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Tao's removal lemma

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I give an account of a theorem from Tao p06 on removable intersections.

1 Definition (a) Let (X,Σ,μ) be a probability space, T a subalgebra of Σ (*not* necessarily a σ-subalgebra) and $\langle\Sigma_i\rangle_{i\in I}$ a family of σ-subalgebras of Σ. I will say that $\langle\Sigma_i\rangle_{i\in I}$ has **T-removable intersections** if whenever $J\subseteq I$ is finite and not empty, $E_i\in\Sigma_i$ for $i\in J$, $\mu(\bigcap_{i\in J}E_i)=\emptyset$ and $\epsilon>0$, there is a family $\langle F_i\rangle_{i\in J}$ such that $F_i\in\mathrm{T}\cap\Sigma_i$ and $\mu(E_i\setminus F_i)\le\epsilon$ for each $i\in J$, and $\bigcap_{i\in J}F_i=\emptyset$. (This is a stronger version of what Tao p06 calls the 'uniform intersection property'.)

(b) If X is a set and Σ, Σ' are two σ-algebras of subsets of X, $\Sigma\vee\Sigma'$ will be the σ-algebra generated by $\Sigma\cup\Sigma'$. If $\langle\Sigma_i\rangle_{i\in I}$ is a family of σ-algebras of subsets of X, I will write $\bigvee_{i\in I}\Sigma_i$ for the σ-algebra generated by $\bigcup_{i\in I}\Sigma_i$.

(c) If (X,Σ,μ) is a probability space and $\mathcal{A}\subseteq\mathcal{E}\subseteq\Sigma$, I will say that $\mathcal{A}$ is **metrically dense** in $\mathcal{E}$ if for every $E\in\mathcal{E}$ and $\epsilon>0$ there is an $F\in\mathcal{A}$ such that $\mu(E\triangle F)\le\epsilon$.

2 Lemma Let (X,Σ,μ) be a probability space and T a subalgebra of Σ. Let $\langle\Sigma_i\rangle_{i\in I}$ be a family of σ-subalgebras of Σ.

(a) $\langle\Sigma_i\rangle_{i\in I}$ has T-removable intersections iff $\langle\Sigma_i\rangle_{i\in J}$ has T-removable intersections for every $J\subseteq I$.

(b) Suppose there is a family $\langle\Sigma'_j\rangle_{j\in J}$ of σ-subalgebras of Σ which has T-removable intersections, and a function $f:I\to J$ such that $\Sigma'_{f(i)}=\Sigma_i$ for every $i\in I$. Suppose that $\mathrm{T}\cap\Sigma_i$ is metrically dense in Σ_i for every $i\in I$. Then $\langle\Sigma_i\rangle_{i\in I}$ has T-removable intersections.

(c) Suppose that, for each $i\in I$, we have a σ-subalgebra Σ'_i of Σ_i such that for every $E\in\Sigma_i$ there is an $E'\in\Sigma'_i$ such that $E\triangle E_i$ is negligible. If $\langle\Sigma'_i\rangle_{i\in I}$ has T-removable intersections, so does $\langle\Sigma_i\rangle_{i\in I}$.

proof (a) is trivial.

(b) Suppose that $K\subseteq I$ is finite and not empty, that $\langle E_i\rangle_{i\in K}\in\prod_{i\in K}\Sigma_i$ is such that $\mu(\bigcap_{i\in K}E_i)=0$, and $\epsilon>0$. Set $n=\#(K)$ and $\eta=\frac{\epsilon}{n+2}>0$. Set $E'_j=\bigcap_{i\in K,f(i)=j}E_i\in\Sigma'_j$ for $j\in f[K]$; then $\bigcap_{j\in f[K]}E'_j$ is negligible, so we have $F'_j\in\mathrm{T}\cap\Sigma'_j$, for $j\in f[K]$, such that $\bigcap_{j\in f[K]}F'_j=\emptyset$ and $\mu(E'_j\setminus F'_j)\le\eta$ for every $j\in f[K]$. Now $\mathrm{T}\cap\Sigma_i$ is metrically dense in Σ_i for each i, so we can find $G_i\in\mathrm{T}\cap\Sigma_i$ such that $\mu(E_i\triangle G_i)\le\eta$ for each i. Set $G'_j=\bigcap_{i\in K,f(i)=j}G_i$ for each $j\in f[K]$. Then

$$\mu(G'_j\setminus F'_j)\le\mu(G'_j\setminus E'_j)+\mu(E'_j\setminus F'_j)\le\sum_{i\in K,f(i)=j}\mu(G_i\setminus E_i)+\eta\le(n+1)\eta.$$

