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Talagrand's example

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I rewrite the construction from Talagrand 06 of an exhaustive submeasure which is not uniformly exhaustive. For basic definitions, see Fremlin 02. Interjections of the form '(T2.3)' or '(T§2.3)' indicate the approximate equation number or section number containing the corresponding idea in Talagrand p06.

1A I begin with two elementary combinatorial facts.

Lemma (T§2.3) Suppose that $\langle I_k\rangle_{k\in K}$ is a non-empty finite family of subsets of $\mathbb{N}$ and $r\in\mathbb{N}$ is such that $\#(I_k)\ge r\#(K)$ for every $k\in K$. Then we have an enumeration $\langle k_i\rangle_{i<m}$ of K and a strictly increasing family $\langle n_i\rangle_{i\le m}$ such that $\#(I_{k_i}\cap n_{i+1}\setminus n_i)=r$ for every $i<m$.

proof Choose n_i, k_i inductively, as follows. Start with $n_0=0$. Given $j<\#(K)$, $n_j\in\mathbb{N}$ and $\langle k_i\rangle_{i<j}$ such that $\#(I_k\setminus n_j)\ge r(\#(K)-j)$ for every $k\in K_j=K\setminus\{k_i:i<j\}$, set

$$n_{j+1}=\min\{n:\#(I_k\cap n\setminus n_j)\ge r\text{ for some }k\in K_j\}$$

and choose $k_j\in K_j$ such that $\#(I_{k_j}\cap n_{j+1}\setminus n_j)\ge r$. Observe that $\#(I_k\cap n_{j+1}\setminus n_j)\le r$ for every $k\in K_j$, so that $\#(I_{k_j}\cap n_{j+1}\setminus n_j)$ must in fact be equal to r and $\#(I_k\setminus n_{j+1})\ge r(\#(K)-j-1)$ for every $k\in K_j$; thus the induction proceeds.

1B Lemma (T3.17) Suppose that $J_0,\dots,J_s$ are finite subsets of $\mathbb{N}$, all of size at least $r\ge 2$, such that $\max J_j<\min J_{j+1}$ for $j<s$. Let $\langle(I_k,w_k)\rangle_{k\in K}$ be a family in $([\mathbb{N}]^{<\omega}\setminus\{\emptyset\})\times[0,\infty[$, and set $a=\sum_{k\in K}w_k$. Then we can find u_j, $v_j\in J_j$ such that $u_j<v_j$ for each j and, setting $W=\bigcup_{j\le s}v_j\setminus u_j$,

$$\textstyle\sum_{k\in K,\#(I_k\cap W)\ge\frac12\#(I_k)}w_k\le\frac{2a}{r-1}.$$

proof Set $J=\bigcup_{j\le s}J_j$ and for $i\in J\setminus\{\max J\}$ let i' be the next member of J above i. For $z\in Z=\prod_{j\le s}(J_j\setminus\{\max J_j\})$ set

$$W(z)=\textstyle\bigcup_{j\le s}z(j)'\setminus z(j),$$

$$S(z)=\textstyle\sum_{k\in K,\#(I_k\cap W(z))\ge\frac12\#(I_k)}w_k.$$

Consider the uniform probability measure λ on Z. For any set $I\subseteq\mathbb{N}$,

$$\begin{aligned}\int\#(I\cap W(z))\lambda(dz)&=\sum_{j=0}^s\int\#(I\cap z(j)'\setminus z(j))\lambda(dz)\\&=\sum_{j=0}^s\sum_{i\in J_j\setminus\{\max J_j\}}\frac{1}{\#(J_j)-1}\#(I\cap i'\setminus i)\\&=\sum_{j=0}^s\frac{\#(I\cap\max J_j\setminus\min J_j)}{\#(J_j)-1}\le\frac{\#(I)}{r-1}.\end{aligned}$$

So, writing $V_I=\{z:\#(I\cap W(z))\ge\frac12\#(I)\}$, $\lambda V_I\le\frac{2}{r-1}$ for any non-empty I. Now

$$\textstyle\int S(z)\lambda(dz)=\sum_{k\in K}w_k\lambda V_{I_k}\le\frac{2}{r-1}\sum_{k\in K}w_k=\frac{2a}{r-1},$$

so there must be a $z\in Z$ such that $S(z)\le\frac{2a}{r-1}$, and we can take $u_j=z(j)$, $v_j=z(j)'$ for each j.

1C Definitions We are now ready to begin work. The construction is complex and demands a large volume of special notation.

(a) (T2.1) We shall work throughout with $X=\prod_{n\in\mathbb{N}}T_n$ where $\langle T_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-empty finite sets and $\sup_{n\in\mathbb{N}}\#(T_n)$ is infinite. X may be regarded as a compact Hausdorff space with the product of the discrete topologies on the T_n. For each $n\in\mathbb{N}$, $\mathfrak{B}_n$ will be the algebra of subsets of X determined by coordinates less than n and $\mathcal{A}_n$ the set of its atoms, that is, the family of sets of the form $\{x:z\subseteq x\in X\}$ for some $z\in\prod_{i<n}T_i$. $\mathfrak{B}=\bigcup_{n\in\mathbb{N}}\mathfrak{B}_n$ will be the algebra of open-and-closed subsets of X.

(b) We shall need two sequences $\langle\alpha_k\rangle_{k\in\mathbb{N}}$ and $\langle N_k\rangle_{k\in\mathbb{N}}$. It is easy enough to give appropriate formulae but perhaps the ideas will be clearer if instead I declare the properties they must have, without apologizing for the redundancies.

(i) (T3.3) $\langle\alpha_k\rangle_{k\in\mathbb{N}}$ is non-increasing, $\sum_{k=0}^{\infty}\alpha_k\le\frac{1}{2}$, and $\alpha_k>0$ and $(2^{k+5})^{\alpha_k}\le 2$ for every $k\in\mathbb{N}$.

(ii) (T4.2, 5.2) $N_k\in\mathbb{N}$ and $2^{-k}(2^{-2k-12}N_k)^{\alpha_k}\ge 2^4$ for every $k\in\mathbb{N}$.

(c) (T3.1) Now we come to some of the key ideas. For $\mathcal{I}\subseteq\mathcal{P}X\times\mathcal{P}\mathbb{N}\times[0,\infty[$, define its 'spread' $\operatorname{spr}\mathcal{I}$ to be $\bigcup_{(E,I,w)\in\mathcal{I}}E$ and its 'weight' $\operatorname{wt}\mathcal{I}$ to be $\sum_{(E,I,w)\in\mathcal{I}}w$.

(d) (T3.2) For any family $\mathcal{E}\subseteq\mathcal{P}T\times\mathcal{P}\mathbb{N}\times[0,\infty[$ define $\phi_{\mathcal{E}}:\mathfrak{B}\to[0,\infty]$ by setting

$$\phi_{\mathcal{E}}E=\inf\{\operatorname{wt}\mathcal{I}:\mathcal{I}\subseteq\mathcal{E}\text{ is finite, }E\subseteq\operatorname{spr}\mathcal{I}\},$$

counting $\inf\emptyset$ as ∞. So $\phi_\emptyset\emptyset=0$ and $\phi_\emptyset E=\infty$ for $E\in\mathfrak{B}\setminus\{\emptyset\}$.

(e) (T2.8) For $E\subseteq X$ and $I\subseteq\mathbb{N}$ set

$$\theta_I(E)=\{y:y\in X,\,y{\restriction}I=x{\restriction}I\text{ for some }x\in E\}.$$

(f) (T§3.4)**(i)** If $m<n$ in $\mathbb{N}$, $\phi:\mathfrak{B}\to[0,\infty]$ is a function and $E\in\mathfrak{B}$, then E is **ϕ-thin between m and n** if $\phi(X\setminus\theta_{n\setminus m}(A\cap E))\ge 1$ for every $A\in\mathcal{A}_m$.

(ii) If $I\subseteq\mathbb{N}$, $\phi:\mathfrak{B}\to[0,\infty]$ is a function and $E\in\mathfrak{B}$, then E is **ϕ-thin along I** if it is ϕ-thin between m and n whenever m, $n\in I$ and $m<n$.

(g) (T§4.0) For $k\le p\in\mathbb{N}$ define $\mathcal{C}_{kp}$ and ν_{kp} by downwards induction on k, as follows. Start with $\mathcal{C}_{pp}=\emptyset$ for every p. Given $\mathcal{C}_{kp}$, set $\nu_{kp}=\phi_{\mathcal{C}_{kp}}$. Given that $k<p$ and $\mathcal{C}_{k+1,p}$ and $\nu_{k+1,p}=\phi_{\mathcal{C}_{k+1,p}}$ have been defined, set

$$\begin{aligned}\mathcal{E}_{kp}=\{(E,I,w):{}&E\in\mathfrak{B},\,I\subseteq\mathbb{N},\,1\le\#(I)\le N_k,\\&w\ge 2^{-k}\Bigl(\frac{N_k}{\#(I)}\Bigr)^{\alpha_k},\,E\text{ is }\nu_{k+1,p}\text{-thin along }I\},\\\mathcal{C}_{kp}={}&\mathcal{E}_{kp}\cup\mathcal{C}_{k+1,p}\end{aligned}$$

and continue.

(h) (T§5.0) Define $\langle c_k\rangle_{k\in\mathbb{N}}$ by setting $c_0=8$, $c_{k+1}=2^{2\alpha_k}c_k$ for every k.

1D Very elementary facts In the hope of aiding digestion of the definitions here, of which 1Cf and 1Cg are likely to be wholly obscure to anyone who has not worked through this proof before, I run over some obvious facts which will be used below.

(a) (T3.2) If $\mathcal{E}\subseteq\mathcal{P}X\times\mathcal{P}\mathbb{N}\times[0,\infty[$ is such that $\phi_{\mathcal{E}}X$ is finite (that is, there is a finite $\mathcal{I}\subseteq\mathcal{E}$ such that $\operatorname{spr}\mathcal{I}=X$), then $\phi_{\mathcal{E}}:\mathfrak{B}\to[0,\infty[$ is a submeasure. (Subadditivity and monotonicity are written into the definition.)

(b) If I, $J \subseteq \mathbb{N}$ then $\theta_I\theta_J = \theta_{I\cap J}$. If $I \subseteq \mathbb{N}$ then $\theta_I(E \cap \theta_I(F)) = \theta_I(F \cap \theta_I(E))$ for all E, $F \subseteq X$. For any $I \subseteq \mathbb{N}$ and any family $\mathcal{E}$ of subsets of X, $\theta_I(\bigcup\mathcal{E}) = \bigcup_{E\in\mathcal{E}}\theta_I(E)$.

