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Random equivalence relations

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I apply ideas from Fremlin n06 to examine permutation-invariant measures on spaces of equivalence relations.

1 Equivalence relations

1A Let I be a set. We can identify an equivalence relation on I with a set $R \subseteq [I]^{<\omega}$ such that

$\{i\} \in R$ for every $i \in I$,

$a \in R$ whenever b, $c \in R$, $b \cap c \neq \emptyset$ and $a \subseteq b \cup c$.

Let $\mathcal{E}_I$ be the family of such sets R, so that $\mathcal{E}_I$ is a G_I-invariant closed subset of $\mathcal{P}([I]^{<\omega})$, where G_I is the group of bijections from I to itself. Taking P_I to be the set of G_I-invariant probability measures on $\mathcal{P}([I]^{<\omega})$, as in Fremlin n06, write P_I^e for the set of those $\mu \in P_I$ such that $\mu(\mathcal{E}_I) = 1$. Write EP_I^e for the set of those $\mu \in P_I^e$ such that whenever J, $K \subseteq I$ are disjoint finite sets, $S \subseteq \mathcal{P}J$ and $S' \subseteq \mathcal{P}K$, then

$$\mu\{R : R\lceil J = S,\, R\lceil K = S'\} = \mu\{R : R\lceil J = S\} \cdot \mu\{R : R\lceil K = S'\}$$

(cf. Fremlin n06, 1E).

Note that if $\mu \in P_I^e$ and $J \subseteq I$, then the measure on $\mathcal{P}([J]^{<\omega})$ induced by the function $R \mapsto R\lceil J = R \cap [J]^{<\omega}$ (see Fremlin n06, 1B) belongs to P_J^e, since $R\lceil J \in \mathcal{E}_J$ for every $R \in \mathcal{E}_I$.

If I, J are sets, $z : J \to I$ is a function and $R \subseteq \mathcal{P}I$, then $\tilde{z}(R)$ will be $\{a : a \subseteq J,\, z[a] \in R\}$.

1B Theorem Suppose that $\mu \in P_{\mathbb{N}}^e$. Then for μ-almost every $R \in \mathcal{E}_{\mathbb{N}}$ all the equivalence classes of R have asymptotic density.

proof Fix $m \in \mathbb{N}$. For each $R \in \mathcal{E}_{\mathbb{N}}$ let L_R be the R-equivalence class containing m. For each $j \in \mathbb{N} \setminus \{m\}$ set $X_j(R) = \chi L_R(j)$, so that $X_j(R) = 1$ iff $\{m, j\} \in R$. Then $\langle X_j \rangle_{j>m}$ is an exchangeable family of random variables with finite expectation, so by the strong law of large numbers for exchangeable random variables (Fremlin 01, 276Xe, or Fremlin 03, 459Xa) $\lim_{n\to\infty} \frac{1}{n-m} \sum_{j=m+1}^{n} X_j(R)$ is defined for μ-almost every R, that is, L_R almost surely has asymptotic density.

Since this is true for every m, we have the result.

1C Theorem Let I be an infinite set, and $\mu \in P_I^e$. Then the following are equiveridical.

(i) μ is an extreme point of P_I^e.

(ii) $\mu \in EP_I^e$.

(iii) Whenever J, $K \subseteq I$ are disjoint finite sets of the same size and $S \in \mathcal{E}_J$ and $S' \in \mathcal{E}_K$ are isomorphic, then

$$\mu\{R : R\lceil J = S,\, R\lceil K = S'\} = (\mu\{R : R\lceil J = S\})^2.$$

(iv) There is a sequence $\langle (m_n, T_n) \rangle_{n\in\mathbb{N}}$ such that

(α) $\langle m_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{N}$ and $\lim_{n\to\infty} m_n = \infty$,

(β) $T_n \in \mathcal{E}_{m_n}$ for each n,

(γ) whenever $J \subseteq I$ is finite and $S \subseteq \mathcal{P}J$, then $\mu\{R : R\lceil J = S\}$ is

$$\lim_{n\to\infty} \frac{1}{m_n^{\#(J)}} \#(\{z : z \in m_n^J,\, \tilde{z}(T_n) = S\}).$$

(v) There are a sequence $\langle (m_n, T_n) \rangle_{n\in\mathbb{N}}$ and a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ such that

(α) $\langle m_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{N}$ and $\lim_{n\to\mathcal{F}} m_n = \infty$,

(β) $T_n \in \mathcal{E}_{m_n}$ for each n,

(γ) whenever $J \subseteq I$ is finite and $S \subseteq \mathcal{P}J$, then $\mu\{R : R\lceil J = S\}$ is

$$\lim_{n\to\mathcal{F}}\frac{1}{m_n^{\#(J)}}\#(\{z:z\in m_n^J,\ \tilde z(T_n)=S\}).$$

(vi) There is a countably separated probability space (Ω,Σ,ν) such that μ agrees with the image measure $\nu^I\theta^{-1}$ on the Baire σ-algebra $\mathcal{B}\mathfrak{a}(\mathcal{P}([I]^\omega)$, where ν^I is the product measure on Ω^I and for $x\in\Omega^I$ I write $\theta(x)=\{a:a\in[I]^{<\omega},\ x\restriction a\text{ is constant}\}\in\mathcal{E}_I$.

(vii) There is a point-supported probability measure ν on $\mathbb{N}$ such that $\nu\{m\}\ge\nu\{n\}$ whenever $1\le m\le n$ in $\mathbb{N}$ and μ agrees with $\nu^I\tilde\theta^{-1}$ on the Baire σ-algebra $\mathcal{B}\mathfrak{a}(\mathcal{P}([I]^{<\omega}))$, where ν^I is the product measure on $\mathbb{N}^I$ and for $y\in\mathbb{N}^I$ I write

$$\tilde\theta(y)=\{a:a\in[I]^{<\omega},\ y(i)=y(j)>0\text{ whenever }i,\,j\in a\text{ and }i\ne j\}\in\mathcal{E}_I.$$

proof For parts (i)-(v) we can use the corresponding result Theorem 1J of Fremlin n06.

(i)⇒(ii) The point is just that P_I^e is the set of those $\mu\in P_I$ which give zero measure to every open-and-closed subset of $\mathcal{P}([I]^{<\omega})\setminus\mathcal{E}_I$. It is therefore extremal in the sense that if $\mu,\,\nu\in P_I$ and $\frac{1}{2}(\mu+\nu)\in P_I^e$ then both μ and ν belong to P_I^e. It follows that every extreme point of P_I^e must also be an extreme point of P_I and belongs to EP_I, by Fremlin n06.