Note that $G'_j\in\mathrm{T}\cap\Sigma'_j$ for each j. Now set $F_i=G_i\setminus(G'_{f(i)}\setminus F'_{f(i)})$ for $i\in K$. Then $F_i\in\mathrm{T}\cap\Sigma_i$ and

$$\mu(E_i\setminus F_i)\le\mu(E_i\setminus G_i)+\mu(G'_{f(i)}\setminus F'_{f(i)})\le(n+1)\eta=\epsilon.$$

Also

$$\begin{aligned}\bigcap_{i\in K}F_i&=\bigcap_{j\in f[K]}\ \bigcap_{i\in K,f(i)=j}G_i\setminus(G'_j\setminus F'_j)\\&=\bigcap_{j\in f[K]}G'_j\setminus(G'_j\setminus F'_j)\subseteq\bigcap_{j\in f[K]}F'_j=\emptyset.\end{aligned}$$

As $\langle E_i\rangle_{i\in K}$ and ϵ are arbitrary, $\langle\Sigma_i\rangle_{i\in I}$ has T-removable intersections.

(c) If $J\subseteq I$ is finite, $\langle E_j\rangle_{j\in J}\in\prod_{j\in J}\Sigma_j$, $\bigcap_{j\in J}E_j$ is negligible and $\epsilon>0$, then for each $j\in J$ let $E'_j\in\Sigma'_j$ be such that $E'_j\triangle E_j$ is negligible. Then $\bigcap_{j\in J}E'_j$ is negligible, so there are $F_j\in\mathrm{T}\cap\Sigma'_j$, for $j\in J$, such that $\mu(E'_j\setminus F_j)\le\epsilon$ for every $j\in J$ and $\bigcap_{j\in J}F_j$ is empty. Now $\mu(E_j\setminus F_j)\le\epsilon$ for every j. As $\langle E_j\rangle_{j\in J}$ and ϵ are arbitrary, $\langle\Sigma_i\rangle_{i\in I}$ has T-removable intersections.

3 Lemma Let (X,Σ,μ) be a probability space and T a subalgebra of Σ. Let I be a set and A an upwards-directed set, and let $\langle\Sigma_{\alpha i}\rangle_{\alpha\in A,i\in I}$ be a family of σ-subalgebras of Σ such that, setting $\Sigma_i=\bigvee_{\alpha\in A}\Sigma_{\alpha i}$ for each i,

(i) $\Sigma_{\alpha i}\subseteq\Sigma_{\beta i}$ whenever $i\in I$ and $\alpha\le\beta$ in A,

(ii) $\langle\Sigma_{\alpha i}\rangle_{i\in I}$ has T-removable intersections for every $\alpha\in A$,

(iii) Σ_i and $\bigvee_{j\in I}\Sigma_{\alpha j}$ are relatively independent over $\Sigma_{\alpha i}$ for every $i\in I$ and $\alpha\in A$.

Then $\langle\Sigma_i\rangle_{i\in I}$ has T-removable intersections.

proof Take a non-empty finite set $J\subseteq I$, a family $\langle E_i\rangle_{i\in J}$ such that $E_i\in\Sigma_i$ for every $i\in J$ and $\bigcap_{i\in J}E_i$ is negligible, and $\epsilon>0$. Set $\delta=\frac{1}{\#(J)+1}$, $\eta=\delta\sqrt{\frac{\epsilon}{2}}>0$. For each $i\in J$ there are an $\alpha\in A$ and an $E_i'\in\Sigma_{\alpha i}$ such that $\mu(E_i\triangle E_i')\le\eta^2$ (because A is upwards-directed); we can suppose that it is the same α for each i. Let $g_i:X\to[0,1]$ be a $\Sigma_{\alpha i}$-measurable function which is a conditional expectation of χE_i on $\Sigma_{\alpha i}$; then

$$\|\chi E_i-g_i\|_2\le\|\chi E_i-\chi E_i'\|_2\le\eta$$

(cf. Fremlin 01, 244Nb). Set $E_i''=\{x:g_i(x)\ge1-\delta\}\in\Sigma_{\alpha i}$; then

$$\mu(E_i\setminus E_i'')=\mu\{x:\chi E_i(x)-g_i(x)>\delta\}\le\frac{\eta^2}{\delta^2}=\frac{\epsilon}{2}.$$