For $n \in \mathbb{N}$ and $E \subseteq X$, $E \in \mathfrak{B}_n$ iff $\theta_n(E) = E$. If $E \in \mathfrak{B}$ and $I \subseteq \mathbb{N}$, $\theta_I(E) \in \mathfrak{B}$. If $m \le n$ in $\mathbb{N}$, $A \in \mathcal{A}_m$ and $A_1 \in \mathcal{A}_n$, then $A \cap \theta_{n\setminus m}(A_1) \in \mathcal{A}_m$. If $A \in \mathcal{A}_m$ then $E \mapsto \theta_{\mathbb{N}\setminus m}(A \cap E) : \mathfrak{B} \to \mathfrak{B}$ is a Boolean homomorphism.

(c) If $m < n$, $\phi : \mathfrak{B} \to [0,\infty]$ is a function, $E \in \mathfrak{B}$ is ϕ-thin between m and n and $E' \in \mathfrak{B}$ is included in E, then E' is ϕ-thin between m and n' for every $n' \ge n$.

(d) All the classes $\mathcal{E}_{kp}$, $\mathcal{C}_{kp}$ are closed under increases in the scalar variable and decreases in the first variable, that is,
— if $k < p$, $(E, I, w) \in \mathcal{E}_{kp}$, $E' \in \mathfrak{B}$, $E' \subseteq E$ and $w' \ge w$ then $(E', I, w') \in \mathcal{E}_{kp}$,
— if $k \le p$, $(E, I, w) \in \mathcal{C}_{kp}$, $E' \in \mathfrak{B}$, $E' \subseteq E$ and $w' \ge w$ then $(E', I, w') \in \mathcal{C}_{kp}$.

(e) If $k \le p$ in $\mathbb{N}$, $\mathcal{C}_{kp} = \bigcup_{k\le l<p}\mathcal{E}_{lp}$.

(f) If $k < p$ in $\mathbb{N}$, $\nu_{kp} \le \nu_{k+1,p}$, because $\mathcal{C}_{kp} \supseteq \mathcal{C}_{k+1,p}$.

(g) $8 \le c_k \le 16$ for every $k \in \mathbb{N}$, because $\sum_{k=0}^{\infty} 2\alpha_k \le 1$.

(h) If $k < p$ in $\mathbb{N}$, then $(X, \{0\}, 2^{-k}N_k^{\alpha_k}) \in \mathcal{E}_{kp}$ so $\nu_{kp}X \le 2^{-k}N_k^{\alpha_k}$ and ν_{kp} is a submeasure.

1E Lemma (T§5.2) Suppose that $k \le p$, $m \in \mathbb{N}$, $A \in \mathcal{A}_m$ and $(E, I, w) \in \mathcal{C}_{kp}$. Suppose that $n > m$ and $I' = I \cap n \setminus m$ is non-empty. Set $E' = \theta_{n\setminus m}(E \cap A)$. Then $(E', I', w') \in \mathcal{C}_{kp}$ where $w' = (\frac{\#(I)}{\#(I')})^{\alpha_k} w$.

proof Suppose that $(E, I, w) \in \mathcal{E}_{lp}$ where $k \le l < p$. Then E' is $\nu_{l+1,p}$-thin along I'. **P** Suppose that i, $j \in I'$ and $i < j$, so that $m \le i < j < n$. Take any $A_1 \in \mathcal{A}_i$, and set $A_2 = A \cap \theta_{n\setminus m}(A_1)$, so that A_2 also belongs to $\mathcal{A}_i$. Then, using the list in 1Db,

$$\begin{aligned}\theta_{j\setminus i}(E' \cap A_1) &= \theta_{j\setminus i}(A_1 \cap \theta_{n\setminus m}(E \cap A))\\ &= \theta_{j\setminus i}(\theta_{n\setminus m}(A_1 \cap \theta_{n\setminus m}(E \cap A)))\\ &= \theta_{j\setminus i}(\theta_{n\setminus m}(E \cap A \cap \theta_{n\setminus m}(A_1)))\\ &= \theta_{j\setminus i}(\theta_{n\setminus m}(E \cap A_2)) = \theta_{j\setminus i}(E \cap A_2).\end{aligned}$$

So

$$\nu_{l+1,p}(X \setminus \theta_{j\setminus i}(E \cap A_1)) = \nu_{l+1,p}(X \setminus \theta_{j\setminus i}(E \cap A_2)) \ge 1$$

because E is $\nu_{l+1,p}$-thin between i and j. As i, j and A_1 are arbitrary, E' is $\nu_{l+1,p}$-thin along I'. **Q**

Of course $\#(I') \le \#(I) \le N_l$. Finally, we have $w \ge 2^{-l}(\frac{N_l}{\#(I)})^{\alpha_l}$, so that (because $\alpha_k \ge \alpha_l$) $w' \ge 2^{-l}(\frac{N_l}{\#(I')})^{\alpha_l}$. Thus $(E', I', w') \in \mathcal{E}_{lp} \subseteq \mathcal{C}_{kp}$.

1F Corollary (a) Suppose that $n \in \mathbb{N}$ and $k \le p$ and that $\mathcal{I} \subseteq \mathcal{C}_{kp}$ is a finite set such that $\#(I \cap n) \ge \frac{1}{4}\#(I)$ whenever $(E, I, w) \in \mathcal{I}$. Then $\nu_{kp}(\theta_n(\operatorname{spr}\mathcal{I})) \le 4^{\alpha_k}\operatorname{wt}\mathcal{I}$.

(b) Suppose that $m \in \mathbb{N}$, $k \le p$ and $A \in \mathcal{A}_m$. Let $\mathcal{I}$ be a finite subset of $\mathcal{C}_{kp}$ such that $\#(I \setminus m) \ge \frac{1}{4}\#(I)$ whenever $(E, I, w) \in \mathcal{I}$. Then $\nu_{kp}(\theta_{\mathbb{N}\setminus m}(A \cap \operatorname{spr}\mathcal{I}]) \le 4^{\alpha_k}\operatorname{wt}\mathcal{I}$.

(c) Suppose that $m < n$ in $\mathbb{N}$, $k \le p$ and $A \in \mathcal{A}_m$. Let $\mathcal{I}$ be a finite subset of $\mathcal{C}_{kp}$ such that $\#(I \cap n \setminus m) \ge 2^{-k-5}\#(I)$ whenever $(E, I, w) \in \mathcal{I}$. Then $\nu_{kp}(\theta_{n\setminus m}(A \cap \operatorname{spr}\mathcal{I})) \le 2\operatorname{wt}\mathcal{I}$.

proof (a) For each $(E, I, w) \in \mathcal{I}$ set $E' = \theta_n(E) \in \mathcal{B}_n$, $I' = I \cap n$ and

$$w' = (\frac{\#(I)}{\#(I')})^{\alpha_k} w \le 4^{\alpha_k} w.$$

By Lemma 1E, with $m=0$ and $A=X$, $(E',I',w')\in\mathcal{C}_{kp}$. Set $\mathcal{J}=\{(E',I',w'):(E,I,w)\in\mathcal{I}\}$ and $B=\operatorname{spr}\mathcal{J}$. Then

$$B=\bigcup_{(E,I,w)\in\mathcal{I}}\theta_n(E)=\theta_n(\operatorname{spr}\mathcal{I})$$

and

$$\nu_{kp}B\le\operatorname{wt}\mathcal{J}=\sum_{(E,I,w)\in\mathcal{I}}w'\le 4^{\alpha_k}\operatorname{wt}\mathcal{I},$$

as required.

(b) This time, take $n>m$ so large that $I\subseteq n$ whenever $(E,I,w)\in\mathcal{I}$. For $(E,I,w)\in\mathcal{I}$, set

$$E'=\theta_{n\setminus m}(A\cap E),\quad I'=I\setminus m=I\cap n\setminus m,\quad w'=\Bigl(\frac{\#(I)}{\#(I')}\Bigr)^{\alpha_k}w\le 4^{\alpha_k}w.$$

Then Lemma 1E tells us that $(E',I',w')\in\mathcal{C}_{kp}$. Setting $\mathcal{J}=\{(E',I',w'):(E,I,w)\in\mathcal{I}\}$,

$$\theta_{\mathbb{N}\setminus m}(A\cap\operatorname{spr}\mathcal{I}]=\bigcup_{(E,I,w)\in\mathcal{I}}\theta_{\mathbb{N}\setminus m}(A\cap E)\subseteq\bigcup_{(E,I,w)\in\mathcal{I}}E'=\operatorname{spr}\mathcal{J},$$

so

$$\nu_{kp}(\theta_{N\setminus m}(A\cap\operatorname{spr}\mathcal{I})\le\operatorname{wt}\mathcal{J}\le 4^{\alpha_k}\operatorname{wt}\mathcal{I}.$$

(c) For $(E,I,w)\in\mathcal{I}$ set

$$E'=\theta_{n\setminus m}(A\cap E),\quad I'=I\cap n\setminus m,\quad w'=\Bigl(\frac{\#(I)}{\#(I')}\Bigr)^{\alpha_k}w\le(2^{k+5})^{\alpha_k}w\le 2w.$$

Then $(E',I',w')\in\mathcal{C}_{kp}$. Setting $\mathcal{J}=\{(E',I',w'):(E,I,w)\in\mathcal{I}\}$,

$$\theta_{n\setminus m}(A\cap\operatorname{spr}\mathcal{I})=\bigcup_{(E,I,w)\in\mathcal{I}}\theta_{n\setminus m}(A\cap E)=\bigcup_{(E,I,w)\in\mathcal{I}}E'=\operatorname{spr}\mathcal{J},$$

so

$$\nu_{kp}(\theta_{n\setminus m}(A\cap\operatorname{spr}\mathcal{I}))\le\operatorname{wt}\mathcal{J}\le 2\operatorname{wt}\mathcal{I}.$$

1G We are at the centre of the argument.

Lemma (T§5.1) Suppose that $r\in\mathbb{N}$ and $t\in T_n$, and set $Y_{rt}=\{x:x\in X,\ x(r)=t\}$. Then $\nu_{kp}Y_{rt}\ge c_k$ whenever $k\le p$ in $\mathbb{N}$.

proof (a) Induce on $p-k$. If $k=p$ then

$$\mathcal{C}_{pp}=\emptyset,\quad \nu_{pp}Y_{rt}=\infty.$$

For the downwards step to $k<p$, given that $\nu_{k+1,p}Y_{rt}\ge c_{k+1}$, take a finite set $\mathcal{I}\subseteq\mathcal{C}_{kp}$ such that $\operatorname{wt}\mathcal{I}<c_k$.

