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Random hypergraphs

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I examine permutation-invariant probability measures on spaces of hypergraphs, discussing extreme points (1J) and theorems of T.Tao on removable intersections (1N) and hypergraphs (2B).

1 Permutation-invariant measures on $\mathcal{P}([I]^{<\omega})$

1A The context Let I be a set. Then $\mathcal{P}([I]^{<\omega})$ is a compact Hausdorff space. Let G_I be the set of permutations of I, and for $\phi \in G_I$, $R \subseteq [I]^{<\omega}$ set

$$\phi \bullet R = \{\phi[a] : a \in R\},$$

so that $\bullet$ is a continuous action of G_I on $\mathcal{P}([I]^{<\omega})$ when G_I is given its topology of pointwise convergence (compare 441Yj in Fremlin 03). Let P_I be the set of Radon probability measures on $\mathcal{P}([I]^{<\omega})$ which are invariant under the action of G_I. P_I is a closed convex subset of the set of all Radon probability measures on $\mathcal{P}([I]^{<\omega})$ in its narrow topology (Fremlin 03, §437).

For $R \subseteq [I]^{<\omega}$ and $J \subseteq I$ I write $R \restriction J$ for the trace $R \cap \mathcal{P}J \subseteq [J]^{<\omega}$ of R on J. If we take $\mathcal{V}$ to be the family of sets of the form $\{R : R \subseteq [I]^{<\omega},\ R \restriction J = S\}$ where $J \subseteq I$ is finite and $S \subseteq \mathcal{P}J$, then $\mathcal{V}$ is a base for the topology of $\mathcal{P}([I]^{<\omega})$ and every open-and-closed subset of $\mathcal{P}([I]^{<\omega})$ is expressible as a finite disjoint union of members of $\mathcal{V}$. Consequently two Baire measures on $\mathcal{P}([I]^{<\omega})$ which agree on $\mathcal{V}$ will be the same. Observe also that if we take $\mathcal{W}$ to be the family of sets of the form $\{R : R \subseteq [I]^{<\omega},\ R \restriction J \in \mathcal{S}\}$ where $J \subseteq I$ is countable and $\mathcal{S} \subseteq \mathcal{P}J$, then $\mathcal{W}$ is a σ-algebra of subsets of $\mathcal{P}([I]^{<\omega})$ including $\mathcal{V}$, so contains every Baire set.

If I, J are sets and $f : I \to J$ is a function, I define $\tilde{f} : \mathcal{P}([J]^{<\omega}) \to \mathcal{P}([I]^{<\omega})$ by setting $\tilde{f}(R) = \{a : a \in [I]^{<\omega},\ f[a] \in R\}$ for $R \subseteq [J]^{<\omega}$. Note that $\tilde{f}$ is continuous, since $\{R : a \in \tilde{f}(R)\} = \{R : f[a] \in R\}$ is a basic open-and-closed set in $\mathcal{P}([J]^{<\omega})$. If $I \subseteq J$ and f is the identity function, then $\tilde{f}(R) = R \restriction I$ for every $R \subseteq [J]^{<\omega}$.

If X is a set and T, T' are two σ-algebras of subsets of X, $\mathrm{T} \vee \mathrm{T}'$ will be the σ-algebra generated by $\mathrm{T} \cup \mathrm{T}'$. If $\langle \mathrm{T}_i \rangle_{\gamma \in \Gamma}$ is a family of σ-algebras of subsets of X, I will write $\bigvee_{\gamma \in \Gamma} \mathrm{T}_\gamma$ for the σ-algebra generated by $\bigcup_{\gamma \in \Gamma} \mathrm{T}_\gamma$.

1B Proposition Let I, J be sets and $f : I \to J$ an injective function. If $\mu \in P_J$ then the image measure $\mu \tilde{f}^{-1}$ belongs to P_I.

proof $\tilde{f}$ is continuous, so $\mu \tilde{f}^{-1}$ is a Radon probability measure on $\mathcal{P}([I]^{<\omega})$ (Fremlin 03, 418I). If $\phi \in G_I$, then there is a $\psi \in G_J$ such that $f\phi^{-1} = \psi^{-1} f$. (We can take $\psi(j) = j$ for $j \in J \setminus f[I]$.) Writing $\hat{\phi} : \mathcal{P}([I]^{<\omega}) \to \mathcal{P}([I]^{<\omega})$ for the map $R \mapsto \phi \bullet R$,

$$\begin{aligned}\hat{\phi}(\tilde{f}(R)) &= \hat{\phi}(\{a : f[a] \in R\}) = \{\phi[a] : f[a] \in R\} = \{b : f[\phi^{-1}[b]] \in R\} \\ &= \{b : \psi^{-1}[f[b]] \in R\} = \{b : f[b] \in \psi \bullet R = \hat{\psi}(R)\} = \tilde{f}(\hat{\psi}(R))\end{aligned}$$

for every $R \subseteq [I]^{<\omega}$, so $\hat{\phi}\tilde{f} = \tilde{f}\hat{\psi}$. Now

$$(\mu \tilde{f}^{-1})\hat{\phi}^{-1} = \mu(\hat{\phi}\tilde{f})^{-1} = \mu(\tilde{f}\hat{\psi})^{-1} = (\mu\hat{\psi}^{-1})\tilde{f}^{-1} = \mu\tilde{f}^{-1}.$$

As ϕ is arbitrary, $\mu \tilde{f}^{-1} \in P_I$.

1C Proposition For any set I, P_I is metrizable in its narrow topology.

proof If I is countable, then $[I]^{<\omega}$ is countable and $\mathcal{P}([I]^{<\omega})$ is metrizable, so the set of all Radon probability measures on $\mathcal{P}([I]^{<\omega})$ is metrizable in its narrow topology (Fremlin 03, 437Ra). If I is infinite, let $J \subseteq I$

be a countably infinite set. Applying 1B to the identity function f from J to I, we have a continuous map $R \mapsto \tilde{f}(R) = R\lceil J : \mathcal{P}([I]^{<\omega}) \to \mathcal{P}([J]^{<\omega})$ and a corresponding map $\mu \mapsto \mu\tilde{f}^{-1} : P_I \to P_J$. But observe now that this is injective. **P** Suppose that μ, $\mu' \in P_I$ and $\mu\tilde{f}^{-1} = \mu'\tilde{f}^{-1}$. Let $K \subseteq I$ be a finite set and $S \subseteq \mathcal{P}K$. Then there are a finite set $K' \subseteq J$ and an $S' \subseteq [K']^{<\omega}$ such that (K, S) is isomorphic to (K', S'). Now

$$\begin{aligned}\mu\{R : R \subseteq [I]^{<\omega},\, R\lceil K = S\} &= \mu\{R : R \subseteq [I]^{<\omega},\, R\lceil K' = S'\}\\ &= \mu\tilde{f}^{-1}[\{R : R \subseteq [I]^{<\omega},\, R\lceil K' = S'\}]\\ &= \mu'\tilde{f}^{-1}[\{R : R \subseteq [I]^{<\omega},\, R\lceil K' = S'\}]\\ &= \mu'\{R : R \subseteq [I]^{<\omega},\, R\lceil K = S\}.\end{aligned}$$

As K and S are arbitrary, μ and μ' agree on the Baire σ-algebra of $\mathcal{P}([I]^{<\omega})$ and must be identical. **Q**

So in fact we have a homeomorphism between P_J and P_I, and P_I is metrizable (indeed, homeomorphic to $P_{\mathbb{N}}$).

1D Lemma Suppose that I is infinite and that $\mu \in P_I$. Let J, $K \subseteq I$ be disjoint finite sets of the same size; suppose that $S \subseteq \mathcal{P}J$ and $S' \subseteq \mathcal{P}K$ are isomorphic. Then

$$\mu\{R : R\lceil J = S,\, R\lceil K = S'\} \ge (\mu\{R : R\lceil J = S\})^2.$$

proof Let $\langle J_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence of subsets of I all of the same size as J and K; for each $n \in \mathbb{N}$ let $S_n \subseteq [J_n]^{<\omega}$ be isomorphic to S, and set $E_n = \{R : R \subseteq [I]^{<\omega},\, R\lceil J_n = S_n\}$. Set $\alpha = \mu\{R : R\lceil J = S\}$ and $\beta = \mu\{R : R\lceil J = S,\, R\lceil K = S'\}$. Then $\mu(E_m) = \alpha$ and $\mu(E_m \cap E_n) = \beta$ for all m, $n \in \mathbb{N}$. But $\sup_{m\ne n} \mu(E_m \cap E_n) \ge \alpha^2$ (Fremlin 88, or 525U of Fremlin 08?). So $\beta \ge \alpha^2$, as claimed.

1E Definition For any set I, write EP_I for the set of those $\mu \in P_I$ such that whenever J, $K \subseteq I$ are disjoint finite sets, $S \subseteq \mathcal{P}J$ and $S' \subseteq \mathcal{P}K$, then

$$\mu\{R : R\lceil J = S,\, R\lceil K = S'\} = \mu\{R : R\lceil J = S\} \cdot \mu\{R : R\lceil K = S'\};$$

that is, the random variables $R\lceil J$, $R\lceil K$ are μ-independent whenever J, $K \subseteq I$ are disjoint. Observe that EP_I is closed for the narrow topology on P_I.

1F Lemma Let I be a set and $\mu \in EP_I$.

(a) If $J \subseteq I$ then the measure ν on $\mathcal{P}([J]^{<\omega})$ induced by μ belongs to EP_J.