Set $E=\bigcap_{i\in J}E_i''$. Then $\mu E=0$. **P** Since $\mu(\bigcap_{i\in J}E_i)=0$, $\mu E\le\sum_{i\in J}\mu(E\setminus E_i)$. For any $i\in J$, setting $H_i=X\cap\bigcap_{j\in J\setminus\{i\}}E_j''$,

$$\begin{aligned}\mu(E\setminus E_i)=\mu((E_i''\setminus E_i)\cap H_i)&=\int\chi(E_i''\setminus E_i)\times\chi H_i d\mu\\&=\int h_i\times h_i'd\mu=\int h_i\times\chi H_i d\mu\end{aligned}$$

where $h_i=(1-g_i)\chi E_i''$, h_i' are conditional expectations of $\chi(E_i''\setminus E_i)$, χH_i on $\Sigma_{\alpha i}$, because Σ_i and $\bigvee_{j\in J}\Sigma_{\alpha j}$ are relatively independent over $\Sigma_{\alpha i}$. But $h_i\le\delta\chi E_i''$, so

$$\mu(E\setminus E_i)\le\delta\mu(E_i''\cap H_i)=\delta\mu E.$$

Summing, we have

$$\mu E\le\delta\#(J)\mu E;$$

but $\delta\#(J)<1$, so $\mu E=0$. **Q**

Because $\langle\Sigma_{\alpha i}\rangle_{i\in I}$ has T-removable intersections, there are $F_i\in\mathrm{T}\cap\Sigma_{\alpha i}\subseteq\mathrm{T}\cap\Sigma_i$, for $i\in J$, such that $\bigcap_{i\in I}F_i=\emptyset$ and $\mu(E_i''\setminus F_i)\le\frac{\epsilon}{2}$ for each i; in which case $\mu(E_i\setminus F_i)\le\epsilon$ for each i. As $\langle E_i\rangle_{i\in J}$ and ϵ are arbitrary, $\langle\Sigma_i\rangle_{i\in I}$ has T-removable intersections.

4 Lemma Let (X,Σ,μ) be a probability space, T a subalgebra of Σ, and $\langle\Sigma_i\rangle_{i\in I}$ a finite family of σ-subalgebras of Σ which has T-removable intersections; suppose that $\mathrm{T}\cap\Sigma_i$ is metrically dense in Σ_i for each i. Set $\Sigma^*=\bigvee_{i\in I}\Sigma_i$. Suppose that we have a finite set J, a function $g:J\to I$ and a family $\langle\Lambda_j\rangle_{j\in J}$ of σ-subalgebras of Σ such that

$\langle\Lambda_j\rangle_{j\in J}$ is relatively independent over Σ^*,

for each $j\in J$, Λ_j and Σ^* are relatively independent over $\Sigma_{g(j)}$,

for each $j\in J$, $\mathrm{T}\cap\Lambda_j$ is metrically dense in Λ_j.

Let A be a finite set and $f:A\to I$, $\phi:A\to\mathcal{P}J$ functions such that $\Sigma_{g(j)}\subseteq\Sigma_{f(\alpha)}$ whenever $\alpha\in A$ and $j\in\phi(\alpha)$. Suppose that

for each $\alpha\in A$, $\bigvee_{j\in\phi(\alpha)}\Lambda_j$ and $\Sigma^*\vee\bigvee_{j\in J\setminus\phi(\alpha)}\Lambda_j$ are relatively independent over $\Sigma_{f(\alpha)}$.

Set $\tilde\Sigma_\alpha=\Sigma_{f(\alpha)}\vee\bigvee_{j\in\phi(\alpha)}\Lambda_j$ for $\alpha\in A$. Then $\langle\tilde\Sigma_\alpha\rangle_{\alpha\in A}$ has T-removable intersections.

proof Of course we can suppose that A is non-empty, and that $J=\bigcup_{\alpha\in A}\phi(\alpha)$.

(a) To begin with, suppose that every Λ_j is actually a finite subalgebra of T.

(i) Take a non-empty set $B \subseteq A$, $\langle E_\alpha \rangle_{\alpha \in B} \in \prod_{\alpha \in B} \tilde{\Sigma}_\alpha$ such that $\bigcap_{\alpha \in B} E_\alpha$ is negligible, and $\epsilon > 0$. Set $K = \bigcup_{\alpha \in B} \phi(\alpha)$. Let $\mathcal{A}$ be the set of atoms of $\bigvee_{j \in K} \Lambda_j$ and set $\eta = \frac{\epsilon}{\#(\mathcal{A})} > 0$.

