that $S_\xi \prec S$ for every $\xi < \zeta$.

proof (a) If $S \prec S' \prec S''$ then there are n_0, $n_1 \in \mathbb{N}$ such that

whenever $\tau \in S'$ and $\min \tau \ge n_0$ then there is a $\sigma \in S$ such that $\sigma \triangleleft \tau$,

whenever $\upsilon \in S''$ and $\min \upsilon \ge n_1$ then there is a $\tau \in S'$ such that $\tau \triangleleft \upsilon$.

Set

$$n_2 = \max(n_1, \sup_{\tau\in S', \min\tau<n_0} \max(\tau));$$

then whenever $\upsilon \in S''$ and $\min \upsilon > n_2$ there is a $\sigma \in S$ such that $\sigma \triangleleft \upsilon$, so $S \prec S''$.

(b) For $n \in \mathbb{N}$ set $B_n = \{\sigma : \sigma \in S^*, \min \sigma \ge n\}$ and for $\xi < \zeta$ set

$$A_\xi = \{\tau : \tau \in S^*, \sigma \triangleleft \tau \text{ for some } \sigma \in S_\xi\}.$$

If $\eta < \xi < \zeta$ then there is some $n_0 \in \mathbb{N}$ such that $S_\xi \cap B_{n_0} \subseteq A_\eta$; taking n such that $\max \sigma < n$ whenever $\sigma \in S_\xi \setminus B_{n_0}$, $A_\xi \cap B_n \subseteq A_\eta$. So Proposition 1D tells us that there is an infinite S such that $S \setminus B_n$ and $S \setminus A_\xi$ are finite for all $n \in \mathbb{N}$ and $\xi < \zeta$. Now $S_\xi \prec S$ for every $\xi < \zeta$.

2D Corollary (Piotrowski & Szymański 87) The additivity of the meager ideal of $\mathbb{R}$ is at least $\mathfrak{t}$.

proof It is enough to show that the union of fewer than $\mathfrak{t}$ nowhere dense subsets of $\{0,1\}^\mathbb{N}$ is meager in $\{0,1\}^\mathbb{N}$ (Fremlin 08?, 522Vb).

(a) For $\sigma \in S^*$ set

$$W_\sigma = \{x : x \in \{0,1\}^\mathbb{N} \text{ and } \sigma \text{ is a segment of } \{i : x(i) = 1\}\};$$

for $S\in\mathcal{S}$ set

$$\tilde W_S=\bigcap_{n\in\mathbb{N}}\bigcup\{W_\sigma:\sigma\in S,\,\min\sigma\ge n\}.$$

Then $\tilde W_S$ is a dense G_δ set, and $\tilde W_{S'}\subseteq\tilde W_S$ whenever $S\prec S'$.

(b) If $S\in\mathcal{S}$ and $F\subseteq\{0,1\}^{\mathbb{N}}$ is nowhere dense there is an $S'\in\mathcal{S}$ such that $S\prec S'$ and $F\cap\tilde W_{S'}$ is empty. **P** For each $\sigma\in S$ let $\tau_\sigma\in S^*$ be such that $\sigma\triangleleft\tau$ and $W_{\tau_\sigma}\cap F$ is empty. Set $S'=\{\tau_\sigma:\sigma\in S\}$. **Q**

(c) Now if $\kappa<\mathfrak{t}$ and $\langle F_\xi\rangle_{\xi<\kappa}$ is a family of nowhere dense closed subsets of $\{0,1\}^{\mathbb{N}}$, we can choose inductively a family $\langle S_\xi\rangle_{\xi\le\kappa}$ in $\mathcal{S}$ such that $S_\eta\prec S_\xi$ if $\eta<\xi\le\kappa$ and $\tilde W_{S_{\xi+1}}\cap F_\xi$ is empty for every $\xi<\kappa$, using 2Cb for the inductive step to a non-zero limit ordinal. In this case, $\tilde W_{S_\kappa}$ is a dense G_δ set disjoint from $\bigcup_{\xi<\kappa}F_\xi$, so $\bigcup_{\xi<\kappa}F_\xi$ is meager.

2G Lemma Assume that $\mathfrak{p}<\mathfrak{t}$. Let $\langle A_\xi\rangle_{\xi<\mathfrak{p}}$ be a family exemplifying $\mathfrak{p}$. Let $\langle B_\alpha\rangle_{\alpha<\kappa}$ be a family as in Proposition 1C. Then we can find a family $\langle S_\xi\rangle_{\xi\le\mathfrak{p}}$ such that
(i) $S_\xi\in\mathcal{S}$ for every $\xi\le\mathfrak{p}$;
(ii) $S_\xi\setminus[B_\alpha]^{<\omega}$ is finite for every $\xi<\mathfrak{p}$ and $\alpha<\kappa$;
(iii) $S_\eta\prec S_\xi$ whenever $\eta<\xi\le\mathfrak{p}$;
(iv) $\max\sigma\in A_\xi$ whenever $\xi<\mathfrak{p}$ and $\sigma\in S_{\xi+1}$.

proof Build $\langle S_\xi\rangle_{\xi\le\mathfrak{p}}$ inductively, as follows.