(b) If $J_0, \dots, J_n$ are disjoint finite subsets of I then the random variables $R \mapsto R\lceil J_0, \dots, R \mapsto R\lceil J_n$ are independent.

proof (a) If K, $L \subseteq J$ are disjoint and finite, $S \subseteq \mathcal{P}K$ and $S' \subseteq \mathcal{P}L$ then

$$\begin{aligned}&\nu\{R : R \subseteq [J]^{<\omega},\, R\lceil K = S,\, R\lceil L = S'\}\\ &\quad= \mu\{R : R \subseteq [I]^{<\omega},\, (R\lceil J)\lceil K = S,\, (R\lceil J)\lceil L = S'\}\\ &\quad= \mu\{R : R \subseteq [I]^{<\omega},\, R\lceil K = S,\, R\lceil L = S'\}\\ &\quad= \mu\{R : R \subseteq [I]^{<\omega},\, R\lceil K = S\} \cdot \mu\{R : R \subseteq [I]^{<\omega},\, R\lceil L = S'\}\\ &\quad= \nu\{R : R \subseteq [J]^{<\omega},\, R\lceil K = S\} \cdot \nu\{R : R \subseteq [J]^{<\omega},\, R\lceil L = S'\}.\end{aligned}$$

(b) Induce on n. For the inductive step to n, suppose that $S_i \subseteq [J_i]^{<\omega}$ for $i \le n$. Set $J = J_0 \cup \ldots \cup J_{n-1}$, and let

$$\mathcal{T} = \{T : T \subseteq \mathcal{P}J,\, T\lceil J_i = S_i \text{ for every } i < n\}.$$

Then

$$\begin{aligned}\mu\{R : R\lceil J_i = S_i \text{ for every } i \le n\} &= \sum_{T\in\mathcal{T}} \mu\{R : R\lceil J = T,\, R\lceil J_n = S_n\}\\ &= \sum_{T\in\mathcal{T}} \mu\{R : R\lceil J = T\}\cdot\mu\{R : R\lceil J_n = S_n\}\\ &= \mu\{R : R\lceil J_i = S_i \text{ for } i < n\}\cdot\mu\{R : R\lceil J_n = S_n\}\\ &= \prod_{i\le n}\mu\{R : R\lceil J_i = S_i\}\end{aligned}$$

by the inductive hypothesis.

1G Theorem If I is infinite, then P_I is the closed convex hull of EP_I.

proof We may suppose that $\mathbb{N}$ is a subset of I.

(a) For each $S \subseteq [I]^{<\omega}$ and $n \ge 1$ define $\theta_{Sn} : n^I \to \mathcal{P}([I]^{<\omega})$ by saying that

$$\theta_{Sn}(x) = \{a : a \in [I]^{<\omega},\, x[a] \in S\}.$$

Let λ_n be the usual product measure on n^I where each copy of n is given the uniform measure, and let ν_{Sn} be the image measure $\lambda_n\theta_{Sn}^{-1}$ on $\mathcal{P}([I]^{<\omega})$. Because λ_n is a Radon measure (Fremlin 03, 416Ua) and θ_{Sn} is continuous, ν_{Sn} is a Radon measure (Fremlin 03, 418I). Because λ_n is invariant under permutations of I, so is θ_{Sn}. If J, K are disjoint (finite) subsets of I, then the maps $x \mapsto x\restriction J$ and $x \mapsto x\restriction K$ are independent random variables for λ_n, so the maps $R \mapsto R\lceil J$ and $R \mapsto R\lceil K$ are independent random variables for ν_{Sn}; thus $\nu_{Sn} \in EP_I$.

(b) If $E \subseteq \mathcal{P}([I]^{<\omega})$ is of the form $\{R : R\lceil J = T\}$ where $J \subseteq I$ is finite and $T \subseteq \mathcal{P}J$, then $\mu E = \lim_{n\to\infty}\int \nu_{Sn}E\,\mu(dS)$. **P** Take $\epsilon > 0$, and let n be so large that $\frac{n!}{(n-m)!n^m} \ge 1-\epsilon$, where $m = \#(J)$. For each $x \in n^I$ such that $x\restriction J$ is injective, we have

$$\{S : \theta_{Sn}(x) \in E\} = \{S : S\lceil x[J] = T_x\}$$

where $T_x \subseteq \mathcal{P}(x[J])$ is the image of T under x. Because $\mu \in P_I$ and (J,T), $(x[J],T_x)$ are isomorphic,

$$\mu\{S : \theta_{Sn}(x) \in E\} = \mu E.$$

If we set

$$F = \{(x,S) : x \in n^I,\, S \subseteq [I]^{<\omega},\, \theta_{Sn}(x) \in E\},$$

then F is a finite union of measurable cylinders in $n^I \times \mathcal{P}([I]^{<\omega})$, so is measured by the product measure $\lambda_n \times \mu$, and

$$(\lambda_n \times \mu)(F) = \int \mu\{S : (x,S) \in F\}\lambda_n(dx).$$

But if we set $W = \{x : x \in n^I,\, x\restriction J \text{ is injective}\}$, then $\lambda_n W = \frac{n!}{(n-m)!n^m} \ge 1-\epsilon$, and $\mu\{S : (x,S) \in F\} = \mu E$ for $x \in W$; so $|\mu E - (\lambda_n \times \mu)(F)| \le \epsilon$.

On the other hand,

$$(\lambda_n \times \mu)(F) = \int \lambda_n\{x : (x,S) \in F\}\mu(dS) = \int \lambda_n\theta_{Sn}^{-1}[E]\mu(dS) = \int \nu_{Sn}E\,\mu(dS).$$

Thus $|\mu E - \int \nu_{Sn}E\,\mu(dS)| \le \epsilon$. This is true for any n large enough, so we have the result. **Q**

(c) It follows that $\int f\,d\mu = \lim_{n\to\infty}\iint f\,d\nu_{Sn}\mu(dS)$ for every $f \in C(\mathcal{P}([I]^{<\omega}))$. **P** Consider first the case in which $f = \chi V$ for some open-and-closed set $V \subseteq \mathcal{P}([I]^{<\omega})$. Then there is a finite $J \subseteq I$ such that V is determined by coordinates in J in the sense that if $R \in V$, $R' \subseteq [I]^{<\omega}$ and $R\lceil J = R'\lceil J$ then $R' \in V$. So V is expressible in the form

$$\bigcup_{T\in\mathcal{T}}\{R : R\lceil J = T\}$$

for some $\mathcal{T} \subseteq \mathcal{P}J$, and

$$\begin{aligned}\mu V &= \sum_{T\in\mathcal{T}} \mu\{R : R\restriction J = T\}\\ &= \sum_{T\in\mathcal{T}} \lim_{n\to\infty} \nu_{Sn}\{R : R\restriction J = T\}\mu(dS) = \lim_{n\to\infty} \nu_{Sn}(V)\mu(dS).\end{aligned}$$

Since finite linear combinations of such characteristic functions are $\| \|_\infty$-dense in $C(\mathcal{P}([I]^{<\omega}))$, we have the result. **Q**

(d) Consequently μ belongs to the closed convex hull of $\{\nu_{Sn} : S \subseteq [I]^{<\omega},\ n \in \mathbb{N}\} \subseteq EP_I$.

1H Lemma Suppose that $k \geq 1$ is finite, and $\epsilon > 0$. Then there is an $n_0 \in \mathbb{N}$ such that whenever $n \geq n_0$ and X is a set of cardinal k^n, there is a family $\mathcal{Q}$ of partitions Q of X such that

$$\bigcup \mathcal{Q} \subseteq [X]^k, \quad \mathcal{Q} \text{ is disjoint}, \quad \#([X]^k \setminus \bigcup \mathcal{Q}) \leq \epsilon\#([X]^k).$$

proof Since $\lim_{n\to\infty} \frac{(k^n)!}{k^{nk}(k^n-k)!} = 1$, we can find an n_0 so large that

$$\big(1 - \tfrac{k!}{k^k}\big)^n \leq \tfrac{\epsilon(k^n)!}{k^{nk}(k^n-k)!k!}$$

for every $n \geq n_0$. Suppose that $n \geq n_0$, and consider $X = \mathbb{Z}_k^n$ with its usual group operation $+$. For $a \in [X]^k$, $i < n$ set $J_i(a) = \{x(i) : x \in a\} \subseteq \mathbb{Z}_k$; set

$$A_i = \{a : a \in [X]^k,\ \#(J_i(a)) = k\}, \quad B_i = A_i \setminus \bigcup_{j<i} A_j$$

for $i < n$. Note that, writing $a+y = \{x+y : x \in a\}$ and $A+x = \{a+x : x \in A\}$, $A_i+x = A_i$ and $B_i+x = B_i$ for every $i < n$ and $x \in X$. Set $H_i = \{x : x \in X,\ x(i) = 0\}$ for $i < n$, and let $\mathcal{Q}$ be the set of partitions of X of the form $\{a + x : x \in H_i\}$ where $i < n$ and $a \in B_i$. The point is that if $a \in B_i$ and x, $x' \in H_i$ and $y \in (a + x) \cap (a + x')$, then $y - x$ and $y - x'$ both belong to $a \in A_i$; but $(y - x)(i) = y(i) = (y - x')(i)$ so $y - x = y - x'$ and $x = x'$; secondly, if $a \in B_i$, $x \in H_i$, $a' \in B_j$, $x' \in H_j$ and $a + x = a' + x'$ then $a + x$ and $a' + x'$ both belong to $B_i \cap B_j$ and $i = j$, so that a, a' belong to the same member of $\mathcal{Q}$. So $\mathcal{Q}$ is a disjoint family of partitions of X into k-element sets. Finally, consider

$$\{w : w \in X^k,\ w[k] \notin \bigcup \mathcal{Q}\} = \{w : w \in (\mathbb{Z}_k^n)^k : \#(\{w(j)(i) : j < k\}) < k \text{ for every } i < n\}.$$

This has $(k^k - k!)^n$ members, so

$$\#([X]^k \setminus \bigcup \mathcal{Q}) \leq (k^k - k!)^n = k^{kn}\big(1 - \tfrac{k!}{k^k}\big)^n \leq \epsilon\tfrac{(k^n)!}{(k^n-k)!k!} = \epsilon\#([X]^k),$$

as required.

1I Lemma Let $k \geq 1$ be an integer, $S \subseteq \mathcal{P}k$ and $\epsilon > 0$. For a non-empty set X and $R \subseteq [X]^{<\omega}$, set $\gamma(R) = \frac{1}{\#(X)^k}\#(\{z : z \in X^k,\ \tilde{z}(R) = S\})$. Then there is an $n_0 \in \mathbb{N}$ such that whenever $n \geq n_0$, $\#(X) = k^n$, $k \subseteq X$, $\mu \in EP_X$ and $\mu\{R : R\restriction k = S\} = \alpha$, then

$$\mu\{R : \gamma(R) \geq \alpha - 2\epsilon\} \geq 1 - \epsilon.$$

proof (a) Take n_0 so large that whenever $n \geq n_0$ then

(α) whenever X is a set of cardinal k^n, there is a family $\mathcal{Q}$ of partitions Q of X such that $\bigcup \mathcal{Q} \subseteq [X]^k$, $\mathcal{Q}$ is disjoint and $\frac{\#(X)^k}{k!} - \#(\bigcup \mathcal{Q}) \leq \frac{\epsilon\#(X)^k}{k!}$

(this is possible by Lemma 1H, recalling that $\frac{\#(X)^k}{k!} \doteq \#([X]^k)$ whenever X is big enough),

(β) $k!k^{nk}\exp(-2k^{n-1}\epsilon^2) \leq \epsilon$

(this is possible because $\lim_{m\to\infty} m^k \exp(-2m\epsilon^2/k) = 0$).