(ii) For each $H \in \mathcal{A}$ and $\alpha \in B$, let $C(H, \alpha)$ be the atom of $\bigvee_{j \in \phi(\alpha)} \Lambda_j$ including H. Then there is a family $\langle F_{H\alpha} \rangle_{\alpha \in B}$, with empty intersection, such that $F_{H\alpha} \in \mathrm{T} \cap \Sigma_\alpha$ and $\mu(E_\alpha \cap C(H, \alpha) \setminus F_{H\alpha}) \le \eta$ for each $\alpha \in B$. **P** For each $j \in K$, let H_j be the atom of Λ_j including H, $h_j : X \to [0, 1]$ a $\Sigma_{g(j)}$-measurable function which is a conditional expectation of χH_j on $\Sigma_{g(j)}$, and $G_j = \{x : h_j(x) > 0\}$. Because Λ_j and Σ^* are relatively independent over $\Sigma_{g(j)}$, and $\Sigma_{g(j)} \subseteq \Sigma^*$, h_j is a conditional expectation of χH_j on Σ^* for each j. Because $\langle \Lambda_j \rangle_{j \in K}$ is relatively independent over Σ^*, $h = \prod_{j \in K} h_j$ is a conditional expectation of $\chi(X \cap \bigcap_{j \in K} H_j)$ on Σ^*. (For the trivial case in which $K = \emptyset$, take $h = \chi X$.) Now for each $\alpha \in B$ we have an $E'_\alpha \in \Sigma_{f(\alpha)}$ such that $E_\alpha \cap C(H, \alpha) = E'_\alpha \cap C(H, \alpha)$. We see that

$$\int_{\bigcap_{\alpha \in B} E'_\alpha} h = \mu(\textstyle\bigcap_{\alpha \in B} E'_\alpha \cap \bigcap_{j \in K} H_j) \le \mu(\bigcap_{\alpha \in B}(E_\alpha \cap C(H, \alpha)))$$

is zero, so $\bigcap_{\alpha \in B} E'_\alpha \cap \bigcap_{j \in K} G_j$ is negligible. Set $E''_\alpha = E'_\alpha \cap \bigcap_{j \in \phi(\alpha)} G_j$ for each α; then $E''_\alpha \in \Sigma_{f(\alpha)}$, because $G_j \in \Sigma_{g(j)} \subseteq \Sigma_{f(\alpha)}$ whenever $j \in \phi(\alpha)$. Also $\bigcap_{\alpha \in B} E''_\alpha$ is negligible.

Because $\langle \Sigma_i \rangle_{i \in I}$ has T-removable intersections and $\mathrm{T} \cap \Sigma_i$ is metrically dense in Σ_i for each i, $\langle \Sigma_{f(\alpha)} \rangle_{\alpha \in B}$ has T-removable intersections (Lemma 2b). So we have $F_{H\alpha} \in \mathrm{T} \cap \Sigma_{f(\alpha)}$, for $\alpha \in B$, such that $\bigcap_{\alpha \in A} F_{H\alpha} = \emptyset$ and $\mu(E''_\alpha \setminus F_{H\alpha}) \le \eta$ for every α.

If $\alpha \in B$ and $j \in \phi(\alpha)$,

$$0 = \int_{E'_\alpha \setminus G_j} h_j = \mu(H_j \cap E'_\alpha \setminus G_j)$$

because h_j is a conditional expectation of χH_j on Σ^*. So if $\alpha \in B$,

$$\begin{aligned}\mu(E_\alpha \cap C(H, \alpha) \setminus F_{H\alpha}) &= \mu(E'_\alpha \cap C(H, \alpha) \setminus F_{H\alpha}) \\ &\le \mu(E''_\alpha \setminus F_{H\alpha}) + \sum_{j \in \phi(\alpha)} \mu(E'_\alpha \cap H_j \setminus G_j)\end{aligned}$$

(because $C(H, \alpha) = X \cap \bigcap_{j \in \phi(\alpha)} H_j$)

$$\le \eta,$$

as required. **Q**

(iii) For $\alpha \in B$ let $\mathcal{A}_\alpha$ be the set of atoms of $\bigvee_{j \in \phi(\alpha)} \Lambda_j$ and set

$$F_\alpha = \textstyle\bigcup_{G \in \mathcal{A}_\alpha}(G \cap \bigcap_{H \in \mathcal{A}, H \subseteq G} F_{H\alpha}).$$

Then $F_\alpha \in \mathrm{T} \cap \tilde{\Sigma}_\alpha$ and

$$\begin{aligned}\mu(E_\alpha \setminus F_\alpha) &= \sum_{G \in \mathcal{A}_\alpha} \mu(E_\alpha \cap G \setminus F_\alpha) \\ &\le \sum_{G \in \mathcal{A}_\alpha} \sum_{H \in \mathcal{A}, H \subseteq G} \mu(E_\alpha \cap G \setminus F_{H\alpha}) \\ &= \sum_{H \in \mathcal{A}} \mu(E_\alpha \cap C(H, \alpha) \setminus F_{H\alpha}) \le \eta\#(\mathcal{A}) = \epsilon.\end{aligned}$$