(a) Because $\kappa<\mathfrak{t}$ and $B_\beta\setminus B_\alpha$ is finite whenever $\alpha\le\beta<\kappa$, there is an infinite $C\subseteq\mathbb{N}$ such that $C\setminus B_\alpha$ is finite for every $\alpha<\kappa$; so we can start the induction with $S_0=\{\{n\}:n\in C\}$.

(b) Given S_ξ where $\xi<\mathfrak{p}$, take an infinite $C\subseteq A_\xi$ such that $C\setminus B_\alpha$ is finite for every $\alpha<\kappa$; for $\sigma\in S_\xi$ let $\sigma'\in S^*$ be such that $\sigma\triangleleft\sigma'\subseteq\sigma\cup C$. Set $S_{\xi+1}=\{\sigma':\sigma\in S_\xi\}$. Then $S_\xi\prec S_{\xi+1}$, $\bigcup S_{\xi+1}\subseteq C\cup\bigcup S_\xi$ so $S_{\xi+1}\setminus[B_\alpha]^{<\omega}$ is finite for every $\alpha<\kappa$, and $\max\sigma\in C\subseteq A_\xi$ for every $\sigma\in S_{\xi+1}$.

(c) Given $\langle S_\xi\rangle_{\eta<\xi}$ where $\xi<\mathfrak{p}$ is a non-zero limit ordinal, set $\zeta=\max(\xi,\kappa)$, and let $\mathcal{I}$ be the family of finite subsets I of S^* such that $\min\sigma\ne\min\tau$ for all distinct σ, $\tau\in I$. Consider the partially ordered set $P=\mathcal{I}\times[\zeta]^{<\omega}$ ordered by saying that $(I,J)\le(I',J')$ if $I\subseteq I'$, $J\subseteq J'$ and for every $\sigma\in I'\setminus I$

for every $\eta\in J\cap\xi$ there is a $\tau\in S_\eta$ such that $\tau\triangleleft\sigma$,

$\sigma\subseteq B_\alpha$ for every $\alpha\in J\cap\kappa$.

It is easy to check that $\le$ is a partial order on P and that P is σ-centered upwards (because $\mathcal{I}$ is countable). Now $Q'_\eta=\{(I,J):\eta\in J\}$ is cofinal with P for every $\eta<\zeta$. In addition, $Q_k=\{(I,J):$ there is a $\sigma\in I$ such that $\sigma\not\subseteq k\}$ is cofinal with P, for every $k\in\mathbb{N}$. **P** Take $(I,J)\in P$. It is enough to consider the case in which $0\in J$; set $\eta^*=\max(J\cap\xi)$. Set $B^*=\bigcap_{\alpha\in J\cap\kappa}B_\alpha$; then there is a $k'\ge\max(n,k)$ such that $(\bigcup S_{\eta^*})\setminus B^*\subseteq k'$. Next, because $S_\eta\prec S_{\eta^*}$ for every $\eta\in J\cap\eta^*$, there is a $\sigma\in S_{\eta^*}$ such that $k'\le\min\sigma$ and for every $\eta\in J\cap\eta^*$ there is a $\tau\in S_\eta$ such that $\tau\triangleleft\sigma$. Because B^* is infinite, we can find an $m\in B^*$ such that $m>\max\sigma$; set $\sigma'=\sigma\cup\{m\}$; note that $\sigma'\subseteq B^*$. Now $(I,J)\le(I\cup\{\sigma'\},J)\in Q_k$. **Q**

Let $R\subseteq P$ be an upwards-directed set meeting every Q'_η and Q_k; such exists because $\zeta<\mathfrak{p}$ and P is σ-centered upwards. Set $S=\bigcup\{I:(I,J)\in R\}$. Because R meets every Q_k, S is infinite. Because R meets Q'_η, $\{\sigma:\sigma\in S,\,\tau\not\triangleleft\sigma$ for every $\tau\in S_\eta\}$ is finite for every $\eta<\xi$, that is, $S_\eta\prec S$ for every $\eta<\xi$. Because R meets Q'_α, $S\setminus[B_\alpha]^{<\omega}$ is finite for every $\alpha<\kappa$. So we can take $S_\xi=S$, and the induction proceeds.