(ii)⇒(i) If $\mu\in EP_I^e$ then (by Fremlin n06 again) it is an extreme point of P_I and therefore must be extreme in P_I^e.

(ii)⇒(iv) We may suppose that $\mathbb{N}\subseteq I$. If (ii) is true, then by Fremlin n06 we can find a $\langle(m_n,T_n)\rangle_{n\in\mathbb{N}}$ such that $T_n\subseteq\mathcal{P}m_n$ for each n and the formula of (iv) is satisfied. But in fact the proof in Fremlin n06 shows that T_n can be chosen from a set of positive measure for the measure $\mu_{m_n}\in P_{m_n}$ induced by the embedding $m_n\subseteq I$. Since μ_{m_n}-almost every $T\subseteq\mathcal{P}m_n$ is an equivalence relation, we can arrange that every T_n is an equivalence relation, as required here.

(iv)⇒(v) is trivial.

(v)⇒(iii)⇒(ii) is just a special case of the corresponding chain in Fremlin n06.

(vi)⇒(vii) (For the definition and basic properties of countably separated measure spaces, see Fremlin 03, §343.) Suppose that (Ω,Σ,ν) is a countably separated probability space such that μ agrees with $\nu^I\theta^{-1}$ on Baire sets. Because (Ω,Σ) is countably separated, all singleton subsets of Ω belong to Σ; let D be the countable set $\{t:t\in\Omega,\ \nu\{t\}>0\}$. Enumerate D as $\langle t_n\rangle_{n\in N}$ where either $N\in\mathbb{N}$ or $N=\mathbb{N}$, in such a way that $\nu\{t_m\}\ge\nu\{t_n\}$ if $m\le n\in N$. Define $f:\Omega\to\mathbb{N}$ by setting $f(t_n)=n+1$ if $n\in N$ and $f(t)=0$ if $t\in\Omega\setminus D$; let ν_0 be the image measure νf^{-1}, so that ν_0 is a measure of the type considered in (vii).

The point is that $\nu^2\{(t,t):t\in\Omega\setminus D\}=0$. **P** Because (Ω,Σ) is countably separated, $\{(t,t):t\in\Omega\}$ and $W=\{(t,t):t\in\Omega\setminus D\}$ belong to the product σ-algebra $\Sigma\widehat{\otimes}\Sigma$ and are measured by ν^2. Now Fubini's theorem tells us that

$$\nu^2W=\int_{\Omega\setminus D}\nu\{t\}\nu(dt)=0.\ \mathbf{Q}$$

So if $a\subseteq I$ is finite,

$$\begin{aligned}\{x:x\restriction a\text{ is constant}\}&\triangle\{x:f(x(i))=f(x(j))>0\text{ for all distinct }i,\,j\in a\}\\&\subseteq\{x:f(x(i))=f(x(j))=0\text{ for some distinct }i,\,j\in a\}\end{aligned}$$

is ν^I-negligible.

If $J\subseteq I$ is finite and $S\subseteq\mathcal{P}J$, then

$$\begin{aligned}\mu\{R:R\lceil J=S\}&=\nu^I\{x:x\in\Omega^I,\ S=\{a:a\subseteq J,\ x\restriction a\text{ is constant}\}\}\\&=\nu^I\{x:x\in\Omega^I,\ S=\{a:a\subseteq J,\ f(x(i))=f(x(j))>0\\&\qquad\qquad\text{whenever }i,j\in a\text{ and }i\ne j\}\}\\&=\nu^I\{x:x\in\Omega^I,\ S=\tilde\theta(\tilde f(x))\lceil J\}\end{aligned}$$

(writing $\tilde f(x)=\langle f(x(i))\rangle_{i\in I}$ for $x\in\Omega^I$)

$$=\nu_0^I\{y:y\in\mathbb{N}^I,\ S=\tilde\theta(y)\lceil J\}$$

because $\tilde{f} : \Omega^I \to \mathbb{N}^I$ is inverse-measure-preserving (Fremlin 01, 254H). So μ agrees with $\nu_0^I\tilde{\theta}$ on the basic sets $\{R : R\lceil J = S\}$ and therefore on the Baire σ-algebra. Thus (vii) is true.

(vii)⇒(vi) is trivial.

(iv)⇒(vii) **(α)** The family Q of probability measures ν on $\mathbb{N}$ satisfying the condition in (vii) is compact if we give it the topology generated by the functionals $\nu \mapsto \nu\{n\}$ for $n \geq 1$. (This is the narrow topology on the set of Radon measures on $\mathbb{N}$ if we give $\mathbb{N}$ the compact metrizable topology generated by the sets $\{n\}$, for $n \geq 1$, and their complements.) Next, the map $\nu \mapsto \nu^I\tilde{\theta}^{-1} : Q \to P_I$ is continuous for the narrow topology on P_I. **P** The point is that for any $n \geq 1$ and $\nu \in Q$,

$$\nu^2\{(k,k) : k \geq n\} \leq \max\{\nu\{k\} : k \geq n\} \leq \frac{1}{n}.$$

So if i, $j \in I$ are distinct,

$$\nu^I\{y : y(i) = y(j) \geq n\} \leq \frac{1}{n}.$$

Now if $V \subseteq \mathcal{P}([I]^{<\omega})$ is of the form $\{R : R\lceil J = S\}$ where $J \subseteq I$ is finite and has more than one member, then

$$|\nu^I\tilde{\theta}^{-1}[V] - \nu^I H_n| \leq \frac{\#([J]^2)}{n},$$

where

$$H_n = \{y : y \in \mathbb{N}^I,\ S = \{a : a \subseteq J,\ 0 < y(i) = y(j) < n \text{ whenever } i,\ j \in a \text{ are distinct}\}.$$

Next, the functional $\nu \mapsto \nu^I H_n$ is continuous for each n, so the functional $\nu \mapsto \nu^I\tilde{\theta}^{-1}[V]$, being a uniform limit of these, is continuous. This is enough to show that $\nu \mapsto \nu^I\tilde{\theta}^{-1}$ is continuous for the vague topology on P_I^e. **Q**

So $Q^* = \{\nu^I\tilde{\theta}^{-1} : \nu \in Q\}$ is compact.