If $H \in \mathcal{A}$ then

$$H \cap F_\alpha \subseteq F_\alpha \cap C(H, \alpha) = \textstyle\bigcap_{H' \in \mathcal{A}, H' \subseteq C(H, \alpha)} C(H, \alpha) \cap F_{H'\alpha} \subseteq F_{H\alpha}$$

for each α. So $H \cap \bigcap_{\alpha \in A} F_\alpha$ is empty. But $X = \bigcup \mathcal{A}$ so $\bigcap_{\alpha \in A} F_\alpha = \emptyset$. As $\langle E_\alpha \rangle_{\alpha \in B}$ and ϵ are arbitrary, $\langle \tilde{\Sigma}_\alpha \rangle_{\alpha \in A}$ has T-removable intersections.

(b) Next, suppose that each Λ_j is the σ-algebra generated by $\mathrm{T} \cap \Lambda_j$.

(i) For $L \in [\mathrm{T}]^{<\omega}$, $j \in J$ and $\alpha \in A$ write Λ_{Lj} for the algebra generated by $\Lambda_j \cap L$ and $\Sigma'_{L\alpha} = \Sigma_{f(\alpha)} \vee \bigvee_{j \in \phi(\alpha)} \Lambda_{Lj}$. Then

$$\Sigma'_{K\alpha} \subseteq \Sigma'_{L\alpha} \text{ whenever } \alpha \in A \text{ and } K \subseteq L \in [\mathrm{T}]^{<\omega},$$

$$\bigvee_{L\in[\mathrm{T}]^{<\omega}} \Sigma'_{L\alpha} = \Sigma_{f(\alpha)} \vee \bigvee_{j\in\phi(\alpha)} \bigvee_{L\in[\mathrm{T}]^{<\omega}} \Lambda_{Lj} = \Sigma_{f(\alpha)} \vee \bigvee_{j\in\phi(\alpha)} \Lambda_j = \tilde{\Sigma}_\alpha$$

because each Λ_j is the σ-algebra generated by $\mathrm{T} \cap \Lambda_j = \bigcup_{L\in[\mathrm{T}]^{<\omega}} \Lambda_{Lj}$. By (a), $\langle\Sigma_{L\alpha}\rangle_{\alpha\in A}$ has T-removable intersections for every $L \in [\mathrm{T}]^{<\omega}$.

(ii) Suppose that $\alpha \in A$ and $L \in [\mathrm{T}]^{<\omega}$. Then $\bigvee_{j\in\phi(\alpha)} \Lambda_j$ and $\Sigma^* \vee \bigvee_{j\in J\setminus\phi(\alpha)} \Lambda_j$ are relatively independent over $\Sigma_{f(\alpha)}$, by hypothesis. So $\tilde{\Sigma}_\alpha = \Sigma_{f(\alpha)} \vee \bigvee_{j\in\phi(\alpha)} \Lambda_j$ and $\Sigma^* \vee \bigvee_{j\in J\setminus\phi(\alpha)} \Lambda_j$ are relatively independent over $\Sigma_{f(\alpha)}$. Because $\Sigma_{f(\alpha)} \subseteq \Sigma'_{L\alpha} \subseteq \tilde{\Sigma}_\alpha$, $\tilde{\Sigma}_\alpha$ and $\Sigma^* \vee \bigvee_{j\in J\setminus\phi(\alpha)} \Lambda_j$ are relatively independent over $\Sigma'_{L\alpha}$. Because $\bigvee_{j\in\phi(\alpha)} \Lambda_{Lj} \subseteq \Sigma'_{L\alpha}$, $\tilde{\Sigma}_\alpha$ and $\Sigma^* \vee \bigvee_{j\in J} \Lambda_{Lj}$ are relatively independent over $\Sigma'_{L\alpha}$. So $\tilde{\Sigma}_\alpha$ and

$$\bigvee_{\beta\in A} \Sigma'_{L\beta} = \bigvee_{\beta\in A}(\Sigma_{f(\beta)} \vee \bigvee_{j\in\phi(\beta)} \Lambda_{Lj}) \subseteq \Sigma^* \vee \bigvee_{j\in J} \Lambda_{Lj}$$

are relatively independent over $\Sigma'_{L\alpha}$.

(iii) With (i), this shows that the family $\langle\Sigma'_{L\alpha}\rangle_{L\in[\mathrm{T}]^{<\omega},\alpha\in A}$ satisfies the conditions of Lemma 3, and $\langle\tilde{\Sigma}_\alpha\rangle_{\alpha\in A}$ has T-removable intersections.