(b) Suppose that $k \subseteq X$ and $\#(X) = k^n$ for some $n \geq n_0$, and that $\mu \in EP_X$; set $\alpha = \mu\{R : R\restriction k = S\}$. Take $\mathcal{Q}$ as in (a-α). Give X a total ordering and for each bijection $\pi : k \to k$ let Z_π be the set of those

$z \in X^k$ such that $z\pi : k \to X$ is strictly increasing. Observe that if $z \in Z_\pi$ then z is determined by $z[k] \in [X]^k$, while $\mu\{R : \tilde{z}(R) = S\} = \alpha$.

For $Q \in \mathcal{Q}$, $\pi \in G_k$ and $R \subseteq [X]^{<\omega}$ set

$$M_{Q\pi}(R) = \#(\{z : z \in Z_\pi,\, z[k] \in Q,\, \tilde{z}(R) = S\}).$$

Then $M_{Q\pi}$ is a sum of $\{0,1\}$-valued random variables all with expectation α, and by 1Fb these variables are independent. By Hoeffding's inequality (Hoeffding 63, or Fremlin 01, 272V[1],

$$\mu\{R : M_{Q\pi}(R) \le (\alpha - \epsilon)\#(Q)\} \le \exp(-2\epsilon^2\#(Q)).$$

So if we set

$$E = \{R : R \subseteq [X]^k,\, M_{Q\pi}(R) \ge (\alpha-\epsilon)\#(Q) \text{ for every } Q \in \mathcal{Q} \text{ and every } \pi \in G_k\},$$

we have

$$\mu E \ge 1 - \#(\mathcal{Q})\#(G_k)\exp(-2k^{n-1}\epsilon^2) \ge 1 - k!k^{nk}\exp(-2k^{n-1}\epsilon^2) \ge 1 - \epsilon$$

by (a-β); of course $\#(Q) = k^{n-1}$ for every $Q \in \mathcal{Q}$. But if $R \in E$ then

$$\begin{aligned}\#(\{z : z \in X^k,\, \tilde{z}(R) = S\}) &\ge \sum_{\pi \in G_k, Q \in \mathcal{Q}} M_{Q\pi}(R)\\ &\ge (\alpha - \epsilon)\sum_{\pi \in G_k, Q \in \mathcal{Q}} \#(Q) = (\alpha-\epsilon)k!\#(\bigcup \mathcal{Q})\end{aligned}$$

and

$$\gamma(R) \ge (\alpha - \epsilon)\frac{k!\#(\bigcup\mathcal{Q})}{\#(X)^k} \ge (\alpha-\epsilon)(1-\epsilon) \ge \alpha - 2\epsilon.$$

So n has the required property.

1J Theorem Let I be an infinite set, and $\mu \in P_I$. Then the following are equiveridical.

(i) μ is an extreme point of P_I.

(ii) $\mu \in EP_I$.

(iii) Whenever J, $K \subseteq I$ are disjoint finite sets of the same size and $S \subseteq \mathcal{P}J$ and $S' \subseteq \mathcal{P}K$ are isomorphic, then

$$\mu\{R : R\restriction J = S,\, R\restriction K = S'\} = (\mu\{R : R\restriction J = S\})^2.$$

(iv) There is a sequence $\langle(m_n, T_n)\rangle_{n\in\mathbb{N}}$ such that

(α) $\langle m_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{N}$ and $\lim_{n\to\infty} m_n = \infty$,

(β) $T_n \subseteq [m_n]^{<\omega}$ for each n,

(γ) whenever $J \subseteq I$ is finite and $S \subseteq \mathcal{P}J$, then $\mu\{R : R\restriction J = S\}$ is

$$\lim_{n\to\infty} \frac{1}{m_n^{\#(J)}}\#(\{z : z \in m_n^J,\, \tilde{z}(T_n) = S\}).$$

(v) There are a sequence $\langle(m_n, T_n)\rangle_{n\in\mathbb{N}}$ and a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ such that

(α) $\langle m_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{N}$ and $\lim_{n\to\mathcal{F}} m_n = \infty$,

(β) $T_n \subseteq [m_n]^{<\omega}$ for each n,

(γ) whenever $J \subseteq I$ is finite and $S \subseteq \mathcal{P}J$, then $\mu\{R : R\restriction J = S\}$ is

$$\lim_{n\to\mathcal{F}} \frac{1}{m_n^{\#(J)}}\#(\{z : z \in m_n^J,\, \tilde{z}(T_n) = S\}).$$

proof (a)(i)$\Rightarrow$(ii) Because EP_I is closed for the narrow topology and P_I is included in the closed convex hull of EP_I (1G), every extreme point of P_I must belong to EP_I (Fremlin 03, 461Xe).

(b)(ii)$\Rightarrow$(iii) is trivial (because we are supposing that $\mu \in P_I$, of course).

[1]Later editions only.

(c)(iii)⇒(i) Suppose that μ', $\mu'' \in P_I$ and that $\mu = \frac{1}{2}(\mu'+\mu'')$ satisfies the condition (iii). Take $J \in [I]^{<\omega}$ and $S \subseteq \mathcal{P}J$; because I is infinite, there is a $K \subseteq I \setminus J$ with $\#(K) = \#(J)$; take $S' \subseteq \mathcal{P}K$ such that (K, S') and (J, S) are isomorphic. Set

$$\alpha = \mu'\{R : R\lceil J = S\} = \mu'\{R : R\lceil K = S'\},$$

$$\beta = \mu'\{R : R\lceil J = S,\ R\lceil K = S'\} \geq \alpha^2$$

(Lemma 1D),

$$\gamma = \mu''\{R : R\lceil J = S\} = \mu''\{R : R\lceil K = S'\},$$

$$\delta = \mu''\{R : R\lceil J = S,\ R\lceil K = S'\} \geq \gamma^2.$$

Then

$$\tfrac{\alpha+\gamma}{2} = \mu\{R : R\lceil J = S\} = \mu\{R : R\lceil K = S'\},$$

$$\tfrac{\beta+\delta}{2} = \mu\{R : R\lceil J = S,\ R\lceil K = S'\} = \tfrac{(\alpha+\gamma)^2}{4}.$$

But this means that

$$\alpha^2 + \gamma^2 \leq \beta + \delta = \frac{1}{2}(\alpha+\gamma)^2$$

and $\alpha = \beta$. As J and S are arbitrary, $\mu' = \mu''$; as μ' and μ'' are arbitrary, μ is extreme. **Q**

(d)(ii)⇒(iv)($\boldsymbol{\alpha}$) We may suppose that $\mathbb{N} \subseteq I$. For $m \in \mathbb{N}$ let μ_m be the image of μ on $\mathcal{P}(\mathcal{P}m)$, as in 1B. Then $\mu_m \in EP_m$, by 1Fa. If $k \in \mathbb{N}$ and $\epsilon > 0$ then there are an $m \geq k$ and a $T \subseteq [m]^{<\omega}$ such that whenever $S \subseteq [k]^{<\omega}$ then

$$|\mu_m\{R : R \subseteq [m]^{<\omega},\ R\lceil k = S\} - \tfrac{1}{m^k}\#(\{z : z \in m^k,\ \tilde{z}(T) = S\})| \leq \epsilon.$$

P Let $\eta > 0$ be such that $2\eta\#(\mathcal{P}(\mathcal{P}k)) \leq \min(\epsilon, 1)$. By Lemma 1I, there is an $m \geq k$ such that whenever $\nu \in EP_m$, $S \subseteq \mathcal{P}k$ and $\nu\{R : R\lceil k = S\}$ then $\nu\{R : \gamma_S(R) \geq \alpha - 2\eta\} \geq 1 - \eta$, where $\gamma_S(R) = \frac{1}{m^k}\#(\{z : z \in m^k,\ \tilde{z}(R) = S\})$. In particular, this is true for $\nu = \mu_m$. Because $\eta\#(\mathcal{P}(\mathcal{P}k)) < 1$ there is a $T \subseteq \mathcal{P}m$ such that $\gamma_S(T) \geq \mu_m\{R : R\lceil k = S\} - 2\eta$ for every $S \subseteq \mathcal{P}k$. Because $2\eta\#(\mathcal{P}(\mathcal{P}k)) \leq \epsilon$, and

$$\textstyle\sum_{S\subseteq\mathcal{P}k} \gamma_S(T) \leq 1 = \sum_{S\subseteq\mathcal{P}k} \mu_m\{R : R\lceil k = S\},$$

T must have the required property. **Q**

It follows now that if $j \leq k$ and $S \subseteq \mathcal{P}j$ then

$$|\mu_m\{R : R \subseteq \mathcal{P}m,\ R\lceil j = S\} - \tfrac{1}{m^j}\#(\{z : z \in m^j,\ \tilde{z}(T) = S\})| \leq \epsilon\#(\mathcal{P}(\mathcal{P}k)).$$

P

$$\begin{aligned}
\frac{1}{m^j}\#(\{z : z \in m^j,\ \tilde{z}(T) = S\}) &= \frac{1}{m^k}\#(\{z : z \in m^k,\ (z\restriction j)^{\sim}(T) = S\}) \\
&= \frac{1}{m^k}\#(\{z : z \in m^k,\ \tilde{z}(T)\lceil j = S\}) \\
&= \sum_{S'\subseteq\mathcal{P}k, S'\lceil j = S} \frac{1}{m^k}\#(\{z : z \in m^k,\ \tilde{z}(T) = S'\}) \\
&= \sum_{S'\subseteq\mathcal{P}k, S'\lceil j = S} \frac{1}{m^k}\#(\{z : z \in m^k,\ \tilde{z}(T) = S'\}) \\
&\simeq \sum_{S'\subseteq\mathcal{P}k, S'\lceil j = S} \mu_m\{R : R\lceil k = S'\} = \mu_m\{R : R\lceil j = S\}
\end{aligned}$$

with an error of at most $\epsilon\#(\mathcal{P}(\mathcal{P}k))$ at '$\simeq$'. **Q**

(β) It follows that for every $n \in \mathbb{N}$ we can find an $m_n \ge n$ and a $T_n \subseteq [m_n]^{<\omega}$ such that whenever $k \le n$ and $S \subseteq \mathcal{P}k$ then

$$|\mu_{m_n}\{R : R \subseteq \mathcal{P}m_n,\, R\lceil k = S\} - \frac{1}{m_n^k}\#(\{z : z \in m_n^k,\, \tilde{z}(T_n) = S\})| \le 2^{-n}.$$

Since

$$\mu_{m_n}\{R : R \subseteq \mathcal{P}m_n,\, R\lceil k = S\} = \mu\{R : R \subseteq [I]^{<\omega},\, R\lceil k = S\}$$

whenever $k \le m_n$, we have

$$|\mu\{R : R \subseteq [I]^{<\omega},\, R\lceil k = S\} - \frac{1}{m_n^k}\#(\{z : z \in m_n^k,\, \tilde{z}(T_n) = S\})| \le 2^{-n}$$

whenever $k \le n$ and $S \subseteq \mathcal{P}k$; consequently

$$|\mu\{R : R \subseteq [I]^{<\omega},\, R\lceil J = S\} - \frac{1}{m_n^{\#(J)}}\#(\{z : z \in m_n^J,\, \tilde{z}(T_n) = S\})| \le 2^{-n}$$

whenever $J \subseteq I$, $\#(J) \le n$ and $S \subseteq \mathcal{P}J$. So $\langle (m_n, T_n)\rangle_{n\in\mathbb{N}}$ witnesses (iv).