(β) At this point, take a sequence $\langle(m_n, T_n)\rangle_{n\in\mathbb{N}}$ as in (iv); of course we can suppose that $m_n \geq 1$ for every n. For each $n \in \mathbb{N}$, let λ_n be the uniform measure on m_n, Ω_n the set of maximal elements of T_n (that is, equivalence classes of the equivalence relation represented by T_n), and ν_n the probability measure on Ω_n such that $\nu_n\{\omega\} = \frac{\#(\omega)}{m_n}$ for each $\omega \in \Omega_n$; that is, $\nu_n = \lambda_n h_n^{-1}$ where $h_n(i) = \omega$ for $i \in \omega \in \Omega_n$. For $x \in \Omega_n^I$ set $\theta_n(x) = \{a : a \in [I]^{<\omega},\ x\restriction a \text{ is constant}\}$. Then $\mu_n = \nu_n^I\theta_n^{-1}$ belongs to Q^* because (vi)⇒(vii). If $J \subseteq I$ is finite and $S \subseteq \mathcal{P}J$, then

$$\begin{aligned}\mu_n\{R : R\lceil J = S\} &= \nu_n^I\theta_n^{-1}\{R : R\lceil J = S\}\\ &= \nu_n^I\{x : x \in \Omega_n^I,\ \{a : a \subseteq J,\ x\restriction a \text{ is constant}\} = S\}\\ &= \nu_n^J\{y : y \in \Omega_n^J,\ \{a : a \subseteq J,\ x\restriction a \text{ is constant}\} = S\}\\ &= \lambda_n^J\{z : z \in m_n^J,\ \{a : a \subseteq J,\ h_n z\restriction a \text{ is constant}\} = S\}\end{aligned}$$

(because the map $z \mapsto \langle h_n(z(j))\rangle_{i\in J}$ is inverse-measure-preserving)

$$= \lambda_n^J\{z : z \in m_n^J,\ \tilde{z}(T_n) = S\}.$$

This is true for every n. On the other hand,

$$\begin{aligned}\mu\{R : R\lceil J = S\} &= \lim_{n\to\infty} \lambda_n^J\{z : z \in m_n^J,\ \tilde{z}(T_n) = S\}\\ &= \lim_{n\to\infty} \mu_n\{R : R\lceil J = S\}.\end{aligned}$$

As J and S are arbitrary, this is enough to ensure that $\mu = \lim_{n\to\infty} \mu_n$ for the narrow topology. Because Q^* is compact, we see that $\mu \in Q^*$, as required.

(vii)⇒(ii) If (vii) is true, J, $K \subseteq I$ are disjoint finite sets, $S \subseteq \mathcal{P}J$ and $S' \subseteq \mathcal{P}K$, let $\mathcal{T}$ be the family of those $T \subseteq \mathcal{P}(J \cup K)$ such that $T\lceil J = S$ and $T\lceil K = S'$. Then

$$\begin{aligned}\mu\{R : R\lceil J = S,\, R\lceil K = S'\} &= \sum_{T\in\mathcal{T}} \mu\{R : R\lceil J\cup K = T\}\\ &= \sum_{T\in\mathcal{T}} \nu^I\{y : \tilde\theta(y)\lceil J\cup K = T\}\\ &= \nu^I\{y : \tilde\theta(y)\lceil J = S,\, \tilde\theta(y)\lceil K = S'\}\\ &= \nu^I\{y : \tilde\theta(y)\lceil J = S\}\cdot\nu^I\{y : \tilde\theta(y)\lceil K = S'\}\end{aligned}$$

(because the sets $\{y : \tilde\theta(y)\lceil J = S\}$, $\{y : \tilde\theta(y)\lceil K = S'\}$ are determined by coordinates in J, K respectively)

$$= \mu\{R : R\lceil J = S\}\cdot\mu\{R : R\lceil K = S'\}.$$

check proof from paper copy!

1D Theorem Suppose that I is uncountable and $\mu \in P^e_I$. Let μ' be the subspace measure on the conegligible G_I-invariant closed set $\mathcal{E}_I$. Then μ' is completion regular iff

(α) $\mu\{R : \{i\}$ is an equivalence class of $R\} = 0$ for some (therefore every) $i \in I$

(β) $\mu\{R : R$ has finitely many equivalence classes$\} = 0$.

proof (a) I had better begin by checking that we are dealing with measurable sets. If $i \in I$, then

$$\{R : R \in \mathcal{E}_I,\, \{i\} \text{ is an equivalence class of } R\} = \{R : R \in \mathcal{E}_I,\, \{i,j\} \notin R \text{ for every } j \ne i\}$$

is closed. Next, for any $n \in \mathbb{N}$,

$$\begin{aligned}&\{R : R \text{ has at most } n \text{ equivalence classes}\}\\ &\qquad= \bigcap_{J\in[I]^{<\omega}} \{R : R\lceil J \text{ has at most } n \text{ equivalence classes}\}\end{aligned}$$

is closed, so $\{R : R$ has finitely many equivalence classes$\}$ is F_σ in $\mathcal{E}_I$.

(b) Suppose that μ' is completion regular. Let $E \subseteq \mathcal{E}_I$ be a non-empty Baire set, $i \in I$ and $n \in \mathbb{N}$. There must be a countable set $J \subseteq I$ such that E is determined by coordinates in J in the sense that if $R \in E$, $S \in \mathcal{E}_I$ and $R\lceil J = S\lceil J$ then $S \in E$.

? If $E \subseteq \{R : \{i\}$ is an equivalence class of $R\}$, take $R \in E$ and set $S = (R\lceil J) \cup [(I \setminus J) \cup \{i\}]^{<\omega}$; then $S \in E$ but $\{i\}$ is not an equivalence class of S. **X**

? If $E \subseteq \{R : R$ has finitely many equivalence classes$\}$, set $S = (R\lceil J) \cup \{\{j\} : j \in I \setminus J\}$. Then $S \in E$ but S has infinitely many equivalence classes. **X**

As E is arbitrary,

$$\begin{aligned}&\mu\{R : \{i\} \text{ is an equivalence class of } R\}\\ &\qquad= \mu\{R : R \text{ has finitely many equivalence classes}\} = 0.\end{aligned}$$

Thus (α) and (β) are true.