(b) Express $\mathcal{I}$ as $\mathcal{I}_1\cup\mathcal{I}_2$ where $\mathcal{I}_1\subseteq\mathcal{E}_{kp}$ and $\mathcal{I}_2\subseteq\mathcal{C}_{k+1,p}$. Set $s=\#(\mathcal{I}_1)$. For $(E,I,w)\in\mathcal{I}_1$ we have $w\ge 2^{-k}$, so $s\le 2^kc_k\le 2^{k+4}$ (Fact Dg). Also, for $(E,I,w)\in\mathcal{I}_1$,

$$2^{-k}\Bigl(\frac{N_k}{\#(I)}\Bigr)^{\alpha_k}\le w\le c_k\le 2^4,$$

so $\#(I)\ge 2^{2k+12}\ge 2^{k+8}s$, by the choice of N_k. By Lemma 1A, we can find $m_0<m_1<\ldots<m_s$ and an enumeration $\langle(E_i,I_i,w_i)\rangle_{i<s}$ of $\mathcal{I}_1$ such that $\#(I_i\cap m_{i+1}\setminus m_i)\ge 2^{k+8}$ for $i<s$.

For each $i<s$, set $u'_i=\sup(I_i\cap(r+1))$ and $I'_i=(I_i\setminus\{u'_i\})\cap m_{i+1}\setminus m_i$, so that $\#(I'_i)\ge 2^{k+8}-1$. By Lemma 1B we can find $u_i<v'_i$ in I'_i, for $i<s$, such that, setting $W'=\bigcup_{i<s}v'_i\setminus u_i$,

$$S'=\sum_{\substack{(E,I,w)\in\mathcal{I}_2\\ \#(I\cap W')\ge\frac{1}{2}\#(I)}}w\le\frac{2}{2^{k+8}-2}\sum_{(E,I,w)\in\mathcal{I}_2}w\le\frac{2^5}{2^{k+8}-2}\le 2^{-k-2}.$$

Now, for $i<s$, let v_i be the least element of I_i greater than u_i, and set $W=\bigcup_{i<s}v_i\setminus u_i$; then

$$S=\sum_{\substack{(E,I,w)\in\mathcal{I}_2\\ \#(I\cap W)\ge\frac{1}{2}\#(I)}}w\le S'\le 2^{-k-2},$$

and $r \notin W$, because $u_i \neq u'_i$ for each i.

(c) Set

$$\mathcal{I}_3 = \{(E, I, w) : (E, I, w) \in \mathcal{I}_2,\ \#(I \cap W) \geq \frac{1}{2}\#(I)\}, \quad \mathcal{I}_4 = \mathcal{I}_2 \setminus \mathcal{I}_3;$$

then

$$\operatorname{wt} \mathcal{I}_3 = S \leq 2^{-k-2} \leq \frac{1}{4}.$$

For $i < s$ set

$$\mathcal{I}_{3i} = \{(E, I, w) : (E, I, w) \in \mathcal{I}_3,\ \#(I \cap v_i \setminus u_i) \geq 2^{-k-5}\#(I)\}.$$

Since $s \leq 2^{k+4}$, $\mathcal{I}_3 = \bigcup_{i<s} \mathcal{I}_{3i}$.

(d) Suppose that $i < s$ and $A \in \mathcal{A}_{u_i}$. Then there is an $A' \in \mathcal{A}_{v_i}$ such that $A' \subseteq A \setminus (E_i \cup \operatorname{spr} \mathcal{I}_{3i})$. **P** Set $C = \theta_{v_i \setminus u_i}(A \cap \operatorname{spr} \mathcal{I}_{3i}) \in \mathfrak{B}_{v_i}$. By Corollary 1Fc, applied to $\mathcal{C}_{k+1,p}$,

$$\nu_{k+1,p} C \leq 2 \operatorname{wt} \mathcal{I}_{3i} < 1.$$

As $(E_i, I_i, w_i) \in \mathcal{E}_{kp}$, E_i is $\nu_{k+1,p}$-thin between u_i and v_i, $\nu_{k+1,p}(X \setminus \theta_{v_i \setminus u_i}(A \cap E_i)) \geq 1$ and C does not include $X \setminus \theta_{v_i \setminus u_i}(A \cap E_i)$. Since these sets both belong to $\mathfrak{B}_{v_i}$ there is an $A_1 \in \mathcal{A}_{v_i}$ disjoint from both C and $\theta_{v_i \setminus u_i}(A \cap E_i)$, that is, disjoint from $\theta_{v_i \setminus u_i}(A \cap (E_i \cup \operatorname{spr} \mathcal{I}_{3i}))$. Now $A' = A \cap \theta_{v_i \setminus i_i}(A_1)$ belongs to $\mathcal{A}_{v_i}$, is included in A and is disjoint from $E_i \cup \operatorname{spr} \mathcal{I}_{3i}$. **Q**

(e) We can therefore find a function $\Gamma : X \to X$ such that

for $j \in \mathbb{N} \setminus W$, $\Gamma(x)(j) = x(j)$ for every $x \in X$,

for each $i < s$ and $A \in \mathcal{A}_{u_i}$ there is an $A' \in \mathcal{A}_{v_i}$, disjoint from $E_i \cup \operatorname{spr} \mathcal{I}_{3i}$, such that $\Gamma(x) \in A'$ for every $x \in A$.

Observe that $\Gamma[X]$ is disjoint from $\operatorname{spr}(\mathcal{I}_1 \cup \mathcal{I}_3)$, while $\Gamma(x) \restriction r$ is determined by $x \restriction r$ for every $r \in \mathbb{N}$.

(f) Take $(E, I, w) \in \mathcal{I}_4$ and consider $\nu_{k+1,p}(\Gamma^{-1}[E])$. There is an l such that $k + 1 \leq l < p$ and $(E, I, w) \in \mathcal{E}_{lp}$. Now if $m, n \in I$ are such that $m < n$ and $n \setminus m$ is disjoint from W, $\Gamma^{-1}[E]$ is $\nu_{l+1,p}$-thin between m and n. **P** Take any $A \in \mathcal{A}_m$. Then we can find an $A' \in \mathcal{A}_m$ such that $\Gamma[A] \subseteq A'$. In this case,

$$A \cap \Gamma^{-1}[E] \subseteq \Gamma^{-1}[\Gamma[A] \cap E] \subseteq \Gamma^{-1}[A' \cap E] \subseteq \theta_{n \setminus m}(A' \cap E)$$

because $\Gamma(x)(i) = x(i)$ whenever $x \in X$ and $i \in n \setminus m$. So $\theta_{n \setminus m}(A \cap \Gamma^{-1}[E]) \subseteq \theta_{n \setminus m}(A' \cap E)$ and

$$\nu_{l+1,p}(X \setminus \theta_{n \setminus m}(A \cap \Gamma^{-1}[E])) \geq \nu_{l+1,p}(X \setminus \theta_{n \setminus m}(A' \cap E)) \geq 1$$

because E is $\nu_{l+1,p}$-thin between m and n. **Q**

Note that

$$2^{-l}\left(\frac{N_l}{\#(I)}\right)^{\alpha_l} \leq w \leq 16$$

so that, as in (b),

$$\#(I) \geq 2^{2l+12} \geq 4s.$$

For each $i < s$ such that $I \cap (u_i + 1) \neq \emptyset$ let u'_i be the largest element of I which is less than or equal to u_i. Set $I' = I \setminus (W \cup \{u'_i : i < s,\ I \cap (u_i + 1) \neq \emptyset\})$. Then

$$\#(I') \geq \frac{\#(I)}{2} - s \geq \frac{\#(I)}{4}.$$

Now $\Gamma^{-1}[E]$ is $\nu_{l+1,p}$-thin along I'. **P** Suppose that $m, n \in I'$ and $m < n$. Let n' be the least element of I such that $m < n'$. Then $n' \leq n$. **?** If $W \cap n' \setminus m \neq \emptyset$, there is an $i < s$ such that $n' \setminus m$ meets $v_i \setminus u_i$, that is, $m < v_i$ and $u_i < n'$. Since $m \in I' \subseteq \mathbb{N} \setminus W$, $m \leq u_i$ and u'_i is defined; now $m \neq u'_i$ so $m < u'_i \in I$ and $n' \leq u'_i \leq u_i$. **X** Thus $W \cap n' \setminus m$ is empty and (ii) tells us that $\Gamma^{-1}[E]$ is $\nu_{l+1,p}$-thin between m and n', therefore $\nu_{l+1,p}$-thin between m and n (Fact 1Dc). As m and n are arbitrary, $\Gamma^{-1}[E]$ is $\nu_{l+1,p}$-thin along I'. **Q**

If we now set $w' = 4^{\alpha_l} w$, we see that

$$1 \leq \#(I') \leq N_l, \quad w' \geq 2^{-l} 4^{\alpha_l} \big(\frac{N_l}{\#(I)}\big)^{\alpha_l} \geq 2^{-l} \big(\frac{N_l}{\#(I')}\big)^{\alpha_l},$$

so

$$(\Gamma^{-1}[E], I', w') \in \mathcal{E}_{lp} \subseteq \mathcal{C}_{k+1,p}$$

and

$$\nu_{k+1,p}(\Gamma^{-1}[E]) \leq w' \leq 4^{\alpha_k} w.$$

(g) We are nearly done. Applying (f) to each member of $\mathcal{I}_4$,

$$\nu_{k+1,p}(\Gamma^{-1}[\operatorname{spr} \mathcal{I}_4]) \leq 4^{\alpha_k} \operatorname{wt} \mathcal{I}_4 < 4^{\alpha_k} c_k = c_{k+1} \leq \nu_{k+1,p} Y_{rt}$$

by the inductive hypothesis in its full strength. So there is a $z \in Y_{rt} \setminus \Gamma^{-1}[\operatorname{spr} \mathcal{I}_4]$. With (e), this means that $\Gamma(z)$ does not belong to

$$\operatorname{spr}(\mathcal{I}_1 \cup \mathcal{I}_3 \cup \mathcal{I}_4) = \operatorname{spr} \mathcal{I}.$$

On the other hand, $\Gamma(z) \in Y_{rt}$ because $r \notin W$. As $\mathcal{I}$ was arbitrary, $\nu_{kp} Y_{rt}$ must be at least c_k, which is what we need to know to proceed with the induction.

1H Definitions (T4.1) I present the last two definitions required. Fix on a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. For $k \in \mathbb{N}$, set

$$\nu_k E = \lim_{p \to \mathcal{U}} \nu_{kp} E \in [0, \infty]$$

for every $E \in \mathfrak{B}$; finally, write ν for ν_0.