(d) Given $\langle S_\eta\rangle_{\xi<\mathfrak{p}}$, let $S_{\mathfrak{p}}\in\mathcal{S}$ be such that $S_\xi\prec S$ for every $\xi<\mathfrak{p}$ (Proposition 2C, because $\mathfrak{p}<\mathfrak{t}$). This completes the construction.

2H Theorem (Shelah 885, 1.12) Assume that $\mathfrak{p}<\mathfrak{t}$. Then there is a regular uncountable $\kappa<\mathfrak{p}$ such that there is a peculiar $(\mathfrak{p},\kappa^*)$-cut in $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$.

proof (a) Let $\langle A_\xi\rangle_{\xi<\mathfrak{p}}$, $\langle B_\alpha\rangle_{\alpha<\kappa}$, $<$, $\prec$ and $\langle S_\xi\rangle_{\xi\le\mathfrak{p}}$ be as in Lemma 2G; recall from 1C that κ is regular and uncountable. Set $A=\{\min\sigma:\sigma\in S_{\mathfrak{p}}\}$; for $n\in A$ take $\sigma_n\in S_{\mathfrak{p}}$ such that $n=\min\sigma_n$. For $\alpha<\kappa$ and $n\in A$ take $f_\alpha(n)$ to be $\min(\sigma_n\setminus B_\alpha)$, if this is defined, and otherwise $f_\alpha(n)=1+\max\sigma_n$. For $\xi<\mathfrak{p}$, $n\in A$ take $g_\xi(n)$ to be $\max\sigma$ for that $\sigma\in S_{\xi+1}$ such that $\sigma\triangleleft\sigma_n$, if there is one, and otherwise $g_\xi(n)=1+\max\sigma_n$.

Observe that if $\alpha<\kappa$ and $n\in A$ then $\sigma_n\cap f_\alpha(n)\subseteq B_\alpha$. Also, given $\xi<\mathfrak{p}$, $g_\xi(n)\in A_\xi$ for all but finitely many $n\in A$, and $g_\xi(n)\ge n$ for every $n\in A$.

(b) If $\alpha \leq \beta < \kappa$ then $B_\beta \setminus B_\alpha$ is finite, so there is an n_0 such that $\sigma_n \cap B_\beta \setminus B_\alpha$ is empty for every $n \in A \setminus n_0$. Accordingly $\sigma_n \setminus B_\alpha \subseteq \sigma_n \setminus B_\beta$ and $f_\alpha(n) \geq f_\beta(n)$ for every $n \in A \setminus n_0$.

(c) If $\alpha < \kappa$ there is a $\beta < \kappa$ such that $f_\beta(n) < f_\alpha(n)$ for infinitely many $n \in A$. **P** Set $D = \bigcup_{n \in A} \sigma_n \cap f_\alpha(n)$.

Take any $\xi < \mathfrak{p}$. Recall that $S_{\xi+1} \setminus [B_\alpha]^{<\omega}$ is finite, while σ is non-empty for every $\sigma \in S_{\xi+1}$. As $S_{\xi+1} \prec S_\mathfrak{p}$, all but finitely many members of $S_\mathfrak{p}$ have proper initial segments belonging to $S_{\xi+1}$, and for all but finitely many n there is an initial segment σ of σ_n such that σ is non-empty and included in B_α. In this case, $\sigma \subseteq f_\alpha(n)$, so $D \supseteq \sigma$. At the same time, $\max \sigma \in A_\xi$, by (iii) of 2G. As this is true for infinitely many n, $D \cap A_\xi$ is infinite.

There must therefore be a $\beta < \kappa$ such that $D \setminus B_\beta$ is infinite. Now

$$D \setminus B_\beta \subseteq \bigcup_{n \in A} \sigma_n \cap f_\alpha(n) \setminus f_\beta(n)$$

so $\{n : n \in A,\ f_\beta(n) < f_\alpha(n)\}$ must be infinite. **Q**

(d) If $\xi < \eta < \mathfrak{p}$ then $g_\xi(n) < g_\eta(n)$ for all but finitely many $n \in A$. **P** $S_{\xi+1} \prec S_{\eta+1} \prec S_\mathfrak{p}$ so for all but finitely many $n \in A$ we have $\tau \in S_{\xi+1}$, $\tau' \in S_{\eta+1}$ such that $\tau \vartriangleleft \tau' \vartriangleleft \sigma_n$. Now, for such n,

$$g_\xi(n) = \max \tau < \max \tau' = g_\eta(n). \text{ } \mathbf{Q}$$

(e) If $\xi < \mathfrak{p}$ and $\alpha < \kappa$ then $g_\xi \leq^* f_\alpha$. **P** Look back at (c). For all but finitely many $n \in A$, there is an initial segment σ of σ_n, belonging to $S_{\xi+1}$, such that $\sigma \subseteq B_\alpha$. But for any such n we have $g_\xi(n) \leq f_\alpha(n)$. **Q**