(c) Suppose that (α) and (β) are true.

(i) If $i \in I$ and $L \subseteq I$ is infinite, then $\bigcup_{j\in L}\{R : \{i,j\} \in R\}$ is conegligible. **P** Of course it is open, therefore measurable. For finite $K \subseteq I \setminus \{i\}$, set $F_K = \{R : \{i,j\} \notin R$ for every $j \in K\}$. Because μ and μ' are G_I-invariant, $\mu' F_K$ depends only on $\#(K)$; say $\mu' F_K = \alpha_n$ whenever $K \in [I \setminus \{i\}]^n$. Next, $\{F_K : i \in K \in [I \setminus \{i\}]^{<\omega}\}$ is a downwards-directed family of closed sets with intersection the μ-negligible set $\{R : \{i\}$ is an equivalence class of $R\}$, so

$$0 = \inf\nolimits_{i\in K\in[I]^{<\omega}} \mu F_K = \inf\nolimits_{n\ge 1} \alpha_n.$$

It follows that

$$0 = \inf\nolimits_{K\in[L\setminus\{i\}]^{<\omega}} \mu' F_K = \mu'(\mathcal{E}_I \setminus \textstyle\bigcup_{j\in L}\{R : \{i,j\} \in R\}),$$

and $\mu'(\bigcup_{j\in L}\{R : \{i,j\} \in R\}) = 1$. **Q**

(ii) If $J \in [I]^{<\omega}$ and $L \subseteq I$ is infinite, then $\bigcap_{i\in L}\bigcup_{j\in J}\{R : \{i,j\} \in R\}$ is negligible. **P** For $K \in [I]^{<\omega}$ and $m \in \mathbb{N}$ set $F'_{Km} = \{R : R\lceil K$ has at most m equivalence classes$\}$. Then $\mu' F'_{Km}$ depends only on m and $\#(K)$; say $\mu' F'_{Km} = \beta_{mn}$ if $\#(K) = n$. Next, $\{F'_{Km} : K \in [I]^{<\omega}\}$ is a downwards-directed family of closed sets with intersection the μ'-negligible set $\{R : R$ has at most m equivalence classes$\}$, so $\inf_{n\in\mathbb{N}} \beta_{mn} = 0$, for each m. Consequently, taking $m = \#(J)$,

$$\bigcap_{i\in L}\bigcup_{j\in J}\{R : \{i,j\} \in R\} = \{R : R\lceil(L \cup J) \text{ has the same number of equivalence classes as } R\lceil J\}$$

$$\subseteq \bigcap_{K\in[L]^{<\omega}} \{R : R\lceil K \text{ has at most } m \text{ equivalence classes}\}$$

has measure at most $\inf_{n\in\mathbb{N}} \beta_{mn}$ and is negligible.

(iii) Suppose that $J \in [I]^{<\omega}$, $L \subseteq I$ is infinite and $S \in \mathcal{E}_J$. Set $H_L = \{\phi : \phi \in G_I,\ \phi(i) = i$ for $i \in I \setminus L\}$, so that H_L is a subgroup of G_I isomorphic to G_L. If $F \subseteq \{R : R\lceil J \setminus L = S\lceil J \setminus L\}$ and $\mu' F > 0$, there is a $\phi \in H_L$ such that $F \cap \{R : \phi \bullet R\lceil J = S\}$ is non-negligible. **P** We may suppose that L is countable, since $J \cap L$ is finite and shrinking L also shrinks H_L. Induce on $\#(J \cap L)$. If $J \cap L$ is empty we can take ϕ to be the identity. For the inductive step to $\#(J \cap L) = n \geq 1$, take $i \in J \cap L$ and set $J' = J \setminus \{i\}$, $S' = S\lceil J'$. Let $\phi_0 \in H_L$ be such that $F' = F \cap \{R : \phi_0 \bullet R\lceil J' = S'\}$ is non-negligible. Set $L' = L \setminus \phi_0^{-1}[J]$.

case 1 Suppose that $i' \in J'$ and $\{i, i'\} \in S$. By (i) above, there is a $j \in L'$ such that $F_1 = \{R : R \in F'$, $\{j, \phi_0^{-1}(i')\} \in R\}$ is non-negligible. Now let $\phi \in H_L$ be such that $\phi^{-1}(i) = j$ and $\phi^{-1}\restriction J' = \phi_0^{-1}\restriction J'$. In this case, $\phi \bullet R\lceil J = S$ whenever $R \in F_1$. So we have an appropriate ϕ.

case 2 Suppose that $\{i\}$ is an equivalence class of S. Then (ii) tells us that $\bigcup_{j\in L'}\{R : \{j\}$ is an equivalence class of $R\lceil J' \cup \{j\}\}$ is conegligible. So there is a $j \in L'$ such that $F_2 = F' \cap \{R : \{j\}$ is an equivalence class of $R\lceil J' \cup \{j\}\}$ is non-negligible. This time, let $\phi \in H_L$ be such that $\phi^{-1}(i) = j$ and $\phi^{-1}\restriction J' = \phi_0^{-1}\restriction J'$; in this case, $\phi \bullet R\lceil J = S$ whenever $R \in F_2$. So the induction proceeds in this case also. **Q**

(iv) Let $F \subseteq \mathcal{E}_I$ be a non-empty self-supporting closed set. Let $F_1 \supseteq F$ be a zero set with the same measure, and $M \subseteq I$ a countable set such that F_1 is determined by coordinates in M, in the sense that if $R \in F_1$, $R' \in \mathcal{E}_I$ and $R\lceil M = R'\lceil M$ then $R' \in F_1$. Let $\mathcal{V}$ be the family of sets of the form $\{R : R\lceil K = S\}$ where $K \in [M]^{<\omega}$, and $F_2 = F_1 \setminus \bigcup\{V : V \in \mathcal{V},\ V \cap F = \emptyset\}$; then F_2 is also a zero set determined by coordinates in M and $F_2 \supseteq F$. Because I is uncountable, $L = I \setminus M$ is infinite.