1I Proposition (a) (T§6.2) For every $k \in \mathbb{N}$, ν_k is a submeasure and $\nu_k X \geq 8$.
(b) (T§4.1) ν is not uniformly exhaustive.

proof (a) By Fact 1Dh, $\nu_k X \leq 2^{-k} N_k^{\alpha_k}$ is finite. It now follows directly from the definition in Fremlin 02, 392G that ν_k, being a limit of submeasures, is a submeasure. By Lemma 1G,

$$\nu_k X = \lim_{p \to \mathcal{U}} \nu_{kp} X \geq c_k \geq 8.$$

(b) For any $n \in \mathbb{N}$ and $t \in T_n$, set $Y_{nt} = \{x : x(n) = t\}$. Then

$$\nu Y_{nt} = \lim_{p \to \infty} \nu_{0p} Y_{nt} \geq 8$$

by Lemma 1G. As $\sup_{n \in \mathbb{N}} \#(T_n)$ is infinite, and $\langle Y_{nt} \rangle_{t \in T_n}$ is disjoint for every n, ν is not uniformly exhaustive.

1J Lemma Suppose that $k \in \mathbb{N}$, $E \in \mathfrak{B}$, $I \in [\mathbb{N}]^{<\omega}$ and E is $\frac{1}{2}\nu_k$-thin along I. Then

$$\{p : p \geq k,\ E \text{ is } \nu_{kp}\text{-thin along } I\} \in \mathcal{F}.$$

If $k \geq 1$ and $\#(I) \geq N_{k-1}$, then $\nu_{k-1} E \leq 2^{-k+1}$.

proof If m, $n \in I$, $m < n$ and $A \in \mathcal{A}_m$, then $\nu_k(X \setminus \theta_{n \setminus m}(A \cap E)) \geq 2$. So

$$U_{An} = \{p : p \geq k,\ \nu_{kp}(X \setminus \theta_{n \setminus m}(A \cap E)) \geq 1\}$$

belongs to $\mathcal{F}$. Setting $U = \bigcap_{m < n \text{ in } I, A \in \mathcal{A}_m} U_{An}$, $U \in \mathcal{F}$ and E is ν_{kp}-thin along I for every $p \in U$.

If $k \geq 1$ and $\#(I) \geq N_{k-1}$, then $(E, I, 2^{-k+1}) \in \mathcal{E}_{k-1,p}$ for every $p \in U$, so $\nu_{k-1,p} E \leq 2^{-k+1}$ for every $p \in U$ and $\nu_{k-1} E \leq 2^{-k+1}$.

1K Lemma (T§6.5) Let m, $k \in \mathbb{N}$ and let $\langle E_i \rangle_{i \in \mathbb{N}}$ be a sequence in $\mathfrak{B}$ such that
every E_i is determined by coordinates in $\mathbb{N} \setminus m$,
$\nu_k(\bigcup_{i \leq n} E_i) < 2$ for every $n \in \mathbb{N}$.
Then for every $\eta > 0$ there is a $C \in \mathfrak{B}$, determined by coordinates in $\mathbb{N} \setminus m$, such that $\nu_k C \leq 4$ and $\nu_k(E_i \setminus C) \leq \eta$ for each i.

proof (a) For each $n > m$, set

$$\tilde{E}_n = \bigcup\{E_i : i \le n,\ E_i \in \mathfrak{B}_n\},$$

so that $\tilde{E}_n$ is determined by coordinates in $n \setminus m$ and $\nu_k \tilde{E}_n < 2$. Set

$$U_n = \{p : p \ge k,\ \nu_{kp}\tilde{E}_n < 2\} \in \mathcal{U}.$$

For $p \in U_n$ we can find a finite $\mathcal{I}_{np} \subseteq \mathcal{C}_{kp}$ such that $\tilde{E}_n \subseteq \operatorname{spr}\mathcal{I}_{np}$ and $\operatorname{wt}\mathcal{I}_{np} \le 2$. For $r > m$ set

$$\begin{aligned}\mathcal{I}_{npr} = \{(E, I, w) : (E, I, w) \in \mathcal{I}_{np},\\ \#(I \cap (r-1) \setminus m) < \frac{1}{2}\#(I) \le \#(I \cap r \setminus m)\},\end{aligned}$$

and set

$$\mathcal{I}'_{np} = \{(E, I, w) : (E, I, w) \in \mathcal{I}_{np},\ \#(I \cap m) \ge \frac{1}{4}\#(I)\}.$$

Set $B_{np} = \theta_m(\operatorname{spr}\mathcal{I}'_{np})$; then

$$\nu_{kp}B_{np} \le 4^{\alpha_k} \operatorname{wt}\mathcal{I}'_{np} \le 4,$$

by Corollary 1Fa. Since $\nu_{kp}X \ge c_k \ge 8$ (Lemma 1G), $B_{np} \ne X$ and there is an $A_{np} \in \mathcal{A}_m$ disjoint from $\operatorname{spr}\mathcal{I}'_{np}$. Next, for $m < r \le n$ set

$$\mathcal{J}_{npr} = \{(\theta_{r \setminus m}(A_{np} \cap E), I \cap r \setminus m, 2w) : (E, I, w) \in \mathcal{I}_{npr}\}.$$

By Lemma 1E, $\mathcal{J}_{npr} \subseteq \mathcal{C}_{kp}$, and of course $\operatorname{wt}\mathcal{J}_{npr} \le 2\operatorname{wt}\mathcal{I}_{npr}$. Note that $\operatorname{spr}\mathcal{J}_{npr}$ is determined by coordinates in $r \setminus m$. Now if $m < j \le n$, $\tilde{E}_j \subseteq \bigcup_{m<r\le j}\operatorname{spr}\mathcal{J}_{npr}$. **P?** Otherwise, since both sets are determined by coordinates in $j \setminus m$, and since $A_{np} \in \mathcal{A}_m$, there is an $A \in \mathcal{A}_j$ with

$$A \subseteq A_{np} \cap \tilde{E}_j \setminus \bigcup_{m<r\le j}\operatorname{spr}\mathcal{J}_{npr}.$$

Since $\theta_{r\setminus m}(A_{np} \cap E) \supseteq E \cap A_{np}$ for every E, A must be disjoint from $\bigcup_{m<r\le j}\operatorname{spr}\mathcal{I}_{npr}$. Since A is also disjoint from $\operatorname{spr}\mathcal{I}'_{np}$ and $A \subseteq \tilde{E}_j \subseteq \operatorname{spr}\mathcal{I}_{np}$, $A \subseteq \operatorname{spr}\mathcal{I}$, where

$$\begin{aligned}\mathcal{I} &= \mathcal{I}_{np} \setminus (\mathcal{I}'_{np} \cup \bigcup_{m<r\le j}\mathcal{I}_{npr})\\ &\subseteq \{(E, I, w) : (E, I, w) \in \mathcal{I}_{np},\ \#(I \setminus j) \ge \frac{1}{4}\#(I)\}.\end{aligned}$$

Since $\mathcal{I} \subseteq \mathcal{C}_{kp}$,

$$8 \le \nu_{kp}X = \nu_{kp}(\theta_{\mathbb{N}\setminus j}(A \cap \operatorname{spr}\mathcal{I})) \le 4^{\alpha_k}\operatorname{wt}\mathcal{I}$$

(Corollary 1Fb)

$$\le 2\operatorname{wt}\mathcal{I}_{np} \le 4.\ \mathbf{XQ}$$

(b) For $r > m$ we can find $F_r \in \mathfrak{B}$ such that

$\sum_{r=m+1}^{\infty}\nu_k F_r \le 4$,

$\tilde{E}_j \subseteq \bigcup_{m<r\le j}F_r$ for every $j > m$,

F_r is determined by coordinates in $r \setminus m$.

P For $m < r \le n \in \mathbb{N}$ and $p \in U_n$ set

$$\mathcal{K}_{npr} = \{E : (E, I, w) \in \mathcal{J}_{npr}\} \subseteq \mathfrak{B}_r.$$

Because $\mathcal{P}\mathfrak{B}_r$ is finite, there is a family $\mathcal{K}_{nr}$ such that $\{p : p \in U_n,\ \mathcal{K}_{npr} = \mathcal{K}_{nr}\}$ belongs to $\mathcal{U}$. Next, for each $r > m$ there is a family $\mathcal{K}_r$ such that $\{n : n \in \mathbb{N},\ \mathcal{K}_{nr} = \mathcal{K}_r\}$ belongs to $\mathcal{U}$. Take $F_r = \bigcup\mathcal{K}_r$. Then

$$\begin{aligned}\nu_k F_r &\le \sum_{E\in\mathcal{K}_r}\nu_k E=\lim_{n\to\mathcal{U}}\sum_{E\in\mathcal{K}_{nr}}\nu_k E\\ &=\lim_{n\to\mathcal{U}}\sum_{E\in\mathcal{K}_{nr}}\lim_{p\to\mathcal{U}}\nu_{kp}E=\lim_{n\to\mathcal{U}}\lim_{p\to\mathcal{U}}\sum_{E\in\mathcal{K}_{nr}}\nu_{kp}E\\ &=\lim_{n\to\mathcal{U}}\lim_{p\to\mathcal{U}}\sum_{E\in\mathcal{K}_{nrp}}\nu_{kp}E\le\lim_{n\to\mathcal{U}}\lim_{p\to\mathcal{U}}\operatorname{wt}\mathcal{J}_{npr}\end{aligned}$$

(because for any $E\in\mathcal{K}_{nrp}$ there are I, w such that $(E,I,w)\in\mathcal{J}_{nrp}\subseteq\mathcal{C}_{kp}$, in which case $\nu_{kp}E\le w$)

$$\le 2\lim_{n\to\mathcal{U}}\lim_{p\to\mathcal{U}}\operatorname{wt}\mathcal{I}_{npr}.$$

So

$$\begin{aligned}\sum_{r=m+1}^{\infty}\nu_k F_r&=\sup_{s\ge m+1}\sum_{r=m+1}^{s}\nu_k F_r\le 2\sup_{s\ge m+1}\sum_{r=m+1}^{s}\lim_{n\to\mathcal{U}}\lim_{p\to\mathcal{U}}\operatorname{wt}\mathcal{I}_{npr}\\ &\le 2\sup_{s\ge m+1}\lim_{n\to\mathcal{U}}\lim_{p\to\mathcal{U}}\sum_{r=m+1}^{s}\operatorname{wt}\mathcal{I}_{npr}\le 2\lim_{n\to\mathcal{U}}\lim_{p\to\mathcal{U}}\operatorname{wt}\mathcal{I}_{np}\le 4.\end{aligned}$$

If $j>m$, $n>m$ and $p\in U_n$ set $\tilde{\mathcal{K}}_{npj}=\bigcup_{m<r\le j}\mathcal{K}_{npr}$, $\tilde{\mathcal{K}}_{nj}=\bigcup_{m<r\le j}\mathcal{K}_{nr}$ and $\tilde{\mathcal{K}}_j=\bigcup_{m<r\le j}\mathcal{K}_r$. Then, for any $j>m$,

$$\{p:p\in U_n,\,\tilde{\mathcal{K}}_{npj}=\tilde{\mathcal{K}}_{nj}\}\in\mathcal{U}\text{ for every }n>m,\quad\{n:n>m,\,\tilde{\mathcal{K}}_{nj}=\tilde{\mathcal{K}}_j\}\in\mathcal{U}.$$

If $x\in\tilde{E}_j$, then whenever $n\ge j$ and $p\in U_n$,

$$x\in\bigcup_{m<r\le j}\operatorname{spr}\mathcal{J}_{npr}=\bigcup_{m<r\le j}\bigcup\mathcal{K}_{npr}=\bigcup\tilde{\mathcal{K}}_{npj},$$

and there is an $E\in\tilde{\mathcal{K}}_{npj}$ containing x. As $U_n\in\mathcal{U}$, there is an $E\in\tilde{\mathcal{K}}_{nj}$ containing x. This is true for every $n\ge j$; as $\{n:n\ge j\}\in\mathcal{U}$, there is an $E\in\tilde{\mathcal{K}}_j$ such that $x\in E\subseteq\bigcup_{m<r\le j}F_r$. Thus $\tilde{E}_j\subseteq\bigcup_{m<r\le j}F_r$.