(f) If $f : A \to \mathbb{N}$ is such that $f \leq^* f_\alpha$ for every $\alpha < \kappa$, then there is a $\xi < \mathfrak{p}$ such that $f \leq^* g_\xi$. **P** Since $f_0(n) \leq 1 + \max \sigma_n$ for every $n \in A$, we can suppose that $f(n) \leq 1 + \max \sigma_n$ for every $n \in A$. Set $D = \bigcup_{n \in A} \sigma_n \cap f(n)$. Then

$$D \setminus B_\alpha \subseteq \bigcup_{n \in A} \sigma_n \cap f(n) \setminus f_\alpha(n)$$

is finite for every $\alpha < \kappa$. So there must be a $\xi < \mathfrak{p}$ such that $D \cap A_\xi$ is finite. Now if $n \in A$ and $g_\xi(n) < f(n) \leq 1 + \max \sigma_n$, $g_\xi(n) = \max \sigma$ for that $\sigma \in S_{\xi+1}$ such that $\sigma \vartriangleleft \sigma_n$, so

$$g_\xi(n) \in A_\xi \cap f(n) \cap \sigma_n \subseteq D \cap A_\xi.$$

As $g_\xi(n) \geq n$ for every $n \in A$, this can be so for only finitely many n, and $f \leq^* g_\xi$. **Q**

(g) **?** Suppose that $f : A \to \mathbb{N}$ is such that $g_\xi \leq^* f$ for every $\xi < \mathfrak{p}$ but $f_\alpha \not\leq^* f$ for every $\alpha < \kappa$. Then there must be an infinite $C \subseteq A$ such that $C \setminus \{n : f_\alpha(n) > f(n)\}$ is finite for every $\alpha < \kappa$. But now we can repeat the argument of (f) with C in place of A to get a contradiction. **X**

(h) Because κ is regular, (c) tells us that there is a strictly increasing family $\langle \alpha_\gamma \rangle_{\gamma < \kappa}$ in κ such that $\{n : n \in A,\ f_{\alpha_{\gamma+1}}(n) < f_{\alpha_\gamma}(n)\}$ is infinite for every γ. Now let M be the countably infinite set $A \times \mathbb{N}$ and set

$$U_\xi = \{(n, i) : n \in A,\ i \leq g_\xi(n)\}, \quad u_\xi = U_\xi^\bullet \in \mathcal{P}M/[M]^{<\omega}$$

for $\xi < \mathfrak{p}$,

$$V_\gamma = \{(n, i) : n \in A,\ i \leq f_{\alpha_\gamma}(n)\}, \quad v_\gamma = V_\gamma^\bullet$$

for $\gamma < \kappa$. Then $(\langle u_\xi \rangle_{\xi < \mathfrak{p}}, \langle v_\gamma \rangle_{\gamma < \kappa})$ is a peculiar $(\mathfrak{p}, \kappa^*)$-cut in $\mathcal{P}M/[M]^{<\omega}$.

P By (b) and the choice of $\langle \alpha_\gamma \rangle_{\gamma < \kappa}$, $v_\beta \subset v_\alpha$ whenever $\alpha < \beta < \kappa$; by (d), $u_\xi \subset u_\eta$ whenever $\eta < \xi < \mathfrak{p}$; by (e), $u_\xi \subseteq v_\alpha$ whenever $\xi < \mathfrak{p}$ and $\alpha < \kappa$.

Suppose that $v \in \mathcal{P}M/[M]^{<\omega}$ and $v \subseteq v_\gamma$ for every $\alpha < \gamma$. Express v as $V^\bullet$ for $V \subseteq M$, so that $V \setminus V_\gamma$ is finite for every $\gamma < \kappa$. In particular, since $V \setminus V_0$ is finite, all the vertical sections of V are finite; set $f(n) = \sup V[\{n\}]$ for $n \in A$, counting $\sup \emptyset$ as 0. Then $f \leq^* f_{\alpha_\gamma} \leq^* f_\gamma$ for every $\gamma < \kappa$, so $f \leq^* g_\xi$ for some $\xi < \mathfrak{p}$. But this means that $V \setminus U_\xi$ is finite and $v \subseteq u_\xi$.