(c) Finally, for the general case, let Λ'_j be the σ-algebra generated by $\Lambda_j \cap \mathrm{T}$ for $j \in J$, and $\Sigma'_\alpha = \Sigma_{f(\alpha)} \vee \bigvee_{j\in\phi(\alpha)} \Lambda'_j$ for $\alpha \in A$. If $j \in J$ and $F \in \Lambda_j$, there is an $F' \in \Lambda'_j$ such that $F\triangle F'$ is negligible; so if $\alpha \in A$ and $E \in \tilde{\Sigma}_\alpha$, there is an $E' \in \Sigma'_\alpha$ such that $E\triangle E'$ is negligible. By (b), $\langle\Sigma'_\alpha\rangle_{\alpha\in A}$ has T-removable intersections; by Lemma 2c, $\langle\tilde{\Sigma}_\alpha\rangle_{\alpha\in A}$ has T-removable intersections.

5 Theorem (Tao p06, Theorem 4.2) Let (X, Σ, μ) be a probability space, and T a subalgebra of Σ. Let Γ be a partially ordered set such that $\gamma \wedge \delta = \inf\{\gamma, \delta\}$ is defined in Γ for all $\gamma, \delta \in \Gamma$, and $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma}$ a family of σ-subalgebras of Σ such that

(i) $\mathrm{T} \cap \Sigma_\gamma$ is metrically dense in Σ_γ for every $\gamma \in \Gamma$,

(ii) if $\gamma, \delta \in \Gamma$ and $\gamma \le \delta$ then $\Sigma_\gamma \subseteq \Sigma_\delta$,

(iii) if $\gamma \in \Gamma$ and Δ, Δ' are finite subsets of Γ such that $\delta \wedge \gamma \in \Delta'$ for every $\delta \in \Delta$, then Σ_γ and $\bigvee_{\delta\in\Delta} \Sigma_\delta$ are relatively independent over $\bigvee_{\delta\in\Delta'} \Sigma_\delta$.

Then $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma}$ has T-removable intersections.

proof (a) To begin with (down to the end of (d) below) suppose that Γ is finite. In this case, we have a rank function $r : \Gamma \to \mathbb{N}$ such that $r(\gamma) = \min\{n : n \in \mathbb{N}, r(\delta) < n \text{ for every } \delta < \gamma\}$ for each $\gamma \in \Gamma$. For $a \subseteq \Gamma$ set $\tilde{\Sigma}_a = \bigvee_{\gamma\in a} \Sigma_\gamma$; note that $\mathrm{T} \cap \Sigma_a$ is always metrically dense in Σ_a.

Let A be the family of those sets $a \subseteq \Gamma$ such that $\gamma \in a$ whenever $\gamma \le \delta \in a$. For $n \in \mathbb{N}$ set $\Gamma_n = \{\gamma : r(\gamma) = n\}$ and $A_n = \{a : a \in A, r(\gamma) < n \text{ for every } \gamma \in a\}$

(b) Suppose that a, b, c are subsets of Γ and that $\gamma \wedge \delta \in c$ whenever $\gamma \in a$ and $\delta \in b \cup (a \setminus \{\gamma\})$. Then

(i) $\langle\Sigma_\gamma\rangle_{\gamma\in a}$ is relatively independent over $\tilde{\Sigma}_c$,

(ii) $\tilde{\Sigma}_a$ and $\tilde{\Sigma}_b$ are relatively independent over $\tilde{\Sigma}_c$.

P Induce on $\#(a)$. If $a = \emptyset$ then $\tilde{\Sigma}_a = \{\emptyset, X\}$ and the result is trivial. For the inductive step, take $\gamma_0 \in a$ and set $a' = a \setminus \{\gamma_0\}$. Then the inductive hypothesis tells us that $\langle\Sigma_\gamma\rangle_{\gamma\in a'}$ is relatively independent over $\tilde{\Sigma}_c$ and that $\tilde{\Sigma}_{a'}$ and $\tilde{\Sigma}_b$ are relatively independent over $\tilde{\Sigma}_c$. We also see that $\gamma_0 \wedge \delta \in c$ whenever $\delta \in a'$, so that Σ_{γ_0} and $\tilde{\Sigma}_{a'}$ are relatively independent over $\tilde{\Sigma}_c$. But this means that $\langle\Sigma_\gamma\rangle_{\gamma\in a}$ is relatively independent over $\tilde{\Sigma}_c$. Similarly, because in fact $\gamma_0 \wedge \delta \in c$ for every $\delta \in a' \cup b$, Σ_{γ_0} and $\tilde{\Sigma}_{a'} \vee \tilde{\Sigma}_b$ are relatively independent over $\tilde{\Sigma}_c$; so the triple Σ_{γ_0}, $\tilde{\Sigma}_{a'}$ and $\tilde{\Sigma}_b$ are relatively independent over $\tilde{\Sigma}_c$, and $\tilde{\Sigma}_a = \Sigma_{\gamma_0} \vee \tilde{\Sigma}_{a'}$ and $\tilde{\Sigma}_b$ are relatively independent over $\tilde{\Sigma}_c$. Thus the induction continues. **Q**