(e)(iv)⇒(v) is trivial, as we can take any non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$.

(f)(v)⇒(ii) Suppose that (v) is true. Take disjoint finite J, $K \subseteq I$ and $S \subseteq \mathcal{P}J$, $S' \subseteq \mathcal{P}K$. Let $\mathcal{T}$ be the set of those $T \subseteq \mathcal{P}(J \cup K)$ such that $T\lceil J = S$ and $T\lceil K = S'$. Then

$$\begin{aligned}
&\mu\{R : R\lceil J = S,\, R\lceil K = S'\}\\
&\quad= \sum_{T\in\mathcal{T}} \mu\{R : R\lceil J\cup K = T\}\\
&\quad= \sum_{T\in\mathcal{T}} \lim_{n\to\mathcal{F}} \frac{1}{m_n^{\#(J\cup K)}}\#(\{z : z \in m_n^{J\cup K},\, \{a : a \subseteq J\cup K,\, z[a] \in T_n\} = T\})\\
&\quad= \sum_{T\in\mathcal{T}} \lim_{n\to\mathcal{F}} \frac{1}{m_n^{\#(J\cup K)}}\#(\{z : z \in m_n^{J\cup K} \text{ is injective},\\
&\qquad\qquad\qquad\qquad \{a : a \subseteq J\cup K,\, z[a] \in T_n\} = T\})
\end{aligned}$$

(because $\lim_{n\to\mathcal{F}} m_n = \infty$)

$$\begin{aligned}
&\quad= \lim_{n\to\mathcal{F}} \frac{1}{m_n^{\#(J\cup K)}} \sum_{T\in\mathcal{T}} \#(\{z : z \in m_n^{J\cup K} \text{ is injective},\\
&\qquad\qquad\qquad\qquad \{a : a \subseteq J\cup K,\, z[a] \in T_n\} = T\})\\
&\quad= \lim_{n\to\mathcal{F}} \frac{1}{m_n^{\#(J\cup K)}}\#(\{z : z \in m_n^{J\cup K} \text{ is injective},\\
&\qquad\qquad \{a : a \subseteq J,\, z[a] \in T_n\} = S,\, \{a : a \subseteq K,\, z[a] \in T_n\} = S'\})\\
&\quad= \lim_{n\to\mathcal{F}} \frac{1}{m_n^{\#(J\cup K)}}\#(\{z : z \in m_n^{J\cup K},\\
&\qquad\qquad \{a : a \subseteq J,\, z[a] \in T_n\} = S,\, \{a : a \subseteq K,\, z[a] \in T_n\} = S'\})\\
&\quad= \lim_{n\to\mathcal{F}} \frac{1}{m_n^{\#(J)}}\#(\{z : z \in m_n^{J},\, \{a : a \subseteq J,\, z[a] \in T_n\} = S\}\\
&\qquad\qquad \cdot \frac{1}{m_n^{\#(K)}}\#(\{z : z \in m_n^{K},\, \{a : a \subseteq K,\, z[a] \in T_n\} = S'\})\\
&\quad= \lim_{n\to\mathcal{F}} \frac{1}{m_n^{\#(J)}}\#(\{z : z \in m_n^{J},\, \{a : a \subseteq J,\, z[a] \in T_n\} = S\})\\
&\qquad\qquad \cdot \lim_{n\to\mathcal{F}} \frac{1}{m_n^{\#(K)}}\#(\{z : z \in m_n^{K},\, \{a : a \subseteq K,\, z[a] \in T_n\} = S'\})\\
&\quad= \mu\{R : R\lceil J = S\} \cdot \mu\{R : R\lceil K = S'\}.
\end{aligned}$$

As J, K, S and S' are arbitrary, $\mu \in EP_I$.

1K Theorem Let I be an infinite set, and $\mu \in P_I$. Then there is exactly one Radon probability measure λ on EP_I such that μ is the barycenter of λ.

proof Let V be the linear subspace of $C(\mathcal{P}([I]^{<\omega}))^*$ consisting of G_I-invariant linear functionals. Then V is a Riesz subspace of $C(\mathcal{P}([I]^{<\omega}))^*$. **P** Each $\phi \in G_I$ acts on $\mathcal{P}([I]^{<\omega})$ as a homeomorphism and on $C(\mathcal{P}([I]^{<\omega}))$ as a Riesz space isomorphism T_ϕ say. If $h \in V$, then h^+, taken in $C(\mathcal{P}([I]^{<\omega}))^*$, is given by

$$h^+(v) = \sup\{h(u) : u \in C(\mathcal{P}([I]^{<\omega})),\ 0 \le u \le v\}$$

for every $v \in C(\mathcal{P}([I]^{<\omega}))^+$. So

$$\begin{aligned} h^+(T_\phi v) &= \sup\{h(u) : 0 \le u \le T_\phi v\} = \sup\{h(T_\phi u) : 0 \le u \le v\} \\ &= \sup\{h(u) : 0 \le u \le v\} = h^+(v) \end{aligned}$$

for every $v \ge 0$, and $T'_\phi h^+ = h^+$; as ϕ is arbitrary, $h^+ \in V$; as h is arbitrary, V is a Riesz subspace. **Q**

Next, we can identify P_I with $\{h : h \in V^+,\ \|h\| = 1\}$, so that every member of V^+ is uniquely expressible as a scalar multiple of a member of P_I. Since P_I is metrizable (1C), Choquet's theorem (Choquet 56, or Phelps 66, §9, or Fremlin 03, 461P[2]) tells us that every member of P_I is uniquely expressible as the barycenter of a Radon probability measure on the set of extreme points of P_I, which by 1J is EP_I.

1L Lemma Let I be a set, and $\mu \in P_I$. For $J \subseteq I$ write Σ_J for the σ-algebra of subsets of $\mathcal{P}([I]^{<\omega})$ generated by sets of the form $E_a = \{R : a \in R \subseteq [I]^{<\omega}\}$ where $a \in [J]^{<\omega}$. Write $\mathbb{T}$ for the set of σ-subalgebras of $\operatorname{dom}\mu$. For $\mathrm{T} \in \mathbb{T}$, write V_{T} for the norm-closed Riesz subspace of $L^1(\mu)$ consisting of equivalence classes of T-measurable functions, and $A_{\mathrm{T}} : L^1(\mu) \to V_{\mathrm{T}}$ for the conditional expectation operator (Fremlin 01, 242J).

(a) We have an action of G_I on $\mathcal{P}(\mathcal{P}([I]^{<\omega}))$ defined by saying that

$$\phi{\bullet}E = \{\phi{\bullet}R : R \in E\} = \{R : R \subseteq [I]^{<\omega},\ \phi^{-1}{\bullet}R \in E\}$$

for $\phi \in G_I$ and $E \subseteq \mathcal{P}([I]^{<\omega})$. If $a \in [I]^{<\omega}$ and $\phi \in G_I$, $\phi{\bullet}E_a = E_{\phi[a]}$.

(b) We have an action of G_I on $\mathbb{R}^{\mathcal{P}([I]^{<\omega})}$ defined by saying that

$$(\phi{\bullet}f)(R) = f(\phi^{-1}{\bullet}R)$$

for $\phi \in G_I$, $f : \mathcal{P}([I]^{<\omega}) \to \mathbb{R}$ and $R \subseteq [I]^{<\omega}$.

(c)(i) We have an action of G_I on $L^0(\mu)$ defined by saying that

$$\phi{\bullet}f^\bullet = (\phi{\bullet}f)^\bullet$$

whenever $\phi \in G_I$ and $f : \mathcal{P}([I]^{<\omega}) \to \mathbb{R}$ is $(\operatorname{dom}\mu)$-measurable. For any $p \in [1,\infty]$, $u \in L^p(\mu)$ and $\phi \in G_I$, $\phi{\bullet}u \in L^p(\mu)$ and $\|\phi{\bullet}u\|_p = \|u\|_p$.

(ii) If $\phi \in G_I$, $J \subseteq I$ and $\phi(i) = i$ for $i \in J$, then $\phi{\bullet}u = u$ for every $u \in V_{\Sigma_J}$.

(d)(i) We have an action of G_I on $\mathbb{T}$ defined by saying that

$$\phi{\bullet}\mathrm{T} = \{\phi{\bullet}E : E \in \mathrm{T}\}$$

for $\phi \in G_I$ and $\mathrm{T} \in \mathbb{T}$.

(ii) If $\langle \mathrm{T}_\gamma \rangle_{\gamma\in\Gamma}$ is a family in $\mathbb{T}$, then $\phi{\bullet}(\bigvee_{\gamma\in\Gamma} \mathrm{T}_\gamma) = \bigvee_{\gamma\in\Gamma} \phi{\bullet}\mathrm{T}_\gamma$ for every $\phi \in G_I$.

(iii) If $\phi \in G_I$ and $J \subseteq I$ then $\phi{\bullet}\Sigma_J = \Sigma_{\phi{\bullet}J}$.