Suppose that $u \in \mathcal{P}M/[M]^{\leq\omega}$ and $u_\xi \subseteq u$ for every $\xi < \mathfrak{p}$. Express u as $U^\bullet$ for $U \subseteq M$, and for $n \in A$ set

$$\begin{aligned} f(n) &= \min\{i : i \in \mathbb{N},\ (n, i) \notin U\} \text{ if this is defined,} \\ &= 2 + \max \sigma_n \text{ otherwise.} \end{aligned}$$

For every $\xi < \mathfrak{p}$, $U_\xi \setminus U$ is finite so $g_\xi \le^* f$; there is therefore a $\gamma < \kappa$ such that $f_{\alpha_\gamma} \le^* f$. In this case $V_\gamma \setminus U$ is finite and $v_\gamma \subseteq u$.

So all the conditions of 2A are satisfied. **Q**

(i) Finally, of course, because M is countably infinite, we can copy the structure of (h) into $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$.

2I Proposition (a) If $\mathfrak{c} = \omega_1$, there is a peculiar Hausdorff gap.

(b) If λ, κ are cardinals such that $\omega \le \max(\lambda, \kappa) < \mathfrak{p}$, there is no peculiar (λ, κ^*)-cut in $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$.

(c) If $\omega \le \kappa < \mathfrak{t}$ there is no peculiar (κ, κ^*)-cut in $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$.

Remark A **Hausdorff gap** is a pair $\langle A_\xi \rangle_{\xi<\omega_1}$, $\langle B_\xi \rangle_{\xi<\omega_1}$ of families of subsets of $\mathbb{N}$ such that

$A_\eta \setminus A_\xi$, $A_\xi \setminus B_\xi$, $B_\xi \setminus B_\eta$ are finite whenever $\eta \le \xi < \omega_1$,

$\{\eta : \eta < \xi,\ A_\eta \setminus B_\xi \subseteq n\}$ is finite for every $\xi < \omega_1$ and $n \in \mathbb{N}$.

I will call such a gap **peculiar** if $(\langle A_\xi^\bullet \rangle_{\xi<\omega_1}, \langle B_\xi^\bullet \rangle_{\xi<\omega_1})$ is a peculiar (ω_1, ω_1^*)-cut in $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$.

proof (a) Start by enumerating $\mathcal{P}\mathbb{N}$ as $\langle C_\xi \rangle_{\xi<\omega_1}$. Build a Hausdorff gap $(\langle A_\xi \rangle_{\xi<\omega_1}, \langle B_\xi \rangle_{\xi<\omega_1})$ by the standard inductive process (e.g., Fremlin 84, 21L), but with a modification, as follows. When choosing $A_{\xi+1}$ and $B_{\xi+1}$, note that the construction demands only that $A_\xi \subseteq A_{\xi+1} \subseteq B_{\xi+1} \subseteq B_\xi$ and that all the inclusions are non-trivial, that is, $A_{\xi+1} \setminus A_\xi$, $B_{\xi+1} \setminus A_{\xi+1}$, $B_\xi \setminus B_{\xi+1}$ are infinite. So here interpolate the requirements

— if $B_\xi \setminus C_\xi$ is infinite, then $A_{\xi+1} \setminus C_\xi$ is infinite,

— if $C_\xi \setminus A_{\xi+1}$ is infinite, then $C_\xi \setminus B_{\xi+1}$ is infinite.

At the end of the construction,

— if $c \in \mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ is such that $A_\xi^\bullet \subseteq c$ for every $\xi < \omega_1$, then $c = C_\xi^\bullet$ for some ξ; $A_{\xi+1} \setminus C_\xi$ is finite so $B_\xi \setminus C_\xi$ is finite and $B_\xi^\bullet \subseteq c$;

— if $c \in \mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ is such that $c \subseteq B_\xi^\bullet$ for every $\xi < \omega_1$, then $c = C_\xi^\bullet$ for some ξ; $C_\xi \setminus B_{\xi+1}$ is finite so $C_\xi \setminus A_{\xi+1}$ is finite and $c \subseteq A_{\xi+1}^\bullet$.

Thus we have a peculiar cut.

(b) Let $\langle a_\xi \rangle_{\xi<\lambda}$ and $\langle b_\alpha \rangle_{\alpha<\kappa}$ be families in $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ such that $a_\eta \subset a_\xi \subseteq b_\beta \subset b_\alpha$ whenever $\eta < \xi < \lambda$ and $\alpha < \beta < \kappa$; let $\langle A_\xi \rangle_{\xi<\lambda}$ and $\langle B_\alpha \rangle_{\alpha<\kappa}$ be families in $\mathcal{P}\mathbb{N}$ such that $A_\xi^\bullet = a_\xi$ and $B_\alpha^\bullet = b_\alpha$ for $\xi < \lambda$ and $\alpha < \kappa$. Let P be the set of pairs (I, J) where $I \in [\mathbb{N}]^{<\omega}$ and $J \in [\max(\kappa, \lambda)]^{<\omega}$, ordered by saying that $(I, J) \le (I', J')$ if $I \subseteq I'$, $J \subseteq J'$ and for every $n \in I' \setminus I$

if $\xi \in J \cap \lambda$, then $n \notin A_\xi$,

if $\alpha \in J \cap \kappa$, then $n \in B_\alpha$.