? Suppose, if possible, that $F_2 \neq F$. Take $R_0 \in F_2 \setminus F$ and a finite set $J \subseteq I$ such that $\{R : R\lceil J = R_0\lceil J\}$ does not meet F; set $S = R_0\lceil J$. Then $\{R : R\lceil J \setminus L = S\lceil J \setminus L\}$ is a non-empty member of $\mathcal{V}$ which meets F_2, so it meets F_1 in a non-negligible set. By (iii), there is a $\phi \in H_L$ such that $F_3 = F_1 \cap \{R : \phi \bullet R\lceil J = S\}$ is non-negligible. In this case, since μ is invariant under the action of ϕ on $\mathcal{E}_I$, $\{\phi \bullet R : R \in F_3\} = \{\phi \bullet R : R \in F_1\} \cap \{R : R\lceil J = S\}$ is non-negligible. But observe that as F_1 is determined by coordinates in M and $\phi\restriction M$ is the identity, $\phi \bullet R \in F_1$ for every $R \in F_1$, so $\{R : R \in F_1,\ R\lceil J = S\}$ is non-negligible. By the choice of F_1 as a measurable envelope of F, $\emptyset = \{R : R \in F,\ R\lceil J = S\}$ is non-negligible; which is absurd. **X**

Thus $F = F_2$ is a zero set. As F is arbitrary, μ' is inner regular with respect to the zero sets, that is, is completion regular.

check proof from paper copy!

2 A special measure

The original point of this note was the following construction of a special member of P_I^e.

2A Lemma (a) Let I be a finite set. Then there is a (unique) $\mu_I \in P_I^e$ such that whenever $R \in \mathcal{E}_I$ has m equivalence classes of sizes $k_1, \dots, k_m$ then

$$\mu_I\{R\} = \frac{1}{\#(I)!}\prod_{j=1}^{m}(k_j - 1)!.$$

(b) If $J \subseteq I$, then μ_J is the image of μ_I under the map $R \mapsto R\lceil J : \mathcal{E}_I \to \mathcal{E}_J$.

Remark If we assign the value 1 to the empty product, the formula makes sense in the case $I = \emptyset$, $\mathcal{E}_I = \{\emptyset\}$.

proof (a) The point is just that if we set $\alpha(R)=\prod_{j=1}^{m}(k_j-1)!$ when R has equivalence classes of sizes $k_1,\dots,k_m$, then $\sum_{R\in\mathcal{E}_I}\alpha(R)=\#(I)!$. **P** $\alpha(R)$ is just the number of permutations ϕ of I such that R is the orbit equivalence relation of ϕ. **Q**

Thus the given formula defines a probability measure, and of course it is invariant under the action of G_I.

(b) It is enough to consider the case in which $I=J\cup\{i_0\}$ has just one more member than J. Suppose that $R\in\mathcal{E}_J$ has equivalence classes of sizes $k_1,\dots,k_m$. Then $\mathcal{D}_R=\{S:S\in\mathcal{E}_I,\,R=S\restriction J\}$ has just $m+1$ members, and the sizes of their equivalence classes are, respectively,

$$(1,k_1,\dots,k_m),\ (k_1+1,k_2,\dots,k_m),\ (k_1,k_2+1,\dots,k_m),\dots,(k_1,\dots,k_m+1).$$

So

$$\sum_{S\in\mathcal{D}_R}\mu_I\{S\}=\tfrac{1}{(n+1)!}\Big(\prod_{i=1}^{m}(k_i-1)!+\sum_{j=1}^{m}k_j\prod_{i=1}^{m}(k_i-1)!\Big)$$

(where $n=\#(J)$)

$$=\tfrac{1}{(n+1)!}\Big(1+\sum_{j=1}^{m}k_j\Big)\prod_{i=1}^{m}(k_i-1)!=\tfrac{1}{n!}\prod_{i=1}^{m}(k_i-1)!=\mu_J\{R\}.$$

As this is true for every $R\in\mathcal{E}_J$, the restriction map is inverse-measure-preserving, as required.

2B Theorem Let I be any set. Then there is a unique $\mu_I\in P_I^e$ such that whenever $J\subseteq I$ is finite and $R\in\mathcal{E}_J$ has equivalence classes of sizes $k_1,\dots,k_m$ then

$$\mu_I\{S:S\in\mathcal{E}_I,\,S\restriction J=R\}=\tfrac{1}{\#(J)!}\prod_{j=1}^{m}(k_j-1)!.$$

proof Apply Prokhorov's theorem (Fremlin 03, 418M) to the partially ordered set $[I]^{<\omega}$, the restriction maps from $\mathcal{E}_K$ to $\mathcal{E}_J$ for $J\subseteq K\in[I]^{<\omega}$, and the measures on each $\mathcal{E}_J$ described in Lemma 4. This provides a Radon probability measure on $\mathcal{E}_I$ satisfying the given formula, which can now be regarded as a probability measure on $\mathcal{P}([I]^{<\omega})$ if we care about the difference. Because the restriction maps separate the points of $\mathcal{E}_I$, μ_I is uniquely determined.

I have to show that μ_I is invariant under the action of G_I. But it is easy to check that if $\phi\in G_I$ then the corresponding measure $\phi\bullet\mu_I$ satisfies the defining condition for μ_I, so is equal to μ_I.

3C Proposition If I is a set and $J\subseteq I$, then the map $R\mapsto R\restriction J$ is inverse-measure-preserving for the measures μ_I, μ_J described in Theorem 5.

proof The image of μ_I on $\mathcal{E}_J$ satisfies the conditions defining μ_J.

2D Remarks Note that various statistics concerning the measures μ_I are easy to calculate, especially if we remember that if I is finite then μ_I is just the image of the uniform probability measure on G_I under the map taking a permutation ϕ to its orbit equivalence relation.