(iv) If $\mathrm{T} \in \mathbb{T}$ and $f : \mathcal{P}([I]^{<\omega}) \to \mathbb{R}$ is T-measurable, then $\phi{\bullet}f$ is $\phi{\bullet}\mathrm{T}$-measurable for every $\phi \in G_I$. So $\phi{\bullet}u \in V_{\phi{\bullet}\mathrm{T}}$ whenever $\mathrm{T} \in \mathbb{T}$, $u \in V_{\mathrm{T}}$ and $\phi \in G_I$.

(v) If $\phi \in G_I$, $\mathrm{T} \in \mathbb{T}$ and $u \in L^1(\mu)$, then $\phi{\bullet}A_{\mathrm{T}}u = A_{\phi{\bullet}\mathrm{T}}(\phi{\bullet}u)$.

proof (a) It is elementary to check that the formulae for $\phi{\bullet}E$ agree and define an action of G_I on $\mathcal{P}(\mathcal{P}([I]^{<\omega}))$, just because we are starting from an action of G_I on $\mathcal{P}([I]^{<\omega})$.

If $a \in [I]^{<\omega}$ then

$$\phi{\bullet}E_a = \{\phi{\bullet}R : a \in R\} = \{\phi{\bullet}R : \phi[a] \in \phi{\bullet}R\} = E_{\phi[a]}.$$

[2]Later editions only.

(b) Again because we are starting from an action of G_I on $\mathcal{P}([I]^{<\omega})$, the formula defines an action of G_I on $\mathbb{R}^{\mathcal{P}([I]^{<\omega})}$.

(c)(i) Again, this is just because we are starting from an action of G_I on $\mathcal{P}([I]^{<\omega})$ and μ is G_I-invariant (Fremlin 03, 441J-441K).

(ii) If $a \in [J]^{<\omega}$ then $\phi\bullet E_a = E_{\phi[a]} = E_a$, as noted in (a). So $\phi\bullet E = E$ for every $E \in \Sigma_J$, $\phi\bullet f = f$ for every Σ_J-measurable f and $\phi\bullet u = u$ for every $u \in V_{\Sigma_J}$.

(d)(i)-(ii) Actually, of course, the formula

$$\phi\bullet\mathrm{T} = \{\phi\bullet E : E \in \mathrm{T}\}$$

defines an action of ϕ on the set of all σ-algebras on $\mathcal{P}([I]^{<\omega})$ which preserves the operation $\bigvee$; and because μ is G_I-invariant, $\phi\bullet \operatorname{dom}\mu = \operatorname{dom}\mu$, so $\phi\bullet\mathrm{T} \in \mathbb{T}$ for every $\mathrm{T} \in \mathbb{T}$.

(iii) If $\phi \in G_I$ and $a \in [J]^{<\omega}$ then

$$\phi\bullet E_a = \{\phi\bullet R : a \in R\} = \{\phi\bullet R : \phi[a] \in \phi\bullet R\} = E_{\phi[a]} \in \Sigma_{\phi[J]}.$$

It follows easily that $\phi\bullet E \in \Sigma_{\phi[J]}$ for every $E \in \Sigma_J$. Similarly, $\phi^{-1}\bullet F \in \Sigma_J$ for every $F \in \Sigma_{\phi[J]}$, so $\phi\bullet\Sigma_J = \Sigma_{\phi[J]}$.

(iv) If $\alpha \in \mathbb{R}$, then, setting $E = \{R : f(R) \ge \alpha\} \in \mathrm{T}$,

$$\{R : (\phi\bullet f)(R) \ge \alpha\} = \{R : f(\phi^{-1}\bullet R) \ge \alpha\} = \{R : \phi^{-1}\bullet R \in E\} = \phi\bullet E \in \Sigma_{\phi\bullet\mathrm{T}};$$

as α is arbitrary, $\phi\bullet f$ is $(\phi\bullet\mathrm{T})$-measurable. If $u \in V_\mathrm{T}$ then there is some T-measurable μ-integrable $f : \mathcal{P}([I]^{<\omega}) \to \mathbb{R}$ such that $u = f^\bullet$, and now $\phi\bullet u = (\phi\bullet f)^\bullet$ where $\phi\bullet f$ is $(\phi\bullet\mathrm{T})$-measurable and μ-integrable, so that $\phi\bullet u \in V_{\phi\bullet\mathrm{T}}$.

(v) Express u as $f^\bullet$ where $f : \mathcal{P}([I]^{<\omega}) \to \mathbb{R}$ is T-measurable and μ-integrable. Then $\phi\bullet f$ is $(\phi\bullet\mathrm{T})$-measurable, by (iv), and μ-integrable, by (c), so $\phi\bullet u = (\phi\bullet f)^\bullet$ belongs to $V_{\phi\bullet\mathrm{T}}$.

Take any $E \in \phi\bullet\mathrm{T}$. Then

$$\begin{aligned}\int_E \phi\bullet u &= \int_E \phi\bullet f = \int (\phi\bullet f) \times \chi E = \int (\phi\bullet f) \times (\phi\bullet\phi^{-1}\bullet\chi E) \\ &= \int \phi\bullet(f \times (\phi^{-1}\bullet\chi E)) = \int f \times (\phi^{-1}\bullet\chi E) = \int_{\phi^{-1}\bullet E} u = \int_{\phi^{-1}\bullet E} A_\mathrm{T} u\end{aligned}$$

because $\phi^{-1}\bullet E \in \mathrm{T}$. Applying this to $A_\mathrm{T} u$ in place of u, we see that

$$\int_E \phi\bullet(A_\mathrm{T} u) = \int_{\phi^{-1}\bullet E} A_\mathrm{T}(A_\mathrm{T} u) = \int_{\phi^{-1}\bullet E} A_\mathrm{T} u = \int_E \phi\bullet u,$$

and this is true for every $E \in \phi\bullet\mathrm{T}$. Since (iv) tells us that $\phi\bullet(A_\mathrm{T} u) \in V_{\phi[J]}$, $\phi\bullet(A_\mathrm{T} u)$ satisfies the defining characterization of $A_{\phi\bullet\mathrm{T}}(\phi\bullet u)$, and they are equal.

1M I continue with the notation from 1L.

Lemma Let I be a set and $\mu \in P_I$. Let $\langle J_\gamma\rangle_{\gamma\in\Gamma}$ be a non-empty finite family of subsets of I with $\bigcap_{\gamma\in\Gamma} J_\gamma$ infinite, and set $\mathrm{T} = \bigvee_{\gamma\in\Gamma} \Sigma_{J_\gamma}$. Let $K \subseteq I$ be finite, and $S \subseteq \mathcal{P}K$; for each $\gamma \in \Gamma$ let $K_\gamma \subseteq I$ be finite and $S_\gamma \subseteq \mathcal{P}K_\gamma$; suppose that $K \cap K_\gamma \subseteq J_\gamma$ for each $\gamma \in \Gamma$. Set $E = \{R : R \subseteq [I]^{<\omega},\ R\restriction K = S\}$ and $F = \bigcap_{\gamma\in\Gamma}\{R : R \subseteq [I]^{<\omega},\ R\restriction K_\gamma = S_\gamma\}$. Then E and F are relatively independent over T.

Remark For the basic theory of relative independence, see Fremlin 03, §458.

proof (a) For $L \subseteq I$ set $\mathrm{T}_L = \bigvee_{\gamma\in\Gamma} \Sigma_{J_\gamma\cap L} \subseteq \mathrm{T}$. For any $u \in L^2(\mu)$ and $\epsilon > 0$ there is a finite $L \subseteq I$ such that $\|A_{\mathrm{T}'}u - A_\mathrm{T} u\|_2 \le \epsilon$ whenever T' is a σ-subalgebra of T including T_L. **P** Because T is the σ-algebra generated by

$$\bigcup_{L\subseteq I \text{ is finite}} \bigcup_{\gamma\in\Gamma} \Sigma_{J_\gamma\cap L},$$

$\bigcup_{L\subseteq I \text{ is finite}} L^2(\mu) \cap V_{\mathrm{T}_L}$ is norm-dense in $L^2(\mu) \cap V_\mathrm{T}$, and there are a finite $L \subseteq I$ and a $v \in L^2(\mu) \cap V_{\mathrm{T}_L}$ such that $\|v - A_\mathrm{T} u\|_2 \le \epsilon$. If now $\mathrm{T}_L \subseteq \mathrm{T}' \subseteq \mathrm{T}$, $v \in L^2(\mu) \cap V_{\mathrm{T}'}$, while $A_{\mathrm{T}'}\restriction L^2(\mu)$ is the orthogonal projection onto $L^2(\mu) \cap V_{\mathrm{T}'}$ (Fremlin 01, 244Nb), so

$$\|A_{\mathrm{T}'}u - A_{\mathrm{T}}u\|_2 = \|A_{\mathrm{T}'}A_{\mathrm{T}}u - A_{\mathrm{T}}u\|_2 \le \|v - A_{\mathrm{T}}u\|_2 \le \epsilon. \text{ } \mathbf{Q}$$

(b) Let g, $h : \mathcal{P}([I]^{<\omega}) \to [0,1]$ be T-measurable functions which are conditional expectations of χE, χF respectively on T. Set $u = \chi E^\bullet$ and $v = \chi F^\bullet$, so that $g^\bullet = A_{\mathrm{T}}u$ and $h^\bullet = A_{\mathrm{T}}v$. Let $\epsilon > 0$; by (a), there is an $L \in [I]^{<\omega}$ such that

$$\|A_{\mathrm{T}'}u - A_{\mathrm{T}}u\|_2 \le \epsilon, \quad \|A_{\mathrm{T}'}v - A_{\mathrm{T}}v\|_2 \le \epsilon, \quad \|A_{\mathrm{T}'}(u \times v) - A_{\mathrm{T}}(u \times v)\|_2 \le \epsilon$$

whenever T' is a σ-subalgebra of T including T_L. Set $M = \bigcup_{\gamma\in\Gamma} K_\gamma \setminus K$. Because $J = I \cap \bigcap_{\gamma\in\Gamma} J_\gamma$ is infinite, there is a set $M' \subseteq J \setminus (L \cup K \cup \bigcup_{\gamma\in\Gamma} K_\gamma)$ such that $\#(M') = \#(M)$; let $\phi \in G_I$ be such that $\phi[M] = M'$, ϕ^2 is the identity and $\phi(i) = i$ for $i \in I \setminus (M \cup M')$. Then

$$\phi(i) = i \text{ for } i \in K,$$

$$\phi[J_\gamma \cap L] \subseteq J_\gamma \text{ for } \gamma \in \Gamma$$

because if $i \in J_\gamma \cap L$, either $\phi(i) = i \in J_\gamma$ or $i \in M$ and $\phi(i) \in M' \subseteq J_\gamma$,

$$\phi[K_\gamma] \subseteq J_\gamma \text{ for } \gamma \in \Gamma$$

because if $i \in K_\gamma$ then either $i \in K_\gamma \cap K \subseteq J_\gamma$ and $\phi(i) = i \in J_\gamma$ or $i \in M$ and $\phi(i) \in J_\gamma$.