(c) For each $n \in \mathbb{N}$, $\langle\tilde{\Sigma}_a\rangle_{a\in A_n}$ has T-removable intersections. **P** Induce on n. If $n = 0$ then $A_n = \{\emptyset\}$ and the result is trivial. For the inductive step to $n+1 \ge 1$, apply Lemma 4, as follows. The inductive hypothesis tells us that $\langle\tilde{\Sigma}_a\rangle_{a\in A_n}$ has T-removable intersections, and we know that $\mathrm{T} \cap \tilde{\Sigma}_a$ is always metrically dense in $\tilde{\Sigma}_a$. Set

$$\Sigma^* = \bigvee_{a\in A_n} \tilde{\Sigma}_a = \tilde{\Sigma}_d$$

where $d=\bigcup_{m<n}\Gamma_m$ is the largest member of A_n. Define $g:\Gamma_n\to A_n$ by setting $g(\gamma)=\{\delta:\delta<\gamma\}$. Then $\gamma\wedge\delta\in d$ for all distinct γ, $\delta\in\Gamma_n$, so $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma_n}$ is relatively independent over Σ^*, by (b-i) just above. If $\gamma\in\Gamma_n$ and $\delta\in d$, $\gamma\wedge\delta\in g(\gamma)$, so Σ_γ and Σ^* are relatively independent over $\tilde\Sigma_{g(\gamma)}$. Of course $\mathrm{T}\cap\Sigma_\gamma$ is metrically dense in Σ_γ for every $\gamma\in\Gamma_n$.

For $a\in A_{n+1}$, set $\phi(a)=a\cap\Gamma_n$ and

$$f(a)=a\setminus\phi(a)=a\cap\bigcup_{m<n}\Gamma_m\in A_n.$$

If $\gamma\in\phi(a)$ then $g(\gamma)\subseteq a$, by the definition of A, so $g(\gamma)\subseteq= f(a)$ and $\tilde\Sigma_{g(\gamma)}\subseteq\tilde\Sigma_{f(a)}$. Finally, by (b-ii), $\bigvee_{\gamma\in\phi(a)}\Sigma_\gamma$ and $\Sigma^*\vee\bigvee_{\gamma\in\Gamma_n\setminus\phi(a)}\Sigma_\gamma$ are relatively independent over $\tilde\Sigma_{f(a)}$, because if $\gamma\in\phi(a)$ and $\delta\in d\cup(\Gamma_n\setminus\phi(a))$ then $\gamma\wedge\delta\in g(\gamma)\subseteq f(a)$.

So all the hypotheses of Lemma 4 are satisfied, and

$$\langle\tilde\Sigma_{f(a)}\vee\bigvee_{\gamma\in\phi(a)}\Sigma_\gamma\rangle_{a\in A_{n+1}}=\langle\tilde\Sigma_a\rangle_{a\in A_{n+1}}$$

has T-removable intersections. Thus the induction proceeds. **Q**

(d) Because Γ is finite, there is some n such that $A=A_n$. Now, for each $\gamma\in\Gamma$, set $e_\gamma=\{\delta:\delta\le\gamma\}$; then $e_\gamma\in A$ and $\Sigma_\gamma=\tilde\Sigma_{e_\gamma}$. By Lemma 2b, or otherwise, $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma}$ has T-removable intersections, as required.

(e) Thus the theorem is true when Γ is finite. For the general case, take any finite $\Gamma_0\subseteq\Gamma$ and set $\Gamma'=\{\inf a: a\subseteq\Gamma_0\text{ is non-empty}\}$. Then Γ' is finite and closed under $\wedge$, and $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma'}$ satisfies the conditions of the theorem. So $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma'}$ and $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma_0}$ have T-removable intersections. As Γ_0 is arbitrary, $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma}$ has T-removable intersections, and the proof is complete.