Then $\le$ is a partial order on P and P is σ-centered upwards. For $n \in \mathbb{N}$, set $Q_n = \{(I, J) : (I, J) \in P,\ I \not\subseteq n\}$; for $\xi \in \max(\kappa, \lambda)$ set $Q'_\xi = \{(I, J) : (I, J) \in P,\ \xi \in J\}$. These are all cofinal with P. Because $\max(\kappa, \lambda) < \mathfrak{p}$, there is an upwards-directed set $R \subseteq P$ meeting every Q_n and Q'_ξ. Set $C = \bigcup\{I : (I, J) \in R\}$; then C is infinite, $C \cap A_\xi$ is finite for every $\xi < \lambda$ and $C \setminus B_\alpha$ is finite for every $\alpha < \kappa$. Set $c = C^\bullet$; then $c \subseteq b_\alpha$ for every $\alpha < \kappa$ but $c \not\subseteq a_\xi$ for every $\xi < \lambda$. So $(\langle a_\xi \rangle_{\xi<\lambda}, \langle b_\alpha \rangle_{\alpha<\kappa})$ is not a peculiar cut.

(c) Taking $\lambda = \kappa$ in (b), there must be an infinite C such that $C \setminus (B_\xi \setminus A_\xi)$ is finite for every $\xi < \kappa$.

Remark Of course (b) is an immediate corollary of Theorem 21A in Fremlin 84.

3 When must $\mathfrak{p}$ and $\mathfrak{t}$ be equal?

3A Theorem (Rothberger 48) If $\mathfrak{p} < \mathfrak{t}$ then $\mathfrak{p} > \omega_1$.

proof In Proposition 1C, $\omega_1 \le \kappa < \mathfrak{p}$.

3B Proposition Suppose that $\mathfrak{m}_{\mathrm{K}} > \omega_1$. Let $\langle A_\alpha \rangle_{\alpha<\omega_1}$ be a family in $[\mathbb{N}]^\omega$ such that $A_\beta \setminus A_\alpha$ is finite whenever $\alpha \le \beta < \omega_1$, and infinite whenever $\beta < \alpha < \omega_1$. Then there are a cofinal set $C \subseteq \omega_1$ and a family $\langle B_\alpha \rangle_{\alpha \in C}$ in $\mathcal{P}\mathbb{N}$ such that

$B_\alpha \setminus A_\beta$ is finite for all α, $\beta \in C$,

$\{\alpha : \alpha \in C \cap \beta,\ B_\alpha \setminus A_\beta \not\subseteq n\}$ is finite for every $\beta \in C$ and $n \in \mathbb{N}$.

proof (a) Let P be the set of objects $p=(n_p,I_p,\langle J_{p\alpha}\rangle_{\alpha\in I_p})$ such that $n_p\in\mathbb{N}$, $I_p\in[\kappa]^{<\omega}$ and $J_{p\alpha}\subseteq n_p$ for every $\alpha\in I_p$. For p, $q\in P$ say that $p\le q$ if

$n_p\le n_q$,
$I_p\subseteq I_q$,
$J_{p\alpha}=n_p\cap J_{q\alpha}$ for every $\alpha\in I_p$,
$J_{q\alpha}\setminus A_\beta\subseteq n_p$ whenever α, $\beta\in I_p$,
$J_{q\alpha}\setminus A_\beta\not\subseteq n_p$ whenever $\beta\in I_p$, $\alpha\in I_q\setminus I_p$ and $\alpha<\beta$.

Then $\le$ is a partial order on P.

(b) P satisfies Knaster's condition upwards. **P** Let $\langle p_\xi\rangle_{\xi<\omega_1}$ be a family in P. Then there is an uncountable $A\subseteq\omega_1$ such that

$n_{p_\xi}=n_{p_\eta}=m$ say, for all ξ, $\eta\in A$,
$\#(I_{p_\xi})=\#(I_{p_\eta})$ for all ξ, $\eta\in A$,
$\langle I_{p_\xi}\rangle_{\xi\in A}$ is a Δ-system with root I say,
$J_{p_\xi\alpha}=J_{p_\eta\alpha}$ for all $\alpha\in I$ and ξ, $\eta\in A$,
$I\prec I_{p_\xi}\setminus J$ for every $\xi\in A$

(that is, $\alpha<\beta$ whenever $\alpha\in I$ and $\beta\in I_{p_\xi}\setminus I$)

$I_{p_\xi}\prec I_{p_\eta}\setminus I$ whenever ξ, $\eta\in A$ and $\xi<\eta$.