Now

$$\phi\bullet u = u$$

by 1L(c-ii); also

$$\|\phi\bullet(A_{\mathrm{T}_L}u) - A_{\mathrm{T}}u\|_2 \le \epsilon.$$

$\mathbf{P}$ Set

$$\mathrm{T}' = \bigvee_{\gamma\in\Gamma} \Sigma_{J_\gamma\cap L} \vee \Sigma_{\phi[J_\gamma\cap L]}.$$

Then $\phi\bullet\mathrm{T}' = \mathrm{T}'$, by 1L(d-ii) and 1L(d-iii), so $\phi\bullet(A_{\mathrm{T}'}u) = A_{\mathrm{T}'}(\phi\bullet u) = A_{\mathrm{T}'}u$, by 1L(d-v). Because $\phi[J_\gamma \cap L] \subseteq J_\gamma$ for every γ, $\mathrm{T}' \subseteq \mathrm{T}$, and of course $\mathrm{T}_L \subseteq \mathrm{T}'$. But now

$$\begin{aligned}\|\phi\bullet(A_{\mathrm{T}_L}u) - A_{\mathrm{T}}u\|_2 &\le \|\phi\bullet(A_{\mathrm{T}_L}u) - \phi\bullet(A_{\mathrm{T}'}u)\|_2 + \|A_{\mathrm{T}'}u - A_{\mathrm{T}}u\|_2 \\ &= \|A_{\mathrm{T}'}u - A_{\mathrm{T}_L}u\|_2 + \|A_{\mathrm{T}'}u - A_{\mathrm{T}}u\|_2 \\ &\le 2\|A_{\mathrm{T}'}u - A_{\mathrm{T}}u\|_2 + \|A_{\mathrm{T}}u - A_{\mathrm{T}_L}u\|_2 \le 3\epsilon. \text{ } \mathbf{Q}\end{aligned}$$

Set

$$\mathrm{T}^* = \bigvee_{\gamma\in\Gamma} \Sigma_{K_\gamma\cup(J_\gamma\cap L)\cup\phi[J_\gamma\cap L]}.$$

Then T^* and

$$\phi\bullet\mathrm{T}^* = \bigvee_{\gamma\in\Gamma} \Sigma_{\phi[K_\gamma]\cup\phi[J_\gamma\cap L]\cup(J_\gamma\cap L)}$$

include T_L, while $\phi\bullet\mathrm{T}^* \subseteq \mathrm{T}$. Also $F \in \mathrm{T}^*$, because $\{R : R \subseteq [I]^{<\omega}, R\lceil K_\gamma = S_\gamma\}$ belongs to $\Sigma_{K_\gamma} \subseteq \mathrm{T}^*$ for every γ. Now

$$\begin{aligned}&\|A_{\mathrm{T}}u \times A_{\mathrm{T}}v - A_{\mathrm{T}_L}u \times A_{\mathrm{T}_L}v\|_2 \\ &\qquad \le \|A_{\mathrm{T}}u \times (A_{\mathrm{T}}v - A_{\mathrm{T}_L}v)\|_2 + \|(A_{\mathrm{T}}u - A_{\mathrm{T}_L}v) \times A_{\mathrm{T}_L}v\|_2 \\ &\qquad \le \|A_{\mathrm{T}}v - A_{\mathrm{T}_L}v\|_2 + \|A_{\mathrm{T}}u - A_{\mathrm{T}_L}v\|_2\end{aligned}$$

(because $\|A_{\mathrm{T}}u\|_\infty$ and $\|A_{\mathrm{T}_L}v\|_\infty$ are both at most 1)

$$\le 2\epsilon,$$

and

$$\begin{aligned}\|A_{\mathrm{T}_L}(u \times v) - A_{\mathrm{T}_L}u \times A_{\mathrm{T}_L}v\|_2 &= \|A_{\mathrm{T}_L}A_{\mathrm{T}^*}(u \times v) - A_{\mathrm{T}_L}u \times A_{\mathrm{T}_L}v\|_2 \\ &= \|A_{\mathrm{T}_L}(v \times A_{\mathrm{T}^*}u) - A_{\mathrm{T}_L}(v \times A_{\mathrm{T}_L}u)\|_2\end{aligned}$$

(because $v \in V_{\mathrm{T}^*}$, see Fremlin 01, 242L)

$$\le \|v \times A_{\mathrm{T}^*}u - v \times A_{\mathrm{T}_L}u\|_2$$
$$\le \|A_{\mathrm{T}^*}u - A_{\mathrm{T}_L}u\|_2$$

(because $\|v\|_\infty \le 1$)

$$= \|\phi^\bullet(A_{\mathrm{T}^*}u) - \phi^\bullet(A_{\mathrm{T}_L}u)\|_2$$
$$= \|A_{\phi^\bullet\mathrm{T}^*}u - \phi^\bullet(A_{\mathrm{T}_L}u)\|_2$$
$$\le \|A_{\phi^\bullet\mathrm{T}^*}u - A_{\mathrm{T}}u\|_2 + \|A_{\mathrm{T}}u - \phi^\bullet(A_{\mathrm{T}_L}u)\|_2 \le 4\epsilon.$$

Putting these together,

$$\begin{aligned}\|A_{\mathrm{T}}(u \times v) - A_{\mathrm{T}}u \times A_{\mathrm{T}}v\|_2 &\le \|A_{\mathrm{T}}(u \times v) - A_{\mathrm{T}_L}(u \times v)\|_2 \\ &\qquad + \|A_{\mathrm{T}_L}(u \times v) - A_{\mathrm{T}_L}u \times A_{\mathrm{T}_L}v\|_2 \\ &\qquad + \|A_{\mathrm{T}}u \times A_{\mathrm{T}}v - A_{\mathrm{T}_L}u \times A_{\mathrm{T}_L}v\|_2 \\ &\le 7\epsilon.\end{aligned}$$

As ϵ is arbitrary, $A_{\mathrm{T}}(u \times v) = A_{\mathrm{T}}u \times A_{\mathrm{T}}v$, that is, $g \times h$ is a conditional expectation of $\chi(E \cap F)$ on T, and E and F are relatively independent over T.

1N Still in the same notation, we come to a theorem of T.Tao.

Theorem (Tao p06) Let I be a set and $\mu \in P_I$. Let $\mathcal{J}$ be a filter on I not containing any finite set. Then whenever $J_0 \in \mathcal{J}$ and $\mathcal{K}, \mathcal{K}' \subseteq \mathcal{J}$ are such that $J \cap J_0 \in \mathcal{K}'$ for every $J \in \mathcal{K}$, Σ_{J_0} and $\bigvee_{J\in\mathcal{K}} \Sigma_J$ are relatively independent over $\bigvee_{J\in\mathcal{K}'} \Sigma_J$.

proof Write T for $\bigvee_{J\in\mathcal{K}'} \Sigma_J$.

(a) Suppose that $\mathcal{K}_0 \subseteq \mathcal{K}$ is finite and not empty, $K \in [J_0]^{<\omega}$, $K_J \in [J]^{<\omega}$ for every $J \in \mathcal{K}_0$, $S \subseteq \mathcal{P}K$, $E = \{R : R \subseteq \mathcal{P}([I]^{<\omega}), R\lceil K = S\}$, $S_J \subseteq \mathcal{P}K_J$ for $J \in \mathcal{K}_0$, and $F = \{R : R \subseteq [I]^{<\omega}, R\lceil K_J = S_J$ for every $J \in \mathcal{K}_0\}$. Then E and F are relatively independent over T. **P** Set $u = \chi E^\bullet$ and $v = \chi F^\bullet$ in $L^2(\mu)$, and take $\epsilon > 0$. Then there is a finite set $\mathcal{K}_1 \subseteq \mathcal{K}'$, including $\{J \cap J_0 : J \in \mathcal{K}_0\}$, such that, setting $\mathrm{T}' = \bigvee_{J\in\mathcal{K}_1} \Sigma_J$,

$$\|A_{\mathrm{T}'}u - A_{\mathrm{T}}u\| \le \epsilon, \quad \|A_{\mathrm{T}'}v - A_{\mathrm{T}}v\| \le \epsilon, \quad \|A_{\mathrm{T}'}(u \times v) - A_{\mathrm{T}}(u \times v)\| \le \epsilon.$$

Set $\Gamma = (\mathcal{K}_0 \times \{0\}) \cup (\mathcal{K}_1 \times \{1\})$, and fix $J_1 \in \mathcal{K}_0$. Define $\langle J'_\gamma\rangle_{\gamma\in\Gamma}$, $\langle K'_\gamma\rangle_{\gamma\in\Gamma}$ and $\langle S'_\gamma\rangle_{\gamma\in\Gamma}$ by saying

—— if $\gamma = (J, 0)$ where $J \in \mathcal{K}_0$, then $J'_\gamma = J \cap J_0$, $K'_\gamma = K_J$, $S'_\gamma = S_J$,

—— if $\gamma = (J, 1)$ where $J \in \mathcal{K}_1$ then $J'_\gamma = J$, $K'_\gamma = \emptyset$ and $S'_\gamma = S_{J_1} \cap \{\emptyset\}$.