Finally, take any $r>m$. Referring back to the definition of $\mathcal{J}_{npr}$, we see that (because $A_{np}\in\mathcal{A}_m$) E is determined by coordinates in $r\setminus m$ whenever $(E,I,w)\in\mathcal{J}_{npr}$. So E is determined by coordinates in $r\setminus m$ whenever

$$E\in\bigcup_{n\in\mathbb{N},p\in U_n}\mathcal{K}_{npr}\supseteq\bigcup_{n\in\mathbb{N}}\mathcal{K}_{nr}\supseteq\mathcal{K}_r,$$

and F_r is also determined by coordinates in $r\setminus m$. **Q**

(c) Let $r_0\ge m$ be such that $\sum_{r=r_0+1}^{\infty}\nu_k F_r<\eta$. Set $C=\bigcup_{m<r\le r_0}F_r$. Then C is determined by coordinates in $\mathbb{N}\setminus m$ and

$$\nu_k C\le\sum_{r=m+1}^{r_0}\nu_k F_r\le 4.$$

For any $i\in\mathbb{N}$, there is some $j>m$ such that $E_i\subseteq\tilde{E}_j$, in which case

$$E_i\setminus C\subseteq\bigcup_{r_0<r\le j}F_r$$

and

$$\nu_k(E_i\setminus C)\le\sum_{r=r_0+1}^{j}\nu_k F_r\le\eta,$$

as required.

1L Lemma (T§6.6) Suppose that $k\in\mathbb{N}$, $\epsilon>0$, $m\in\mathbb{N}$, $B\in\mathfrak{B}_m$ and that $\langle E_i\rangle_{i\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{B}$. Then there are $n>m$ and $B'\in\mathfrak{B}_n$ such that $B'\subseteq B$, B' is $\frac{1}{2}\nu_k$-thin between m and n and $\limsup_{i\to\infty}\nu_k(E_i\cap B\setminus B')\le\epsilon$.

proof Set $\eta=\frac{\epsilon}{\#(\mathcal{A}_m)}$. For those $A\in\mathcal{A}_m$ included in B define $C'_A\subseteq A$ as follows.

case 1 If there is some r such that $\nu_k(\theta_{\mathbb{N}\setminus m}(A\cap\bigcup_{i\le r}E_i))\ge 2$, set $C'_A=A\setminus\bigcup_{i\le r}E_i$, so that $\frac12\nu_k(\theta_{\mathbb{N}\setminus m}(A\setminus C'_A))\ge 1$ and $E_i\cap A\setminus C'_A=\emptyset$ for $i>r$.

case 2 If $\nu_k(\theta_{\mathbb{N}\setminus m}(A\cap\bigcup_{i\le r}E_i))<2$ for every r, then by Lemma 1K, applied to the sequence $\langle\theta_{\mathbb{N}\setminus m}(A\cap E_i)\rangle_{i\in\mathbb{N}}$, we can find a $C\in\mathfrak{B}$, determined by coordinates in $\mathbb{N}\setminus m$, such that $\nu_kC\le 4$ and $\nu_k(\theta_{\mathbb{N}\setminus m}(A\cap E_i)\setminus C)\le\eta$ for every i. Set $C'_A=C\cap A$. Because C is determined by coordinates in $\mathbb{N}\setminus m$ and $A\in\mathcal{A}_m$, $\nu_k(\theta_{\mathbb{N}\setminus m}(C'_A))=\nu_kC\le 4$. Also $E_i\cap A\setminus C'_A\subseteq\theta_{\mathbb{N}\setminus m}(A\cap E_i)\setminus C$ so $\nu_k(E_i\cap A\setminus C'_A)\le\eta$ for every i.

Set

$$B'=\bigcup\{C'_A:A\in\mathcal{A}_m,\,A\subseteq B\}.$$

Then $B'\subseteq B$ and

$$\begin{aligned}\limsup_{i\to\infty}\nu_k(E_i\cap B\setminus B')&\le\sum_{A\in\mathcal{A}_m,A\subseteq B}\limsup_{i\to\infty}\nu_k(E_i\cap A\setminus C'_A)\\&\le\sum_{A\in\mathcal{A}_m,A\subseteq B}\eta\le\epsilon.\end{aligned}$$

Let $n>m$ be such that $C'_A\in\mathfrak{B}_n$ whenever $A\in\mathcal{A}_m$ and $A\subseteq B$. Then B' is $\frac12\nu_k$-thin between m and n. **P** Take any $A\in\mathcal{A}_m$ and set $\tilde C=\theta_{n\setminus m}(A\cap B')$. If $A\not\subseteq B$ then $A\cap B'$ and $\tilde C$ are empty and $\nu_k(X\setminus\tilde C)\ge 8$ (Proposition 1Ia). Otherwise, $A\cap B'=C'_A\in\mathfrak{B}_n$ so $\tilde C=\theta_{\mathbb{N}\setminus m}(A\cap C'_A)$ is disjoint from $\theta_{\mathbb{N}\setminus m}(A\setminus C'_A)$ (see the last remark in 1Db). If C'_A was chosen as in case 1 above,

$$\nu_k(X\setminus\tilde C)\ge\nu_k(\theta_{\mathbb{N}\setminus m}(A\setminus C'_A))\ge 2.$$

If C'_A was chosen as in case 2,

$$\nu_k(X\setminus\tilde C)=\nu_k(X\setminus\theta_{\mathbb{N}\setminus m}(C'_A))\ge\nu_kX-\nu_k(\theta_{\mathbb{N}\setminus m}(C'_A))\ge 8-4.$$

So in all three cases we have $\frac12\nu_k(X\setminus\tilde C)\ge 1$, as required. **Q**

Thus we have an appropriate B'.

1M Theorem (T§6.6) ν is exhaustive.

proof Let $\langle E_i\rangle_{i\in\mathbb{N}}$ be a disjoint sequence in $\mathfrak{B}$. Take any $k\in\mathbb{N}$ and $\epsilon>0$, and choose $\langle B_j\rangle_{j\in\mathbb{N}}$ and $\langle n_j\rangle_{j\in\mathbb{N}}$ inductively, as follows. $B_0=X$ and $n_0=0$. Given that $B_j\in\mathfrak{B}_{n_j}$, take $n_{j+1}>n_j$ and $B_{j+1}\in\mathfrak{B}_{n_{j+1}}$ such that $B_{j+1}\subseteq B_j$, B_{j+1} is $\frac12\nu_{k+1}$-thin between n_j and n_{j+1}, and $\limsup_{i\to\infty}\nu_{k+1}(E_i\cap B_j\setminus B_{j+1})\le\epsilon$ (Lemma 1L). Continue. Note that $\limsup_{i\to\infty}\nu_{k+1}(E_i\setminus B_j)\le j\epsilon$ for every j.

Set $I=\{n_j:j<N_k\}$ and $B=B_{N_k-1}$. Then B is $\frac12\nu_{k+1}$-thin along I (use Fact 1Dc). So $U=\{p:p\ge k+1,\,B$ is $\nu_{k+1,p}$-thin along $I\}$ belongs to $\mathcal{U}$, by Lemma 1J. But now $(B,I,2^{-k})\in\mathcal{E}_{kp}$ for every $p\in U$, so $\nu_{kp}B\le 2^{-k}$ for every $p\in U$ and $\nu_kB\le 2^{-k}$.

Of course $\nu\le\nu_k\le\nu_{k+1}$. So

$$\limsup_{i\to\infty}\nu E_i\le\nu_kB+\limsup_{i\to\infty}\nu_{k+1}(E_i\setminus B)\le 2^{-k}+(N_k-1)\epsilon.$$

As k, ϵ and $\langle E_i\rangle_{i\in\mathbb{N}}$ are arbitrary, ν is exhaustive.

1N Proposition Give X the metric σ defined by setting $\sigma(x,y)=2^{-n}$ if $x{\restriction}n=y{\restriction}n$ and $x(n)\ne y(n)$, and let G be the isometry group of X. Then ν is G-invariant.

proof I show by downwards induction on k that $(g[E],I,w)\in\mathcal{E}_{kp}$ whenever $k\le p$ in $\mathbb{N}$, $(E,I,w)\in\mathcal{E}_{kp}$ and $g\in G$. The case $k=p$ is trivial. For the downwards step to $k<p$, observe first that the inductive hypothesis will ensure that $g[E]\in\mathcal{C}_{k+1,p}$ whenever $g\in G$ and $E\in\mathcal{C}_{k+1,p}$. It follows that $\nu_{k+1,p}$ is G-invariant. **P** If $F\in\mathfrak{B}$, $\nu_{k+1,p}F<\eta$ and $g\in G$, then there is a finite set $\mathcal{I}\subseteq\mathcal{C}_{k+1,p}$ such that $F\subseteq\operatorname{spr}\mathcal{I}$ and $\operatorname{wt}\mathcal{I}<\eta$. Set $\mathcal{J}=\{(g[E],I,w):(E,I,w)\in\mathcal{I}\}$; then $\mathcal{J}\subseteq\mathcal{C}_{k+1,p}$, $g[F]\subseteq g[\operatorname{spr}\mathcal{I}]=\operatorname{spr}\mathcal{J}$ and $\operatorname{wt}\mathcal{J}=\operatorname{wt}\mathcal{I}<\eta$, so $\nu_{k+1,p}g[F]<\eta$. As η is arbitrary, $\nu_{k+1,p}g[F]\le\nu_{k+1,p}F$; applying the same argument to g^{-1}, $\nu_{k+1,p}g[F]=\nu_{k+1,p}F$; as F and g are arbitrary, $\nu_{k+1,p}$ is G-invariant. **Q**

Now suppose that $E\in\mathfrak{B}$ and $m<n$ are such that E is $\nu_{k+1,p}$-thin between m and n. Then $g[E]$ is $\nu_{k+1,p}$-thin between m and n. **P** Suppose that $A\in\mathcal{A}_m$. Set $A'=g^{-1}[A]$ and $F=A'\cap E$; then $A'\in\mathcal{A}_m$.