(a) For instance, if $i,\,j\in I$ are distinct, then

$$\Pr(i,\,j\text{ belong to the same equivalence class})=\frac{1}{2}$$

just because there are only two permutations on $\{i,j\}$. If $i,\,j,\,k\in I$ are distinct, then

$$\Pr(i,\,j,\,k\text{ belong to the same equivalence class})=\frac{2}{6}=\frac{1}{3}$$

(because there are two 3-cycles in $G_{\{i,j,k\}}$), while

$$\Pr(i,\,j,\,k\text{ all belong to different equivalence classes})=\frac{1}{6}.$$

Generally, if $J \subseteq I$ is finite and not empty,

$$\Pr(J \text{ lies within a single equivalence class}) = \frac{1}{\#(J)},$$

$$\Pr(\text{all members of } J \text{ belong to different equivalence classes}) = \frac{1}{\#(J)!}.$$

(b) If $K \in [I]^{<\omega}$, $J_0, \dots, J_m$ are disjoint non-empty subsets of K and $i \in I \setminus K$, then

$$\begin{aligned}&\Pr(i \text{ belongs to the same equivalence class as the members of some } J_j\\ &\qquad \text{given that } J_j \text{ is an equivalence class of the relation restricted to } K \text{ for each } j \le m)\end{aligned}$$

is just $\frac{1}{\#(K)+1}\sum_{l=0}^m \#(J_l)$. **P** Set $j_l = \#(J_l)$ for $l \le m$, $k = \#(K)$. The probability that J_l is an equivalence class of the relation restricted to K for every l is

$$\frac{1}{k!}(k - \sum_{l=0}^m j_l)!\prod_{l=0}^m (j_l - 1)!;$$

if $r \le m$ the probability that, in addition, $J_r \cup \{i\}$ is an equivalence class of the relation restricted to $K \cup \{i\}$ is

$$\frac{1}{(k+1)!}(k - \sum_{l=0}^m j_l)! j_r \prod_{l=0}^m (j_l - 1)!;$$

summing over r, we get

$$\frac{1}{(k+1)!}(\sum_{l=0}^m j_l)(k - \sum_{l=0}^m j_l)!\prod_{l=0}^m (j_l - 1)!;$$

taking the ratio, we get the result. **Q**

(c) If $i \in I$ and $J \subseteq I \setminus \{i\}$ is finite, then

$$\Pr(J \text{ does not meet the equivalence class containing } i) = \frac{1}{\#(J)+1}$$

(count the permutations on $J \cup \{i\}$ which leave i fixed). So if I is infinite and $i \in I$, then

$$\Pr(\text{the equivalence class containing } i \text{ is finite}) = 0.$$

If I is countably infinite, then

$$\Pr(\text{there is a finite equivalence class}) = 0.$$

(d) If I is infinite, then

$$\Pr(\text{there are finitely many equivalence classes}) = 0.$$

P We may suppose that $\mathbb{N} \subseteq I$. Now for any n, $k \in \mathbb{N}$

$$\begin{aligned}&\Pr(R \text{ restricted to } n+1 \text{ has } k \text{ equivalence classes}\\ &\qquad\qquad \text{given that } R \text{ restricted to } n \text{ has } k \text{ equivalence classes}) = \frac{1}{n+1}.\end{aligned}$$

So with probability 1 the number of equivalence classes of R restricted to n increases by infinitely many steps as $n \to \infty$. **Q**

(e) If $J \subseteq I$ is finite, with n members, and $i \in J$, then for $1 \le k \le n$

$$\Pr(\text{the equivalence class containing } i \text{ meets } J \text{ in a set with } k \text{ elements}) = \frac{1}{n}.$$

P We can do the calculation as if $J = I$; now if we generate the cycle containing i of a random permutation on J, we see that the probability of returning to i after k steps (so that the cycle is of length k) is

$$\frac{n-1}{n}\frac{n-2}{n-1}\dots\frac{n-k+1}{n-k+2}\frac{1}{n-k+1} = \frac{1}{n}. \quad \textbf{Q}$$

2E At a fractionally deeper level we have the following computation.

Proposition Suppose that I is a non-empty finite set, with $n = \#(I)$. For $R \in \mathcal{E}_I$ let $\mathcal{C}(R)$ be the set of equivalence classes of R.

(a) For $1 \le k \le n$, set $m_k(R) = \#(\{C : C \in \mathcal{C}(R),\ \#(C) = k\})$. Then

$$\int m_k(R)\mu_I(dR) = \frac{1}{k}.$$

(b) If $f : \{1, \dots, n\} \to \mathbb{C}$ is any function, then

$$\int \sum_{C\in\mathcal{C}(R)} f(\#(C))\mu_I(dR) = \sum_{k=1}^{n} \frac{1}{k} f(k).$$

(c) $\int \#(\mathcal{C}(R))\mu_I(dR) = \sum_{k=1}^{n} \frac{1}{k}$.

proof (a) Consider the set

$$W = \{(R, i) : \text{the equivalence class of } R \text{ containing } i \text{ has } k \text{ members}\}.$$

Then $\mu_I W^{-1}[\{i\}] = \frac{1}{n}$ for every $i \in I$, by 2De. So

$$1 = \int \#(W[R])\mu_I(dR) = \int k\#(\{C : C \in \mathcal{C}(R),\ \#(C) = k\})\mu_I(dR),$$

as required. **Q**

(b) We have

$$\begin{aligned}\int \sum_{C\in\mathcal{C}(R)} f(\#(C))\mu_I(dR) &= \int \sum_{k=1}^{n} \#(\{C : C \in \mathcal{C}(R),\ \#(C) = k\}) f(k)\mu_I(dR)\\ &= \sum_{k=1}^{n} f(k) \int \#(\{C : C \in \mathcal{C}(R),\ \#(C) = k\})\mu_I(dR)\\ &= \sum_{k=1}^{n} \frac{1}{k} f(k)\end{aligned}$$

by (a).

(c) Take f constant in (b). (Or, of course, induce on n.)

2F Alternative construction If $I = \mathbb{N}$, we have an alternative construction of μ_I, as follows. Give $\Omega = \prod_{n\ge1}\{1, \dots, n\}$ its usual product measure λ, each factor being given its uniform measure. Define $h : \Omega \to \mathbb{N}^{\mathbb{N}}$ and $g : \Omega \to \mathcal{E}_{\mathbb{N}}$ by saying that $h(z)(i) = \lim_{n\to\infty} z^n(i + 1) - 1$ and $g(z) = \{a : a \in [\mathbb{N}]^{<\omega}$, $hz{\restriction}a$ is constant$\}$. Note that as $h(z)(i)$ is determined by $z(1), \dots, z(i + 1)$, g is continuous for the product topology on Ω. Also λ is a Radon measure, by Fremlin 03, 416Ua again, so the image measure $\mu = \lambda g^{-1}$ is a Radon measure on $\mathcal{P}([\mathbb{N}]^{<\omega})$. It is easy to check, using the formulae in the proof of Lemma 2A, that μ projects onto μ_n for each $n \in \mathbb{N}$, so that $\mu = \mu_{\mathbb{N}}$.