Then $J'_\gamma \in \mathcal{J}$ for every $\gamma \in \Gamma$, so $\bigcap_{\gamma\in\Gamma} J'_\gamma$ is infinite; $K \cap K'_\gamma \subseteq J'_\gamma$ for every $\gamma \in \Gamma$; $F = \{R : R\lceil K'_\gamma = S'_\gamma$ for every $\gamma \in \Gamma\}$; and $\mathrm{T}' = \bigvee_{\gamma\in\Gamma} \Sigma_{J'_\gamma}$. So Lemma 1M tells us that $A_{\mathrm{T}'}(u \times v) = A_{\mathrm{T}'}u \times A_{\mathrm{T}'}v$. Consequently $\|A_{\mathrm{T}}(u \times v) - A_{\mathrm{T}}u \times A_{\mathrm{T}}v\|_2 \le 3\epsilon$. As ϵ is arbitrary, $A_{\mathrm{T}}(u \times v) = A_{\mathrm{T}}u \times A_{\mathrm{T}}v$ and E, F are relatively independent over T. **Q**

(b) Suppose that $\mathcal{K}_0 \subseteq \mathcal{K}$ is finite and not empty, $K \in [J_0]^{<\omega}$, $K_J \in [J]^{<\omega}$ for every $J \in \mathcal{K}_0$, $E \in \Sigma_K$ and $F \in \bigvee_{J\in\mathcal{K}_0} \Sigma_{K_J}$. Then E and F are relatively independent over T. **P** E and F can be expressed as $\bigcup_{l\in L} E_l$ and $\bigcup_{m\in M} F_m$ where $\langle E_l\rangle_{l\in L}$ is a finite partition of E, $\langle F_m\rangle_{m\in M}$ is a finite partition of F, and E_l, F_m satisfy the conditions of (a) for suitable S, S_J. In this case E_l and F_m are relatively independent over T for all l and m, so E and F also are. **Q**

(c) Suppose that $E \in \Sigma_{J_0}$ and $F \in \bigvee_{J\in\mathcal{K}} \Sigma_J$. Then E and F are relatively independent over T. **P** If $\mathcal{K}$ is empty, this is trivial. Otherwise, for any $\epsilon > 0$ there are E_1, F_1 such that $\mu(E\triangle E_1) \le \epsilon^2$ and $\mu(F\triangle F_1) \le \epsilon^2$ and $E_1 \in \Sigma_K$ for some $K \in [J_0]^{<\omega}$, while $F_1 \in \bigvee_{J\in\mathcal{K}_0} \Sigma_{K_J}$ for some non-empty finite $\mathcal{K}_0 \subseteq \mathcal{K}$ and suitable finite sets $K_J \subseteq J$. In this case, setting $u = \chi E^\bullet$, $v = \chi F^\bullet$, $u_1 = \chi E_1^\bullet$ and $v_1 = \chi F_1^\bullet$,

$$\|u - u_1\|_2 \le \epsilon, \quad \|v - v_1\|_2 \le \epsilon, \quad \|u \times v - u_1 \times v_1\|_2 \le 2\epsilon,$$

$$\|A_{\mathrm{T}}u - A_{\mathrm{T}}u_1\|_2 \le \epsilon, \quad \|A_{\mathrm{T}}v - A_{\mathrm{T}}v_1\|_2 \le \epsilon,$$

$$\|A_{\mathrm{T}}(u \times v) - A_{\mathrm{T}}(u_1 \times v_1)\|_2 \le 2\epsilon,$$

$$\|(A_{\mathrm{T}}u \times A_{\mathrm{T}}v) - (A_{\mathrm{T}}u_1 \times A_{\mathrm{T}}v_1)\|_2 \le 2\epsilon,$$

while

$$A_{\mathrm{T}}(u_1 \times v_1) = A_{\mathrm{T}}u_1 \times A_{\mathrm{T}}v_1$$

by (b); consequently

$$\|A_{\mathrm{T}}(u \times v) - A_{\mathrm{T}}u \times A_{\mathrm{T}}v\|_2 \le 4\epsilon.$$

As ϵ is arbitrary, $A_{\mathrm{T}}(u \times v) = A_{\mathrm{T}}u \times A_{\mathrm{T}}v$ and E, F are relatively independent over T. **Q**

But this is just the definition of relative independence for the σ-algebras Σ_{J_0} and $\bigvee_{J\in\mathcal{K}} \Sigma_J$.

1O Removable intersections For the next result we need another concept. Let (X, Σ, μ) be a probability space, T an algebra of subsets of X and $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma}$ a family of σ-subalgebras of Σ. I will say that $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma}$ has **T-removable intersections**[3] if whenever $\Delta \subseteq \Gamma$ is finite and not empty, $E_\gamma \in \Sigma_\gamma$ for $\gamma \in \Delta$, $\bigcap_{\gamma\in\Delta} E_\gamma$ is negligible and $\epsilon > 0$, we can find $F_\gamma \in \mathrm{T} \cap \Sigma_\gamma$, for $\gamma \in \Delta$, such that $\bigcap_{\gamma\in\Gamma} F_\gamma$ is empty and $\mu(E_\gamma \setminus F_\gamma) \le \epsilon$ for every $\gamma \in \Delta$.

Now Tao's Removal Theorem (Tao p06, Theorem 4.2; Fremlin n06) is the following.

Theorem Let (X, Σ, μ) be a probability space, and T a subalgebra of Σ. Let Γ be a partially ordered set such that $\gamma \wedge \delta = \inf\{\gamma, \delta\}$ is defined in Γ for all γ, $\delta \in I$, and $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma}$ a family of σ-subalgebras of Σ such that

(i) $\mathrm{T} \cap \Sigma_\gamma$ is metrically dense in Σ_γ[4] for every $\gamma \in \Gamma$,

(ii) if γ, $\delta \in \Gamma$ and $\gamma \le \delta$ then $\Sigma_\gamma \subseteq \Sigma_\delta$,

(iii) if $\gamma \in \Gamma$ and Δ, Δ' are finite subsets of Γ such that $\delta \wedge \gamma \in \Delta'$ for every $\delta \in \Delta$, then Σ_γ and $\bigvee_{\delta\in\Delta} \Sigma_\delta$ are relatively independent over $\bigvee_{\delta\in\Delta'} \Sigma_\delta$.

Then $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma}$ has T-removable intersections.

1P The following is now easy.

Theorem (Tao p06) Let I be a set and $\mu \in P_I$. Let $\mathcal{J}$ be a filter on I not containing any finite set. Let T be the algebra of open-and-closed subsets of $\mathcal{P}([I]^{<\omega})$. Then, defining Σ_J as in 1L, $\langle\Sigma_J\rangle_{J\in\mathcal{J}}$ has T-removable intersections.

proof If $J \subseteq I$ and $a \in [J]^{<\omega}$ then $\{R : a \in R\}$ belongs to $\mathrm{T}\cap\Sigma_J$; accordingly Σ_J is the σ-algebra generated by $\mathrm{T}\cap\Sigma_J$ and $\mathrm{T}\cap\Sigma_J$ is metrically dense in Σ_J. Ordering $\mathcal{J}$ by $\subseteq$, condition (ii) of 1O is obviously satisfied, and condition (iii) is just Theorem 1N. So Theorem 1O gives the result.

2 The hypergraph removal lemma

2A Definition If I, J are sets, $R \subseteq \mathcal{P}I$ and $S \subseteq \mathcal{P}J$, I will say for the purposes of this section that an **embedding** of (I, R) in (J, S) is an injective function $f : I \to J$ such that $f[a] \in S$ for every $a \in R$, that is, $R \subseteq \tilde{f}(S)$.

2B Theorem (Nagle Rödl & Schacht 06, Theorem 3) Let L be a finite set with r members, and $T \subseteq \mathcal{P}L$. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever I is a finite set, $R \subseteq \mathcal{P}I$ and the number of embeddings of (L, T) in (I, R) is at most $\delta\#(I)^r$, there is an $S \subseteq \mathcal{P}I$ such that $\#(R \cap [I]^k \setminus S) \le \epsilon\#(I)^k$ for every k and there is no embedding of (L, T) in (I, S).

proof (Tao p06) **?** Suppose, if possible, otherwise.

(a) We have a sequence $\langle(m_n, T_n)\rangle_{n\in\mathbb{N}}$ such that

$m_n \in \mathbb{N}$, $T_n \subseteq \mathcal{P}m_n$; the number of embeddings of (L, T) in (m_n, T_n) is at most $2^{-n}m_n^r$; if $S \subseteq \mathcal{P}m_n$ and $\#(T_n \cap [m_n]^k \setminus S) \le \epsilon m_n^k$ for every k then there is an embedding of (L, T) in (m_n, S)

[3] Tao p06 calls a version of this the 'uniform intersection property' or UIP.

[4] That is, whenever $E \in \Sigma_\gamma$ and $\epsilon > 0$ there is an $F \in \mathrm{T} \cap \Sigma_\gamma$ such that $\mu(F \triangle E) \le \epsilon$.

for every $n \in \mathbb{N}$. Of course (L, T) always has at least one embedding in (m_n, T_n) so $\lim_{n\to\infty} m_n = \infty$. Let I be an infinite set including L and $\mathcal{F}$ a non-principal ultrafilter on $\mathbb{N}$. Let $\mu \in EP_I$ be the measure defined from $\langle (m_n, T_n)\rangle_{n\in\mathbb{N}}$ and $\mathcal{F}$ by the process of 1J(v).

(b) Let $\mathcal{J}$ be $\{J : I \setminus L \subseteq J \subseteq I,\ J \cap L \in T\}$. For $J \in \mathcal{J}$ let Σ_J be the σ-algebra of subsets of $\mathcal{P}([I]^{<\omega})$ generated by sets of the form $E_a = \{R : a \in R \subseteq [I]^{<\omega}\}$ where $a \in [J]^{<\omega}$. By Theorem 1P, $\langle \Sigma_J\rangle_{J\in\mathcal{J}}$ has T-removable intersections.