(If $z \in A$, then $A' = \{x : \sigma(x, g^{-1}(z)) < 2^{-m}\}$.) Define $\pi_A(x)$, for $x \in X$, by saying that $\pi_A(x) \in A$ and $\pi_A(x)(i) = x(i)$ for $i \geq m$ (T§3.3). Define $h : X \to X$ by setting

$$\begin{aligned} h(x)(i) &= x(i) \text{ if } i < m \\ &= g^{-1}(\pi_A(x))(i) \text{ if } i \geq m \end{aligned}$$

for every $x \in X$. Then $h \in G$, and $\pi_{A'}h = g^{-1}\pi_A$.

If $x \in X$, then

$$\begin{aligned} x \in \theta_{\mathbb{N}\setminus m}(A \cap g[E]) &\iff x \in \theta_{\mathbb{N}\setminus m}(g[F]) \\ &\iff \text{ there is a } y \in F \text{ such that } x{\restriction}\mathbb{N} \setminus m = g(y){\restriction}\mathbb{N} \setminus m \\ &\iff \text{ there is a } y \in F \text{ such that } \pi_A(x) = g(y) \end{aligned}$$

(because if $y \in F$ then $g(y) \in A$)

$$\begin{aligned} &\iff g^{-1}(\pi_A(x)) \in F \\ &\iff \pi_{A'}(h(x)) \in F \iff h(x) \in \theta_{\mathbb{N}\setminus m}(F). \end{aligned}$$

Thus

$$\begin{aligned} \theta_{n\setminus m}(A \cap g[E]) &= \theta_{n\setminus m}(\theta_{\mathbb{N}\setminus m}(A \cap g[E])) = \theta_{n\setminus m}(h^{-1}[\theta_{\mathbb{N}\setminus m}(A' \cap E)]) \\ &= h^{-1}[\theta_{n\setminus m}(\theta_{\mathbb{N}\setminus m}(A' \cap E))] \end{aligned}$$

(because $h \in G$)

$$= h^{-1}[\theta_{n\setminus m}(A' \cap E)]$$

and

$$\begin{aligned} \nu_{k+1,p}(X \setminus \theta_{n\setminus m}(A \cap g[E])) &= \nu_{k+1,p}h^{-1}[X \setminus \theta_{n\setminus m}(A' \cap E)] \\ &= \nu_{k+1,p}(X \setminus \theta_{n\setminus m}(A' \cap E)) \end{aligned}$$

(because $\nu_{k+1,p}$ is G-invariant)

$$\geq 1.$$

As A is arbitrary, $g[E]$ is $\nu_{k+1,p}$-thin between m and n. **Q**

It follows that if $(E, I, w) \in \mathcal{E}_{kp}$ then $g[E]$ is $\nu_{k+1,p}$-thin along I and $(g[E], I, w) \in \mathcal{E}_{kp}$. Thus the induction continues.

At the end of the induction, we have, for any $E \in \mathfrak{B}$ and $g \in G$,

$$\nu g[E] = \lim_{p\to\mathcal{U}} \nu_{0p}g[E] = \lim_{p\to\mathcal{U}} \nu_{0p}E = \nu E,$$

so ν also is G-invariant.

1O Corollary (a) If we give each T_n a group structure and X the product group structure, then ν is translation-invariant.

(b) ν is strictly positive.

proof (a) Translations will all belong to the isometry group G as described in Proposition 1N.

(b) If $E \in \mathfrak{B}$ is non-empty, then X can be covered by finitely many translates of E.

1P The Maharam algebra of ν Because ν is strictly positive, we have a metric ρ on $\mathfrak{B}$ defined by setting $\rho(E, F) = \nu(E \triangle F)$ for E, $F \in \mathfrak{B}$, and can form the metric completion $\widehat{\mathfrak{B}}$ of $\mathfrak{B}$; the continuous extension $\hat{\nu}$ of ν to $\widehat{\mathfrak{B}}$ is now a strictly positive Maharam submeasure on the Dedekind complete Boolean algebra $\widehat{\mathfrak{B}}$ (Fremlin 02, 393H[1]). Note that because G acts on $\mathfrak{B}$ as a subgroup of the isometry group of $\mathfrak{B}$, it acts on $\widehat{\mathfrak{B}}$ as a subgroup of the automorphism group of $(\widehat{\mathfrak{B}}, \hat{\nu})$.

2 The exhaustivity rank of ν

In Fremlin 08?, §539, I discuss the exhaustivity ranks of exhaustive submeasures. (See also Fremlin n06, §7.) For the example described in §1 above, we can compute an estimate of its exhaustivity rank. Of course the argument is just a refinement of the ideas in 1K-1M.

2A Well-founded partial orders (a) Recall that if $(P, \le)$ is a partially ordered set which is **well-founded**, that is, every non-empty subset of P has a minimal element, then P has a rank function rank : $P \to \mathrm{On}$ such that $\mathrm{rank}(p) = \sup\{\mathrm{rank}(q) + 1 : q < p\}$ for every $p \in P$. The **height** of P is now $\sup_{p\in P} \mathrm{rank}(p) + 1$.

(b) Conversely, if $(P, \le)$ is a partially ordered set and $r : P \to \mathrm{On}$ is such that $r(p) < r(q)$ whenever $p < q$, then P is well-founded and $\mathrm{rank}(p) \le r(p)$ for every $p \in P$.

(c) Recall that an ordinal ξ is **indecomposable** if it is not expressible as the ordinal sum $\eta + \zeta$ where η and ζ are both less than ξ. In this case, if I is a non-empty finite set and we give the Cartesian product ξ^I its product partial order, the height of ξ^I is ξ. **P** Define $r : \xi^I \to \xi$ as follows. Given that $\eta_i < \xi$ for each $i \in I$, express η_i in its Cantor normal form

$$\eta_i = \omega^{\beta_{i0}} + \omega^{\beta_{i1}} + \ldots$$

where $\langle\beta_{ij}\rangle_{j<m_i}$ is non-increasing (Jech 03, Theorem 2.26). Re-enumerate $\langle\beta_{ij}\rangle_{i\in I,j<m_i}$ as a non-increasing sequence $\langle\beta_j\rangle_{j<m}$ and set

$$r(\langle\eta_i\rangle_{i\in I}) = \omega^{\beta_0} + \omega^{\beta_1} + \ldots;$$

because ξ is indecomposable, this is less than ξ. Now if $\langle\eta_i\rangle_{i\in I}$, $\langle\eta'_i\rangle_{i\in I} \in \xi^I$, $\eta_i \le \eta'_i$ for every i, and there is some i such that $\eta_i < \eta'_i$, we shall have $r(\langle\eta_i\rangle_{i\in I}) < r(\langle\eta'_i\rangle_{i\in I})$. By (b), ξ^I is well-founded, with height at most ξ. Because I is non-empty, the height is exactly ξ. **Q**

(d) If $(P, \le)$ is a partially ordered set, $\langle P_i\rangle_{i\in I}$ is a finite cover of P, and every P_i (with the induced partial order) is well-founded, then P is well-founded; if $\xi \ge \omega$ is an indecomposable ordinal and every P_i has height at most ξ, then P has height at most ξ. **P** For each i, we have a rank function $\mathrm{rank}_i : P_i \to \xi$. Define $f : P \to \xi^I$ by setting $f(p)(i) = \sup\{\mathrm{rank}_i(q) + 1 : q \in P_i,\ q < p\}$ for $p \in P$ and $i \in I$. If $p < q$ in P, then $f(p) < f(q)$ in ξ^I (if $i \in I$ and $p \in P_i$, then $f(p)(i) = \mathrm{rank}_i(p) < f(q)(i)$). Since we have a rank function from ξ^I to ξ, by (c), we can compose this with f to give a function $r : P \to \xi$ and apply (b). **Q**

(e) Note that the ordinal power ω^β is indecomposable for every ordinal β.

(f) If ξ, η are ordinals, then the ordinal product $\xi \cdot \eta$ is the order type of the lexicographic ordering on $\eta \times \xi$ (Jech 03, 2.24). So if $n \in \mathbb{N}$, the order type of the lexicographic ordering on the Cartesian product ξ^n is the ordinal power ξ^n.

2B Lemma (Compare 1K.) Suppose that m, $k \in \mathbb{N}$. Let $\mathcal{V}$ be the set

$$\mathcal{V} = \{V : V \in \mathfrak{B} \text{ is determined by coordinates in } \mathbb{N} \setminus m,\ \nu_k V < 2\}.$$

Then for each $V \in \mathcal{V}$ there is a sequence $\langle F_r\rangle_{r\in\mathbb{N}}$ such that

$F_r \in \mathfrak{B}$ is determined by coordinates in $r \setminus m$ for every $r \in \mathbb{N}$,

$H \subseteq \bigcup_{j\le r} F_j$ whenever $r \in \mathbb{N}$, $H \in \mathfrak{B}_r$ and $H \subseteq V$,

$\sum_{r=0}^{\infty} \nu_k F_r \le 4$.

proof (a)(i) Set $U = \{p : p \in \mathbb{N},\ \nu_{kp} V < 2\} \in \mathcal{F}$. For $p \in U$, choose a finite set $\mathcal{I}_p \subseteq \mathcal{C}_{kp}$ such that $V \subseteq \mathrm{spr}\,\mathcal{I}_p$ and $\mathrm{wt}\,\mathcal{I}_p \le 2$; set

$$\mathcal{I}'_p = \{(E, I, w) : (E, I, w) \in \mathcal{I}_p,\ \#(I \cap m) \ge \tfrac{1}{4}\#(I)\}, \quad B_p = \theta_m(\mathrm{spr}\,\mathcal{I}'_p).$$

[1]Formerly 393B.

By 1Fa, $\nu_{kp}B_p \le 2\operatorname{wt}\mathcal{I}'_p \le 4$. Since $\nu_{kp}X \ge 8$, $B_p \ne X$ and there is an $A_p \in \mathcal{A}_m$ disjoint from $\operatorname{spr}\mathcal{I}'_p$. Next, for $p \in U$ and $r > m$, set

$$\mathcal{I}_{pr} = \{(E, I, w) : (E, I, w) \in \mathcal{I}_p, \#(I \cap (r-1)) < \frac{1}{2}\#(I) \le \#(I \cap r)\},$$

$$\mathcal{J}_{pr} = \{(\theta_{r\setminus m}(A_p \cap E), I \cap r, 2w) : (E, I, w) \in \mathcal{I}_{pr}\}, \quad F_{pr} = \operatorname{spr}\mathcal{J}_{pr}.$$

For completeness, set $F_{pr} = \mathcal{J}_{pr} = \mathcal{I}_{pr} = \emptyset$ if $r \le m$. By 1E, $\mathcal{J}_{pr} \subseteq \mathcal{C}_{kp}$, and of course

$$\nu_{kp}F_{pr} \le \operatorname{wt}\mathcal{J}_{pr} \le 2\operatorname{wt}\mathcal{I}_{pr},$$

so

$$\textstyle\sum_{r=0}^{\infty} \nu_{kp}F_{pr} \le 2\sum_{r=0}^{\infty} \operatorname{wt}\mathcal{I}_{pr} \le 2\operatorname{wt}\mathcal{I}_p \le 4.$$

Note that each F_{pr} is determined by coordinates in $r \setminus m$ and includes $A_p \cap \operatorname{spr}\mathcal{I}_{pr}$.