Take any $\xi<\eta$ in A. If $I_{p_\xi}=I_{p_\eta}=I$ then $p_\xi=p_\eta$ are surely compatible. Otherwise, define $q\in P$ as follows. Set $I_q=I_{p_\xi}\cup I_{p_\eta}$ $\alpha^*=\max I_{p_\xi}$ and $\beta^*=\min(I_{p_\eta}\setminus I)$. Because $A_{\alpha^*}\setminus A_\alpha$ is finite for every $\alpha\le\alpha^*$, $A_\beta\setminus A_{\beta^*}$ is finite for every $\beta\ge\beta^*$, and $A_{\alpha^*}\setminus A_{\beta^*}$ is infinite, there is a $j\ge m$ such that

$$j\in\bigcap\{A_\alpha:\alpha\in I_{p_\xi}\}\setminus\bigcup\{A_\beta:\beta\in I_{p_\eta}\setminus I\}.$$

Set $n_q=j+1$ and define $J_{q\alpha}$, for $\alpha\in I_q$, by saying that

—— if $\alpha\in I_{p_\eta}$ then $J_{q\alpha}=J_{p_\xi\alpha}$;
—— if $\alpha\in I_{p_\xi}\setminus I$ then $J_{q\alpha}=J_{p_\xi\alpha}\cup\{j\}$.

The definition of q ensures that

$n_{p_\xi}\le n_q$, $n_{p_\eta}\le n_q$,
$I_{p_\xi}\subseteq I_q$, $I_{p_\eta}\subseteq I_q$,
$J_{q\alpha}\cap n_{p_\xi}=J_{p_\xi\alpha}$ for every $\alpha\in I_{p_\xi}$, $J_{q\alpha}\cap n_{q_\xi}=J_{p_\eta\alpha}$ for every $\alpha\in I_{p_\eta}$,
$J_{q\alpha}\setminus A_\beta\subseteq n_{p_\xi}$ whenever α, $\beta\in I_{p_\xi}$, $J_{q\alpha}\setminus A_\beta\subseteq n_{q_\xi}$ whenever α, $\beta\in I_{p_\eta}$.

Concerning the final clause in the definition of the partial ordering of P, if $\beta\in I_{p_\xi}$ and $\alpha\in I_q\setminus I_{p_\xi}$ then $\beta<\alpha$ and there is nothing to check. If $\beta\in I_{p_\eta}$ and $\alpha\in I_q\setminus I_{p_\eta}$ and $\alpha<\beta$ then $\alpha\in I_{p_\xi}\setminus I$ and $\beta\in I_{p_\eta}\setminus I$ so $j\in J_{q\alpha}\setminus A_\beta$ and $J_{q_\alpha}\setminus A_\beta\not\subseteq n_{p_\eta}$. So we have $p_\xi\le q$ and $p_\eta\le q$ and p_ξ, p_η are upwards-compatible in P. As ξ and η are arbitrary, $\langle p_\xi\rangle_{\xi\in A}$ is upwards-linked; as $\langle p_\xi\rangle_{\xi<\omega_1}$ is arbitrary, P satisfies Knaster's condition upwards. **Q**

(c) For every $n\in\mathbb{N}$ the set $Q_n=\{p:p\in P,\ n_p\ge n\}$ is cofinal with P. Also $\#(P)=\omega_1$. So P is expressible as $\bigcup_{i\in\mathbb{N}}R_i$ where each R_i is an upwards-directed set meeting every Q_n (Fremlin 84, 31Ba).

For each $\alpha<\omega_1$ we have a $\tilde p_\alpha\in P$ defined by saying that $n_{\tilde p_\alpha}=0$ and $I_{\tilde p_\alpha}=\{\alpha\}$. So there is an $i\in\mathbb{N}$ such that $C=\{\alpha:\tilde p_\alpha\in R_i\}$ is uncountable. Set $B_\alpha=\bigcup\{J_{p\alpha}:p\in R_i\}$ for $\alpha\in C$.

(d)(i) If α, $\beta\in C$ then $B_\alpha\setminus A_\beta$ is finite. **P** There is a $p\in R_i$ such that $\tilde p_\alpha\le p$ and $\tilde p_\beta\le p$ so α, β both belong to I_p. Let q be any member of R_i; then there is an $r\in R_i$ such that $r\ge p$ and $r\ge q$. In this case,

$$J_{q\alpha}\setminus A_\beta\subseteq J_{r\alpha}\setminus A_\beta\subseteq n_p.$$

As q is arbitrary, $B_\alpha\setminus A_\beta\subseteq n_p$ is finite.