2G Theorem If $I = \mathbb{N}$, then μ_I-almost surely every equivalence class has asymptotic density, and the sum of the asymptotic densities of the equivalence classes is 1.

proof From Theorem 1B we know that, almost surely, every equivalence class has asymptotic density.

To estimate the sum of these densities, fix $n \ge 1$ for the moment. For $j \ge n$ set $X_{nj}(R) = 1$ if the R-equivalence class of j meets n, 0 otherwise. Then $\langle X_{nj}\rangle_{j\ge n}$ is an exchangeable family of random variables, so $Y_n(R) = \lim_{m\to\infty} \frac{1}{m-n+1} \sum_{j=n}^{m} X_{nj}(R)$ is defined for almost every R. By Lebesgue's Dominated Convergence Theorem, $\mathbb{E}(Y_n)$ is equal to the common value of $\mathbb{E}(X_{nj})$ for $j \ge n$, which is $\frac{n}{n+1}$, by 2Dc.

Since $Y_n \leq 1$, $\Pr(Y_n \leq 1-\epsilon) \leq \frac{1}{(n+1)\epsilon}$ for every $\epsilon > 0$. But observe also that Y_n is less than or equal to the sum Y of the asymptotic densities of the equivalence classes. So $\Pr(Y \leq 1-\epsilon) \leq \frac{1}{(n+1)\epsilon}$ for every $n \in \mathbb{N}$ and $\epsilon > 0$, and $Y = 1$ almost surely, as claimed.

2H Proposition Suppose that $\mathcal{P}([\mathbb{N}]^{<\omega})$ is given the probability $\mu_{\mathbb{N}}$. For $m \in \mathbb{N}$ and $R \in \mathcal{E}_{\mathbb{N}}$, let $C_m(R)$ be the equivalence class of R containing m.

(a) Writing d for asymptotic density, $\Pr(d(C_m) \leq \alpha) = \alpha$ for every $\alpha \in [0,1]$.

(b) For $\alpha > \frac{1}{2}$, $\Pr(\sup_{m\in\mathbb{N}} d(C_m) \geq \alpha) = -\ln\alpha$.

proof (a) For $n > m$, 2De tells us that

$$|\Pr(\frac{1}{n}\#(C_m \cap n) \leq \alpha| \leq \frac{1}{n}.$$

Since we have $d(C_m) = \lim_{n\to\infty} \frac{\#(C_m\cap n)}{n}$ μ_I-a.e. (Theorem 1B), we get the result.

(b) By Theorem 1K of Fremlin 06, $\mu_{\mathbb{N}}$ is the barycenter of just one Radon probability measure λ on the set $EP_{\mathbb{N}}$ of extreme points of $P_{\mathbb{N}}$; as noted in the proof of 1C here, $P^e_{\mathbb{N}}$ is extremal and $EP^e_{\mathbb{N}}$ must be λ-conegligible. We have

$$\int f\,d\mu_{\mathbb{N}} = \int_{EP^e_{\mathbb{N}}} \int_{\mathcal{E}_{\mathbb{N}}} f d\mu \lambda(d\mu)$$

for every bounded Borel measurable $f : \mathcal{E}_{\mathbb{N}} \to \mathbb{R}$.

Set

$$f(R) = \sup_{m\in\mathbb{N}} d^*(C_m(R))$$

where d^* is upper asymptotic density. If $\mu \in EP^e_{\mathbb{N}}$, there is a measure ν on $\mathbb{N}$ as in Theorem 1C(v). By the ordinary strong law of large numbers, $f(R) = \nu\{1\}$ for μ-almost every R. Moreover, if $\nu\{1\} > \frac{1}{2}$, we can say that the probability that the equivalence class containing 0 has density $\nu\{1\}$ is just $\nu\{1\}$, because this is the probability that $x(0) = 1$ when we take a $\nu^{\mathbb{N}}$-random point x of $\mathbb{N}^{\mathbb{N}}$.

So if $\alpha > \frac{1}{2}$,

$$\begin{aligned} \mu\{R : d^*(C_0(R)) \geq \alpha\} &= \nu\{1\} = \int f d\mu \text{ if } \int f d\mu \geq \alpha, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Setting $g(\mu) = \int f d\mu$, $g : P^e_{\mathbb{N}} \to [0,1]$ is Borel measurable, and

$$\mu_{\mathbb{N}}\{R : f(R) \geq \alpha\} = \int \mu\{R : f(R) \geq \alpha\}\lambda(d\mu) = \lambda\{\mu : g(\mu) \geq \alpha\}.$$

On the other hand, we know that if $\alpha > \frac{1}{2}$ then

$$\begin{aligned} 1-\alpha = \mu_{\mathbb{N}}\{R : d^*(C_0(R)) \geq \alpha\} &= \int \mu\{R : d^*(C_0(R)) \geq \alpha\}\lambda(d\mu) \\ &= \int_{\{\mu : f(\mu)\geq\alpha\}} g(\mu)\lambda(d\mu) = \alpha\lambda\{\mu : g(\mu) \geq \alpha\} + \int_\alpha^1 \lambda\{\mu : g(\mu) > t\}dt \end{aligned}$$

this last integral being with respect to Lebesgue measure on $[0,1]$, of course. So, setting $h(t) = \lambda\{\mu : g(\mu) \leq t\}$,

$$1-\alpha = \alpha(1-h(\alpha)) + \int_\alpha^1 (1-h(t))dt$$

for $\alpha > \frac{1}{2}$. It follows, first, that h is continuous on $\left]\frac{1}{2}, 1\right]$, and next that $h'(\alpha)$ is defined and equal to $\frac{1}{\alpha}$ at any point $\alpha \in \left]\frac{1}{2}, 1\right[$; finally, that $h(\alpha) = 1 + \ln\alpha$ for $\alpha > \frac{1}{2}$. So

$$\mu_{\mathbb{N}}\{R : f(R) \geq \alpha\} = 1 - h(\alpha) = \ln\alpha$$

for $\alpha > \frac{1}{2}$.