(ii) If $H \in \mathfrak{B}_r$ and $H \subseteq V$, then $H \subseteq \bigcup_{j\le r} F_{pj}$. **P?** Otherwise, since V is determined by coordinates in $\mathbb{N}\setminus m$ and is certainly not equal to X, $r > m$. Since V is determined by coordinates in $\mathbb{N}\setminus m$, $H' = \theta_{\mathbb{N}\setminus m}H$ is included in V. Since both H' and $\bigcup_{j\le r} F_{pj}$ are determined by coordinates in $r \setminus m$, there is an $A \in \mathcal{A}_r$ such that $A \subseteq A_p$ and $A \subseteq H' \setminus \bigcup_{j\le r} F_{pj}$. Since A is also disjoint from $\operatorname{spr}\mathcal{I}'_p$ and $A \subseteq \operatorname{spr}\mathcal{I}_p$, $A \subseteq \operatorname{spr}\mathcal{I}$, where

$$\mathcal{I} = \mathcal{I}_p \setminus (\mathcal{I}'_p \cup \textstyle\bigcup_{j\le r} \mathcal{I}_{pr}) \subseteq \{(E, I, w) : (E, I, w) \in \mathcal{I}_p,\ \#(I \setminus r) \ge \frac{1}{4}\#(I)\}.$$

Since $\mathcal{I} \subseteq \mathcal{C}_{kp}$,

$$8 \le \nu_{kp}X = \nu_{kp}(\theta_{\mathbb{N}\setminus r}A) = \nu_{kp}(\theta_{\mathbb{N}\setminus r}(A \cap \operatorname{spr}\mathcal{I})) \le 2\operatorname{wt}\mathcal{I}$$

(Corollary 1Fb)

$$\le 4. \ \mathbf{XQ}$$

(b) Because every $\mathfrak{B}_r$ is finite, there is for each $r \in \mathbb{N}$ an F_r such that $\{p : p \in U,\ F_{pr} = F_r\} \in \mathcal{F}$; of course F_r is determined by coordinates in $r \setminus m$. Now $\sum_{r=0}^{\infty} \nu_k F_r \le 4$. If $r \in \mathbb{N}$, $H \in \mathfrak{B}_r$ and $H \subseteq V$, then there is a $p \in U$ such that $F_{pj} = F_j$ for every $j \le r$, and now $H \subseteq \bigcup_{j\le r} F_{pj} = \bigcup_{j\le r} F_j$.

2C Lemma Let m, $k \in \mathbb{N}$, $\eta > 0$. Set

$$\mathfrak{B}^{(m)} = \{E : E \in \mathfrak{B} \text{ is determined by coordinates in } \mathbb{N} \setminus m\},$$

$$\mathfrak{B}^{(m)}_r = \mathfrak{B}^{(m)} \cap \mathfrak{B}_r = \{E : E \in \mathfrak{B} \text{ is determined by coordinates in } r \setminus m\}$$

for every $r \in \mathbb{N}$,

$$\mathcal{V} = \{E : E \in \mathfrak{B}^{(m)},\ \nu_k(X \setminus E) < 2\}.$$

(a) For $E \in \mathcal{V}$ use Lemma 2B to choose a sequence $\langle F_r(E)\rangle_{r\in\mathbb{N}}$ such that

$F_r(E) \in \mathfrak{B}^{(m)}_r$ for every $r \in \mathbb{N}$,

$H \subseteq \bigcup_{j\le r} F_j(E)$ whenever $r \in \mathbb{N}$, $H \in \mathfrak{B}^{(m)}_r$ and $H \subseteq X \setminus E$,

$\sum_{r=0}^{\infty} \nu_k F_r(E) \le 4$;

set $r^*(E) = \min\{r : E \in \mathfrak{B}^{(m)}_r\}$ and $F(E) = \bigcup_{r\le r^*(E)} F_r(E)$. For $E \in \mathfrak{B}^{(m)} \setminus \mathcal{V}$, set $F(E) = E$. Then $F(E) \in \mathfrak{B}^{(m)}$ and $\nu_k(X \setminus F(E)) \ge 2$ for every E.

(b) For E, $E' \in \mathfrak{B}^{(m)}$ say that $E' \preccurlyeq E$ if either $E' = E$ or $E' \subset E$ and

— *either* $E \in \mathcal{V}$ and $E' \notin \mathcal{V}$

— *or* $E' \in \mathcal{V}$ and $\nu_k(X \setminus (E' \cup F(E))) \ge \eta$.

Then $\preccurlyeq$ is a partial ordering on $\mathfrak{B}^{(m)}$. If E, E', $E'' \in \mathfrak{B}^{(m)}$ and $E'' \subseteq E' \prec E$, then $E'' \prec E$.

(c) $\preccurlyeq$ is well-founded, with height at most the ordinal power ω^{n+1}, where $n = \lfloor 4/\eta \rfloor$.

proof (a) For every $E\in\mathcal{V}$, $F_r(E)\in\mathfrak{B}^{(m)}$ for every r, so $F(E)\in\mathfrak{B}^{(m)}$; if $E\in\mathfrak{B}^{(m)}\setminus\mathcal{V}$, $F(E)=E\in\mathfrak{B}^{(m)}$. If $E\in\mathcal{V}$ then $\nu_kF(E)\le 4$ and $\nu_k(X\setminus F(E))\ge 8-4\ge 2$; if $E\in\mathfrak{B}^{(m)}\setminus\mathcal{V}$ then $\nu_k(X\setminus F(E))=\nu_k(X\setminus E)\ge 2$.

(b) If E, $E'\in\mathfrak{B}^{(m)}$, $E'\subseteq E$ and $E'\in\mathcal{V}$, then of course $E\in\mathcal{V}$. If E, E', $E''\in\mathcal{V}$, $E'\prec E$ and $E''\subseteq E'$, then

$$\nu_k(X\setminus(E''\cup F(E)))\ge\nu_k(X\setminus(E'\cup F(E)))\ge\eta$$

and $E''\prec E$; accordingly $\preccurlyeq$ is a partial order, with every element of $\mathfrak{B}^{(m)}\setminus\mathcal{V}$ below every element of $\mathcal{V}$.

If E, E', $E''\in\mathfrak{B}^{(m)}$ and $E''\subseteq E'\prec E$, then $E''\subset E\in\mathcal{V}$ and

— if $E''\in\mathcal{V}$, then $E''\prec E$, as noted above;

— if $E''\notin\mathcal{V}$ and $E\in\mathcal{V}$, then $E''\prec E$.

(c)(i) Consider first the upper segment $\mathcal{V}$. It is helpful to note immediately that $X\setminus E\subseteq F(E)$ for every $E\in\mathcal{V}$; so that we can suppose that $F_r(E)=\emptyset$ for $r>r^*(E)$. Now the key fact is this: if E, E' and E'' belong to $\mathcal{V}$, $E''\preccurlyeq E'\prec E$, $l\in\mathbb{N}$ and $F_r(E'')\subseteq F(E)$ for every $r\le l$, then $\sum_{l<r\le r^*(E')}\nu_kF_r(E'')\ge\eta$. **P** $X\setminus E'\in\mathfrak{B}^{(m)}_{r^*(E')}$ and $X\setminus E'\subseteq X\setminus E''$, so

$$X\setminus E'\subseteq\bigcup\nolimits_{r\le r^*(E')}F_r(E''),$$

$$(X\setminus(E'\cup F(E)))\subseteq(X\setminus E')\setminus\bigcup\nolimits_{r\le l}F_r(E'')\subseteq\bigcup\nolimits_{l<r\le r^*(E')}F_r(E'')$$

and

$$\sum\nolimits_{l<r\le r^*(E')}\nu_kF_r(E'')\ge\nu_k(X\setminus(E'\cup F(E)))\ge\eta.\ \mathbf{Q}$$

(ii) For $E\in\mathcal{V}$, set $s_E=\max\{j:j\eta\le\sum_{r=0}^{r^*(E)}\nu_kF_r(E)\}$, and for $j\le s_E$ let $r_j(E)$ be minimal subject to $j\eta\le\sum_{r=0}^{r_j(E)}\nu_kF_r(E)$; set $\tilde F_j(E)=\bigcup_{r\le r_j(E)}F_r(E)$. Now the remark in (a) becomes: if E, E' and E'' belong to $\mathcal{V}$, $E''\preccurlyeq E'\prec E$, $j\le\min(s_E,s_{E''})$ and $\tilde F_j(E'')=\tilde F_j(E)$, then $s_{E''}\ge j+1$ and $\tilde F_{j+1}(E'')\in\mathfrak{B}^{(m)}_{r^*(E')}$. **P** $\bigcup_{r\le r_j(E'')}F_r(E'')\subseteq F(E)$, so $\sum_{r_j(E'')<r\le r^*(E')}\nu_kF_r(E'')\ge\eta$,

$$\sum\nolimits_{r\le r^*(E')}\nu_kF_r(E'')\ge\eta+\sum\nolimits_{r\le r_j(E'')}\nu_kF_r(E'')\ge\eta+j\eta=(j+1)\eta$$

and $r_{j+1}(E'')\le r^*(E')$. Accordingly

$$\tilde F_{j+1}(E'')\in\mathfrak{B}^{(m)}_{r_{j+1}(E'')}\subseteq\mathfrak{B}^{(m)}_{r^*(E')}.\ \mathbf{Q}$$

(iii) For $\sigma\in\bigcup_{r\le n+1}(\mathfrak{B}^{(m)})^r$, write $\mathcal{V}_\sigma$ for

$$\{E:E\in\mathcal{V},\ s_E\ge\#(\sigma)-1,\ \tilde F_j(E)=\sigma(j)\text{ for }j<\#(\sigma)\}.$$

I show by downwards induction on $\#(\sigma)$ that V_σ is well-founded under $\preccurlyeq$ and has height at most the ordinal power $\omega^{n-\#(\sigma)+1}$. **P** To start the induction, note that $s_E\le n$ for every $E\in\mathcal{V}$, because $\sum_{r=0}^\infty\nu_kF_r(E)\le 4$. If $\sigma\in(\mathfrak{B}^{(m)})^{n+1}$ and $E\in V_\sigma$, then (c) tells us that E must be minimal in V_σ, so the height of V_σ is at most $1=\omega^0$. For the inductive step down to $\#(\sigma)=r\le n$, take $\sigma\in(\mathfrak{B}^{(m)})^r$ and E, $E'\in V_\sigma$ such that $E'\prec E$. If E'' also belongs to V_σ and $E''\preccurlyeq E'$, then (c) tells us that $s_{E'}\ge r$ and $\tilde F_r(E'')\in\mathfrak{B}^{(m)}_{r^*(E')}$. So

$$\{E'':E''\in V_\sigma,\ E''\preccurlyeq E'\}\subseteq\bigcup\nolimits_{G\in\mathfrak{B}^{(m)}_{r^*(E')}}V_{\sigma^\frown<G>}$$

is a union of finitely many sets, on each of which $\preccurlyeq$ is well-founded with height at most ω^{n-r}, and therefore itself is well-founded with height at most $\max(\omega,\omega^{n-r})$, by 2Ad. Consequently V_σ is also well-founded, with height at most $\max(\omega,\omega^{n-r})+1<\omega^{n-r+1}$. **Q**

(iv) In particular $\mathcal{V}=\mathcal{V}_\emptyset$ is well-founded, with height at most ω^{n+1}. Consequently $\mathfrak{B}^{(m)}$ is well-founded, with height at most the ordinal sum $1+\omega^{n+1}=\omega^{n+1}$.