(ii) If $\beta\in C$ and $n\in\mathbb{N}$ then $\{\alpha:\alpha\in C\cap\beta,\ B_\alpha\setminus A_\beta\not\subseteq n\}$ is finite. **P** There is a $p\in R_i\cap Q_n$ such that $\tilde p_\alpha\le p$. Suppose that $\beta\in C\cap\alpha\setminus I_p$. Then there is a $q\in R_i$ such that $q\ge p$ and $\beta\in I_q$. In this case

$$(B_\alpha\setminus A_\beta)\setminus n\supseteq(J_{q\alpha}\setminus Q_\beta)\setminus n_p\ne\emptyset.$$

Thus $\{\alpha:\alpha\in C\cap\beta,\ B_\alpha\setminus A_\beta\not\subseteq n\}\subseteq I_p$ is finite. **Q**

(e) So we have an appropriate C and $\langle B_\alpha\rangle_{\alpha\in C}$.

3C Corollary (Shelah 885, 2.3) Suppose that $\mathfrak{m}_{\mathrm{K}} > \omega_1$. Then if λ is any cardinal with $\operatorname{cf}\lambda > \omega_1$, there is no peculiar (λ, ω_1^*)-cut in $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$.

proof ? Otherwise, let $(\langle c_\xi \rangle_{\xi<\lambda}, \langle a_\alpha \rangle_{\alpha<\omega_1})$ be such a cut; choose $A_\alpha \subseteq \mathbb{N}$ such that $A_\alpha^\bullet = a_\alpha$ for $\alpha < \omega_1$. By Proposition 3B we have an uncountable $D \subseteq \omega_1$ and a family $\langle B_\alpha \rangle_{\alpha \in D}$ in $\mathcal{P}\mathbb{N}$ such that $B_\alpha \setminus A_\beta$ is finite for all α, $\beta < \omega_1$ and $\{\alpha : \alpha \in D \cap \beta,\ B_\alpha \setminus A_\beta \not\subseteq n\}$ is finite for every $\beta \in D$ and $n \in \mathbb{N}$. Now, for each $\alpha \in D$, $B_\alpha^\bullet \subseteq a_\beta$ for every $\beta \in D$ and therefore for every $\beta < \omega_1$, so there is a $\xi_\alpha < \lambda$ such that $B_\alpha^\bullet \subseteq c_{\xi_\alpha}$. As $\operatorname{cf}\lambda > \omega_1$, $\zeta = \sup_{\alpha \in D} \xi_\alpha$ is less than λ; let $C \subseteq \mathbb{N}$ be such that $C^\bullet = c_\zeta$. Then $B_\alpha \setminus C$ and $C \setminus A_\alpha$ are finite for every $\alpha \in D$. But now there must be an n such that

$$D' = \{\alpha : \alpha \in D,\ (B_\alpha \setminus C) \cup (C \setminus A_\alpha) \subseteq n\}$$

is uncountable. Let $\beta \in D'$ be such that $D' \cap \beta$ is infinite. Then for any $\alpha \in D' \cap \beta$ we have $B_\alpha \setminus A_\beta \subseteq n$; which is supposed to be impossible. **X**

3D Theorem (Shelah 885, 2.5) Assume that $\mathfrak{m}_{\mathrm{K}} > \omega_1$ and that $\mathfrak{p} < \mathfrak{t}$. Then $\mathfrak{p} > \omega_2$.

proof By 3A, $\mathfrak{p} > \omega_1$; but $\mathfrak{p}$ is regular (Fremlin 84, 21K), so $\operatorname{cf}\mathfrak{p} > \omega_1$. By Corollary 3C, there is no peculiar $(\mathfrak{p}, \omega_1^*)$-cut in $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$. But Theorem 2H tells us that there is a peculiar $(\mathfrak{p}, \kappa^*)$-cut for some regular uncountable $\kappa < \mathfrak{p}$, so κ is at least ω_2 and $\mathfrak{p} > \omega_2$.

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References

Fremlin D.H. [84] *Consequences of Martin's Axiom.* Cambridge U.P., 1984.

Fremlin D.H. [08?] *Measure Theory, Vol. 5: Set-theoretic Measure Theory,* in preparation. *Partial drafts available via* `http://www.essex.ac.uk/maths/staff/fremlin/mt.htm`. *Note that paragraph references are subject to fluctuation.*

Piotrowski Z. & Szymański A. [87] 'Some remarks on category in topological spaces', Proc. Amer. Math. Soc. 101 (1987) 156-160.

Rothberger F. [48] 'On some problems of Hausdorff and Sierpiński', Fundamenta Math. 35 (1948) 29-46.

Shelah S. [885] 'A comment on "$\mathfrak{p} < \mathfrak{t}$"', version of 13.12.06.