6 Corollary Let (X,Σ,μ) be a probability space, $\langle\Sigma_i\rangle_{i\in I}$ an independent family of σ-subalgebras of Σ, and T a subalgebra of Σ such that $\mathrm{T}\cap\Sigma_i$ is metrically dense in Σ_i for every $i\in I$. For $J\subseteq I$ set $\tilde\Sigma_J=\bigvee_{i\in J}\Sigma_i$. Then $\langle\tilde\Sigma_J\rangle_{J\subseteq I}$ has T-removable intersections.

proof Of course T is metrically dense in $\tilde\Sigma_J$ for every J, and $\tilde\Sigma_J\subseteq\tilde\Sigma_K$ whenever $J\subseteq K\subseteq I$. If $J\subseteq I$ and $\mathcal{K}$, $\mathcal{K}'$ are families of subsets of I such that $J\cap K\in\mathcal{K}'$ for every $K\in\mathcal{K}$, set $L=\bigcup\mathcal{K}$ and $L'=\bigcup\mathcal{K}'$; then $\bigvee_{K\in\mathcal{K}}\tilde\Sigma_K=\tilde\Sigma_L$ and $\bigvee_{K\in\mathcal{K}'}\tilde\Sigma_K=\tilde\Sigma_{L'}$. Also $J\cap L\subseteq L'$. So $\tilde\Sigma_{J\setminus L'}$, $\tilde\Sigma_{L\setminus L'}$ and $\tilde\Sigma_{L'}$ are independent; consequently $\tilde\Sigma_{J\setminus L'}$ and $\tilde\Sigma_{L\setminus L'}$ are relatively independent over $\tilde\Sigma_{L'}$, $\tilde\Sigma_{J\cup L'}=\tilde\Sigma_J\vee\tilde\Sigma_{L'}$ and $\tilde\Sigma_{L\cup L'}=\tilde\Sigma_L\vee\tilde\Sigma_{L'}$ are relatively independent over $\tilde\Sigma_{L'}$, and $\tilde\Sigma_J$ and $\tilde\Sigma_L$ are relatively independent over $\tilde\Sigma_{L'}$. But this is just what is required to make Theorem 5 applicable.

7 Measure algebras As in Fremlin 03, §458, all the ideas here can be transferred to the measure-algebra setting. Suppose that $(\mathfrak{A},\bar\mu)$ is a probability algebra, $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$, and $\langle\mathfrak{A}_i\rangle_{i\in I}$ a family of closed subalgebras of $\mathfrak{A}$; then we can say that $\langle\mathfrak{A}_i\rangle_{i\in I}$ has **$\mathfrak{B}$-removable intersections** if whenever $J\subseteq I$ is finite and not empty, $a_i\in\mathfrak{A}_i$ for $i\in J$, $\inf_{i\in J}a_i=0$ and $\epsilon>0$, there is a family $\langle b_i\rangle_{i\in J}$ such that $b_i\in\mathfrak{B}\cap\mathfrak{A}_i$ and $\mu(a_i\setminus b_i)\le\epsilon$ for each $i\in J$, and $\inf_{i\in J}b_i=0$. Now Theorem 5 becomes

> Let $(\mathfrak{A},\bar\mu)$ be a probability space, and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$. Let Γ be a partially ordered set such that $\gamma\wedge\delta=\inf\{\gamma,\delta\}$ is defined in Γ for all γ, $\delta\in\Gamma$, and $\langle\mathfrak{A}_\gamma\rangle_{\gamma\in\Gamma}$ a family of closed subalgebras of $\mathfrak{A}$ such that
>
> (i) for every $\gamma\in\Gamma$, $\mathfrak{B}\cap\mathfrak{A}_\gamma$ is dense in $\mathfrak{A}_\gamma$ for the measure-algebra topology on $\mathfrak{A}$ (Fremlin 02, §323),
>
> (ii) if γ, $\delta\in\Gamma$ and $\gamma\le\delta$ then $\mathfrak{A}_\gamma\subseteq\mathfrak{A}_\delta$,
>
> (iii) if $\gamma\in\Gamma$ and Δ, Δ' are finite subsets of Γ such that $\delta\wedge\gamma\in\Delta'$ for every $\delta\in\Delta$, then $\mathfrak{A}_\gamma$ and $\bigvee_{\delta\in\Delta}\mathfrak{A}_\delta$ are relatively independent over $\bigvee_{\delta\in\Delta'}\mathfrak{A}_\delta$,
>
> where $\bigvee_{\delta\in\Delta}\mathfrak{A}_\delta$ is the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{\delta\in\Delta}\mathfrak{A}_\delta$. Then $\langle\mathfrak{A}_\gamma\rangle_{\gamma\in\Gamma}$ has $\mathfrak{B}$-removable intersections.

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