(c) Set $E_J = \{R : R \in [I]^{<\omega},\ J \cap L \in R\}$ for $J \in \mathcal{J}$. Then $E_J \in \Sigma_J$ for every $J \in \mathcal{J}$, and

$$\begin{aligned}\mu(\bigcap_{J\in\mathcal{J}} E_J) = \mu\{R : T \subseteq R\} &= \sum_{T\subseteq T'\subseteq \mathcal{P}L} \mu\{R : R{\restriction}L = T'\}\\
&= \sum_{T\subseteq T'\subseteq \mathcal{P}L} \lim_{n\to\mathcal{F}} \frac{1}{m_n^r}\#(\{z : z \in m_n^L,\ \tilde z(T_n) = T'\})\\
&= \lim_{n\to\mathcal{F}} \frac{1}{m_n^r} \sum_{T\subseteq T'\subseteq \mathcal{P}L} \#(\{z : z \in m_n^L,\ \tilde z(T_n) = T'\})\\
&= \lim_{n\to\mathcal{F}} \frac{1}{m_n^r}\#(\{z : z \in m_n^L,\ T \subseteq \tilde z(T_n)\})\\
&= \lim_{n\to\mathcal{F}} \frac{1}{m_n^r}\#(\{z : z \in m_n^L \text{ is injective},\ T \subseteq \tilde z(T_n)\})\\
&= \lim_{n\to\mathcal{F}} \frac{1}{m_n^r}\#(\{z : z \text{ is an embedding of } (L, T) \text{ in } (m_n, T_n)\}) = 0.\end{aligned}$$

(d) Take $\eta > 0$ such that $2\eta\#(\mathcal{J}) < \epsilon$, and $\langle F_J\rangle_{J\in\mathcal{J}}$ such that $\bigcap_{J\in\mathcal{J}} F_J = \emptyset$ and $F_J \in \mathrm{T} \cap \Sigma_J$ and $\mu(E_J \setminus F_J) \le \eta$ for every $J \in \mathcal{J}$. Every F_J is open-and-closed, so there is an $M \in [I]^{<\omega}$ such that $L \subseteq M$ and every F_J is determined by coordinates in M, that is,

$$F_J = \{R : R \in [I]^{<\omega},\ R{\restriction}M \in H_J\}$$

for some $H_J \subseteq \mathcal{P}M$. Because $\bigcap_{J\in\mathcal{J}} F_J$ is empty, so is $\bigcap_{J\in\mathcal{J}} H_J$.

For $J \in \mathcal{J}$ set $E'_J = \{S : J \cap L \in S \subseteq \mathcal{P}M\}$; note that E'_J, H_J and therefore $E'_J \setminus H_J$ are determined by coordinates in $(J \cap L) \cup (M \setminus L)$.

(e) Let $n \ge r$ be such that

$$\begin{aligned}\frac{1}{m_n^{\#(M)}}\#(\{z : z \in m_n^M,\ \tilde z(T_n) \in E'_J \setminus H_J\}) &\le \eta + \mu\{R : R{\restriction}M \in E'_J \setminus H_J\}\\
&= \eta + \mu\{R : R \in E_J \setminus F_J\} \le 2\eta\end{aligned}$$

for every $J \in \mathcal{J}$. For $w \in m_n^{M\setminus L}$ and $J \in \mathcal{J}$ set

$$Q_{wJ} = \{z{\restriction}J \cap L : w \subseteq z \in m_n^M,\ z{\restriction}L \text{ is injective},\ \tilde z(T_n) \in E'_J \setminus H_J\}.$$

Then

$$\begin{aligned}\sum_{J\in\mathcal{J}, w\in m_n^{M\setminus L}} m_n^{\#(L\setminus J)}\#(Q_{wJ}) &\le \sum_{J\in\mathcal{J}, w\in m_n^{M\setminus L}} \#(\{z{\restriction}L : w \subseteq z \in m_n^M,\ \tilde z(T_n) \in E'_J \setminus H_J\})\\
&= \sum_{J\in\mathcal{J}} \#(\{z : z \in m_n^M,\ \tilde z(T_n) \in E'_J \setminus H_J\})\\
&\le 2\eta\#(\mathcal{J})m_n^{\#(M)},\end{aligned}$$

so there must be a $w \in m_n^{M\setminus L}$ such that

$$\textstyle\sum_{J\in\mathcal{J}} m_n^{\#(L\setminus J)}\#(Q_{wJ}) \le 2\eta\#(\mathcal{J})m_n^r < \epsilon m_n^r.$$

(f) Consider

$$S=T_n\setminus\bigcup_{k\in\mathbb{N}}\bigcup_{J\in\mathcal{J},\#(J\cap L)=k}\{z[J\cap L]:z\in Q_{wJ}\}.$$

Then

$$\#(T_n\cap[m_n]^k\setminus S)=\#(T_n\cap[m_n]^k\cap\bigcup_{\substack{J\in\mathcal{J}\\ \#(J\cap L)=k}}\{z[J\cap L]:z\in Q_{wJ}\})$$

(because $z{\restriction}L$ is injective whenever $z\in\bigcup_{J\in\mathcal{J}}Q_{wJ}$)

$$\begin{aligned}&\le\sum_{J\in\mathcal{J},\#(J\cap L)=k}\#(Q_{wJ})\\ &=\frac{1}{m_n^{r-k}}\sum_{J\in\mathcal{J},\#(J\cap L)=k}m_n^{\#(L\setminus J)}\#(Q_{wJ})\\ &\le\frac{1}{m_n^{r-k}}\sum_{J\in\mathcal{J}}m_n^{\#(L\setminus J)}\#(Q_{wJ})<\epsilon m_n^k\end{aligned}$$

for every k. So by the choice of (m_n,T_n) there is an embedding v of (L,T) in (m_n,S); take $z=v\cup w$, so that $w\subseteq z\in m_n^M$ and $z{\restriction}L=v$ is injective and $z[a]\in S$ for every $a\in T$. There is some $J\in\mathcal{J}$ such that $\tilde z(T_n)\notin H_J$. On the other hand,

$$z[J\cap L]\in S\subseteq T_n,$$

so $J\cap L\in\tilde z(T_n)$ and $\tilde z(T_n)\in E'_J$. But now $z\in Q_{wJ}$ and $z[J\cap L]$ cannot belong to S. **X**

This contradiction proves the theorem.

2C Corollary: the Hypergraph Removal Lemma For every $\epsilon>0$ and $r\ge2$ there is a $\delta>0$ such that whenever I is a finite set, $R\subseteq[I]^r$ and $\#(\{J:J\in[I]^{r+1},\ [J]^r\subseteq R\})\le\delta\#(I)^{r+1}$, there is an $S\subseteq R$ such that $\#(R\setminus S)\le\epsilon\#(I)^r$ and there is no $J\in[I]^{r+1}$ such that $[J]^r\subseteq S$.

proof Take $\#(L)=r+1$, $T=[L]^r$ in Theorem 2B.

2D Corollary: Szemerédi's Theorem (Szemerédi 75) For every $\epsilon>0$ and $r\ge2$ there is an $n_0\in\mathbb{N}$ such that whenever $n\ge n_0$, $A\subseteq n$ and $\#(A)\ge\epsilon n$ there is an arithmetic progression of length $r+1$ in A.

proof (Frankl & Rödl 02) Set $\eta=\frac{1}{r!}(\frac{\epsilon}{2r!})^r$. Take $\delta>0$ such that whenever I is a finite set, $R\subseteq[I]^r$ and $\#(\{J:J\in[I]^{r+1},\ [J]^r\subseteq R\})\le\delta\#(I)^{r+1}$, there is an $S\subseteq R$ such that $\#(R\setminus S)\le\frac{\eta}{2(r+1)^r}\#(I)^r$ and there is no $J\in[I]^{r+1}$ such that $[J]^r\subseteq S$. Let n_0 be such that $\epsilon n\ge2r\cdot r!$ and $n(r+1)^{r+1}\delta\ge1$ whenever $n\ge n_0$. Take $n\ge n_0$ and $A\subseteq n$ such that $\#(A)\ge\epsilon n$.

Let $C\subseteq n^r$ be the set

$$\{(i_0,i_1,\dots,i_{r-1}):\sum_{j=0}^{r-1}(j+1)i_j\in A\}.$$

Then $\#(C)\ge\eta n^r$. **P** For $m<r!$ set $A_m=\{i:i\in A,\ i\equiv m\mod r!\}$. Then there is an m such that $\#(A_m)\ge\frac{\epsilon n}{r!}$. Now we have an injection $\phi:[A_m]^r\to C$ given by saying that if $l_0<\dots l_{r-1}$ in A_m then

$$\begin{aligned}\phi(\{l_0,\dots,l_{r-1}\})(j)&=l_0\text{ if }j=0\\ &=\frac{1}{j+1}(l_j-l_{j-1})\text{ if }0<j<r.\end{aligned}$$

So

$$\#(C)\ge\#([A_m]^r)\ge\frac{1}{r!}(\frac{\epsilon n}{r!}-r)^r\ge\frac{1}{r!}(\frac{\epsilon n}{2r!})^r=\eta n^r.\ \textbf{Q}$$

Let I be $n\times(r+1)$ and for $c=(i_0,\dots,i_{r-1})\in C$ set

$$J_c=\{(i_j,j):j<r\}\cup\{(\sum_{j=0}^{r-1}i_j,r)\}\in[I]^{r+1}.$$

Observe that if c, $c' \in C$ are distinct, then $[J_c]^r \cap [J_{c'}]^r = \emptyset$, since given any face of J_c we can read off all but at most one of the coordinates of c and calculate the last. Set $R = \bigcup_{c\in C}[J_c]^r \subseteq [I]^r$.

? Suppose, if possible, that the only r-simplices $J \in [I]^{r+1}$ such that $[J]^r \subseteq R$ are of the form J_c for some $c \in C$. Then there are at most

$$\#(C) \le n^r \le \delta\#(I)^{r+1}$$

such simplices; by the choice of δ, there is an $S \subseteq R$ such that $R \setminus S$ covers no r-simplices and

$$\#(S) \le \frac{\eta}{2(r+1)^r}\#(I)^r = \frac{\eta}{2}n^r < \#(C).$$

But every J_c must have a face in S, and no two J_c share a face, so this is impossible. **X**

So we have an r-simplex $J \in [I]^{r+1}$, which is not of the form J_c where $c \in C$, such that $[J]^r \subseteq R$. Now since the only faces put into R come from the J_c, and therefore meet each of the $r+1$ levels $n \times \{k\}$ in at most one point, J must be of the form $\{(i_j, j) : j < r\} \cup \{(l, r)\}$. Since $\{(i_j, j) : j < r\}$ is a face of some J_c, $c = (i_0, \dots, i_{r-1}) \in C$. Set $l' = i_0 + \ldots + i_{r-1}$; then $l' \ne l$ because $J \ne J_c$. For each $k < r$, $J \setminus \{(i_k, k)\}$ is a face of J and therefore of $J_{c'}$ for some $c' \in C$; now $J_{c'}$ must be

$$(J \setminus \{(i_k, k)\}) \cup \{(l - \textstyle\sum_{j<r,j\ne k} i_j, k)\} = (J \setminus \{(i_k, k)\}) \cup \{(i_k + l - l', k)\}$$

and

$$\textstyle\sum_{j=0}^{r}(j+1)i_j + (k+1)(l - l')$$

belongs to A. Since this is true for every $k < r$, and we also have $\sum_{j=0}^{r}(j+1)i_j \in A$ because $c \in C$, we have an arithmetic progression in A of length $r+1$, as required.

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