2I Proposition For $R \in \mathcal{E}_{\mathbb{N}}$ define $C_n(R)$ inductively, for $n \in \mathbb{N}$, by saying that

$$C_n(R) \text{ is the empty set if } \bigcup_{i<n} C_i(R) = \mathbb{N},$$

$$C_n(R) \text{ is the equivalence class of } R \text{ containing } \min(\mathbb{N} \setminus \bigcup_{i<n} C_i(R)) \text{ otherwise.}$$

Define $\boldsymbol{\alpha} : \mathcal{E}_{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by setting $\boldsymbol{\alpha}(R)(n) = d_*(C_n(R))$ for each R and n. For $y \in [0,1]^{\mathbb{N}}$ define $h(y)(n) = y(n)(1 - \sum_{i<n} y(i))$ for each n. If we give $\mathcal{P}([\mathbb{N}]^{<\omega})$ the measure $\mu_{\mathbb{N}}$ and $[0,1]^{\mathbb{N}}$ its usual measure, then $\boldsymbol{\alpha}$ and h have the same distribution.

proof For $m \geq 1$ and $R \in \mathcal{E}_{\mathbb{N}}$ define $\langle C_{mn}(R)\rangle_{n\in\mathbb{N}}$ by saying

$$C_{mn}(R) \text{ is the empty set if } \bigcup_{i<n} C_{mi}(R) = m$$

$$\text{is the equivalence class of } R{\restriction}m \text{ containing } \min(m \setminus \bigcup_{i<n} C_{mi}(R)) \text{ otherwise.}$$

Set $\boldsymbol{\alpha}_m(R) = \langle \frac{1}{m}\#(C_{mn}(R)\rangle_{n\in\mathbb{N}}$. If we generate $R{\restriction}m$ as the orbit equivalence relation of a random permutation of m then it is easy to see that $\#(C_{mn}(R))$ is uniformly distributed in $\{1, \ldots, k\}$ where $k = m - \sum_{i<n} \#(C_{mn}(R))$. Consequently the distributions of $\boldsymbol{\alpha}_0, \ldots$ converge vaguely to the distribution of h.

Of course $C_{mn}(R)$ is not always $m \cap C_n(R)$. But for any given R we see that for any n then $C_{mn}(R)$ will be equal to $m \cap C_n(R)$ for any m so large that $m \cap C_i(R) \neq \emptyset$ for every $i < n$. So $\boldsymbol{\alpha}_m(R)(n) = \frac{1}{m}\#(C_n(R))$ for all m large enough, and

$$\boldsymbol{\alpha}(R)(n) = d_*(C_n(R)) = \lim_{m\to\infty} \boldsymbol{\alpha}_m(R)(n)$$

for all R such that $C_n(R)$ has asymptotic density, that is, for almost all R. Thus $\boldsymbol{\alpha} =_{\text{a.e.}} \lim_{m\to\infty} \boldsymbol{\alpha}_m$ and the distribution of $\boldsymbol{\alpha}$ is the same as the distribution of h.

2J Another process Let I be any set. Define $\phi : [0,1]^{\mathbb{N}} \times [0,1]^I \to \mathcal{E}_I$ as follows. For $u \in [0,1]^{\mathbb{N}}$, $v \in [0,1]^I$ and $i \in I$, set $f_{uv}(i) = \min\{n : v(i) > \prod_{k=0}^{n} u(k)\}$ if this is defined. (Allow $f_{uv}(i) = \infty$ if there is no such n.) Set $\phi(u,v) = \{a : a \in [I]^{<\omega},\ f_{uv}{\restriction}a \text{ is constant}\}$. Now if ν is the usual measure on $[0,1]^{\mathbb{N}} \times [0,1]^I$, ϕ is inverse-measure-preserving for ν and μ_I.

sketch of proof It is enough to consider finite I. Induce on $n = \#(I)$. If $n \leq 1$ the result is trivial. For the inductive step to $n \geq 2$, given $u \in [0,1]^{\mathbb{N}}$ and $v \in [0,1]^I$, let m_{uv} be minimal such that $\prod_{k=0}^{m_{uv}} u(k) < \max_{i\in I} v(i)$; this is almost surely defined. Set $J_{uv} = \{i : v(i) \geq \prod_{k=0}^{m_{uv}} u(k)\}$. Then $\Pr(\#(J_{uv}) = l) = \frac{1}{\#(I)}$ for $1 \leq l \leq \#(I)$. **P** Given that $\prod_{k\leq m} u(k) < \max_{i\in I} v(i) \leq \prod_{k<m} u(k)$, then the $v(i)$ are all uniformly distributed in $[0, \prod_{k<m} u(k)]$ and so is $\alpha = \prod_{k\leq m} u(k)$, so there are $n \cdot n!$ equally likely orderings of α and the v_i, of which $n!$ give the right value to $\#(J_{uv})$. **Q**

For any particular value l of $\#(J_{uv})$, the equivalence class J_{uv} of the relation $\phi(u,v)$ is equally likely to be any of the $\frac{n!}{l!(n-l)!}$ members of $[I]^l$. Now restart the process. We know that $\prod_{k\leq m_{uv}} u(k) = \alpha_{uv}$ say, while $v(i) < \alpha_{uv}$ for $i \in I \setminus J_{uv}$; but otherwise these $v(i)$ are independent and uniformly distributed in $[0, \alpha_{uv}[$. So the inductive hypothesis tells us that we have the distribution $\mu_{I\setminus J_{uv}}$ for $\phi(u,v){\restriction}I \setminus J_{uv}$. This means that for any partition $K_1, \ldots, K_m$ of I into non-empty sets and $l \leq m$,

$$\Pr(\phi(u,v) \text{ has equivalence classes } K_1, \ldots, K_m \text{ and } J_{uv} = K_l) = \frac{1}{n}\frac{k_l!(n-k_l)!}{n!}\frac{\prod_{j\neq l}(k_j-1)!}{(n-k_l)!}$$

where $k_j = \#(K_j)$ for each j, and

$$
\begin{aligned}
&\Pr(\phi(u,v) \text{ has equivalence classes } K_1,\dots,K_m)\\
&\qquad= \sum_{l=1}^{m} \frac{1}{n}\frac{k_l!(n-k_l)!}{n!}\frac{\prod_{j\ne l}(k_j-1)!}{(n-k_l)!}\\
&\qquad= \sum_{l=1}^{m} \frac{1}{n}\frac{k_l}{n!}\prod_{j=1}^{m}(k_j-1)! = \frac{1}{n!}\prod_{j=1}^{m}(k_j-1)!,
\end{aligned}
$$

which is what we need.

Reference

Fremlin D.H. [n06] 'Random hypergraphs', note of 6.5.06 (`http://www.essex.ac.uk/maths/staff/fremlin/preprints.htm`).