2D Construction Fix $k\ge 1$. Set $N=N_{k-1}-1$. For $m\in\mathbb{N}$ set $\eta_m=\dfrac{2^{-k}}{N\#(\mathcal{A}_m)}$ and let $\mathcal{V}_m$ be $\{E:E\in\mathfrak{B}^{(m)},\ \nu_k(X\setminus E)<2\}$, $F_m:\mathfrak{B}^{(m)}\to\mathfrak{B}^{(m)}$ the function, and $\preccurlyeq_m$ the well-founded partial order on $\mathfrak{B}^{(m)}$ defined from k, m and η_m by the processes of Lemma 2C. Set

$$T = \bigcup_{n\in\mathbb{N}} \{\tau : \tau \in \mathfrak{B}^n,\ \nu_{k-1}\tau(i) \ge 2^{-k+2} \text{ for every } i < n,$$
$$\tau(i) \cap \tau(j) = \emptyset \text{ whenever } i < j < n\}.$$

For $j \le N$, $\tau \in T$ define $E_j(\tau)$, $m_j(\tau)$, $B_j(\tau)$ inductively, as follows. If $\tau = \emptyset$ then $E_j(\emptyset) = X$ for every j. Given $\langle E_j(\tau)\rangle_{j\le N}$, set $B_0(\tau) = X$, $m_0(\tau) = 0$. If $j < N_k$ and $m_j(\tau)$, $B_j(\tau)$ have been defined, set

$$B_{j+1}(\tau) = \bigcup_{A\in\mathcal{A}_{m_j(\tau)}, A\subseteq B_j(\tau)} A \cap F_{m_j(\tau)}(\theta_{\mathbb{N}\setminus m_j(\tau)}(A \cap E_j(\tau))),$$

and take $m_{j+1}(\tau)$ minimal such that $m_{j+1}(\tau) > m_j(\tau)$ and $B_{j+1}(\tau) \in \mathfrak{B}_{m_{j+1}(\tau)}$. Observe that $B_{j+1}(\tau) \subseteq B_j(\tau)$.

This ensures that $B_{j+1}(\tau)$ is $\frac{1}{2}\nu_k$-thin between $m_j(\tau)$ and $m_{j+1}(\tau)$. **P** If $A \in \mathcal{A}_{m_j}(\tau)$, then $A \cap B_{j+1}(\tau)$ is either empty or $A \cap F_{m_j(\tau)}(\theta_{\mathbb{N}\setminus m_j(\tau)}(A \cap E_j(\tau)))$. In the latter case, setting $m = m_j(\tau)$, $n = m_{j+1}(\tau)$, $E = \theta_{\mathbb{N}\setminus m}(A \cap E_j(\tau))$,

$$\theta_{n\setminus m}(A \cap F_m(E)) = \theta_{\mathbb{N}\setminus m}(A \cap F_m(E)) = F_m(E)$$

because $A \in \mathcal{A}_m$ and $F_m(E) \in \mathcal{B}^{(m)}$. So

$$\nu_k(X \setminus \theta_{n\setminus m}(A \cap B_{j+1}(\tau))) = \nu_k(X \setminus F_m(E) \ge 2. \textbf{ Q}$$

Consequently, setting $I(\tau) = \{m_j(\tau) : j < N\}$, $B_N(\tau)$ is $\frac{1}{2}\nu_k$-thin along I and $\nu_{k-1}B_N(\tau) \le 2^{-k+1}$, by 1J.

Now for the rule defining $E_j(\tau)$ for $\#(\tau) \ge 1$. Induce on $\#(\tau)$. The inductive hypothesis will include the requirement

$$E_j(\tau) \supseteq X \setminus \bigcup_{i<\#(\tau)} \tau(i) \text{ for every } j \le N.$$

Suppose that $\tau' = \tau^\frown\!<\!H\!>$ where $\langle E_j(\tau)\rangle_{j\le N}$ has been defined and $\nu_{k-1}H \ge 2^{-k+2}$. Since $\langle B_j(\tau)\rangle_{j\le N}$ is non-increasing, $B_0(\tau) = X$ and $\nu_{k-1}B_N(\tau) \le 2^{-k+1}$, there must be an $l < N$ such that $\nu_{k-1}(H \cap B_l(\tau) \setminus B_{l+1}(\tau)) \ge \dfrac{2^{-k}}{N}$, and an $\hat{A} \in \mathcal{A}_{m_l(\tau)}$ such that $\hat{A} \subseteq B_l(\tau)$ and

$$\nu_{k-1}(\hat{A} \cap H \setminus B_{l+1}(\tau)) \ge \eta_{m_l(\tau)}.$$

Set $E_l(\tau') = E_l(\tau) \setminus (\hat{A} \cap H)$; for $j \ne l$, set $E_j(\tau') = E_j(\tau)$. Then we certainly have

$$E_j(\tau') \supseteq X \setminus \bigcup_{i<\#(\tau')} \tau'(i)$$

for every $j \le N$, and we can continue the induction.

2E Lemma If $k \ge 1$ and T is the tree described in 2D, there is a function rank from T to the ordinal power $\omega^{\omega\cdot N}$ such that $\mathrm{rank}(\tau) < \mathrm{rank}(\sigma)$ whenever σ, $\tau \in T$ and τ properly extends σ.

proof (a)(i) Let $\langle \mathcal{V}_m\rangle_{m\in\mathbb{N}}$, $\langle \eta_m\rangle_{m\in\mathbb{N}}$, $\langle F_m\rangle_{m\in\mathbb{N}}$, $\langle \preccurlyeq_m\rangle_{m\in\mathbb{N}}$ N, $\langle E_j(\tau)\rangle_{\tau\in T, j\le N}$, $\langle B_j(\tau)\rangle_{\tau\in T, j\le N}$ and $\langle m_j(\tau)\rangle_{\tau\in T, j\le N}$ be as in 2C-2D. By Lemma 2C there is for each $m \in \mathbb{N}$ a rank function $\mathrm{rank}_m : \mathfrak{B}^{(m)} \to \omega^\omega$ such that $\mathrm{rank}_m(E') < \mathrm{rank}_m(E)$ whenever E, $E' \in \mathfrak{B}^{(m)}$ and $E' \prec_m E$.

(ii) For each $m \in \mathbb{N}$, set $Q_m = (\mathfrak{B}^{(m)})^{\mathcal{A}_m}$ with the product partial ordering; then Q_m is well-founded with height at most ω^ω, by 2Ac.

Accordingly, there is a function $\mathrm{rank}'_m : \mathfrak{B} \to \omega^\omega$ such that $\mathrm{rank}'_m(E') < \mathrm{rank}_m(E)$ whenever $\theta_{\mathbb{N}\setminus m}(A \cap E') \preccurlyeq_m \theta_{\mathbb{N}\setminus m}(A \cap E)$ for every $A \in \mathcal{A}_m$ and there is some $A \in \mathcal{A}_m$ such that $\theta_{\mathbb{N}\setminus m}(A \cap E') \ne \theta_{\mathbb{N}\setminus m}(A \cap E)$.

(b)(i) Examining the construction in 2D, we see that whenever $\tau \in T$ and $\tau' = \tau^\frown\!<\!H\!>$ is a one-term extension of τ, then there is exactly one $l < N$ such that $E_l(\tau') \ne E_l(\tau)$. As $E_j(\tau') = E_j(\tau)$ for $j < l$, $B_j(\tau') = B_j(\tau)$ and $m_j(\tau') = m_j(\tau)$ for $j \le l$. Set $m = m_l(\tau) = m_l(\tau')$. Then there is an $\hat{A} \in \mathcal{A}_m$ such that $\hat{A} \subseteq B_l(\tau)$ and $E_l(\tau') = E_l(\tau) \setminus (\hat{A} \cap H)$.

(ii) Now

$$\theta_{\mathbb{N}\setminus m}(\hat{A} \cap E_l(\tau')) \prec_m \theta_{\mathbb{N}\setminus m}(\hat{A} \cap E_l(\tau)).$$

P Set $E=\theta_{\mathbb{N}\setminus m}(\hat A\cap E_l(\tau))$,

$$E'=\theta_{\mathbb{N}\setminus m}(\hat A\cap E_l(\tau'))=\theta_{\mathbb{N}\setminus m}(\hat A\cap E_l(\tau)\setminus H),$$

so that both E and E' belong to $\mathfrak{B}^{(m)}$ and $E'\subseteq E$. **?** If $E\notin\mathcal{V}_m$, $F_m(E)=E$. In this case,

$$\hat A\cap B_{l+1}(\tau)=\hat A\cap F_m(E)=\hat A\cap E=\hat A\cap E_l(\tau).$$

But

$$E_l(\tau)\supseteq X\setminus\bigcup_{i<\#(\tau)}\tau(i)\supseteq H,$$

so $\hat A\cap B_{l+1}(\tau)\supseteq\hat A\cap H$, which is impossible, since $\nu_{k-1}(\hat A\cap H\setminus B_{l+1}(\tau))>0$. **X**

Thus $E\in\mathcal{V}_m$. If $E'\notin\mathcal{V}_m$, then certainly $E'\prec_m E$. Otherwise, $\hat A\cap H$ is disjoint from $\hat A\cap E_l(\tau')$, so is also disjoint from E'. Accordingly $\hat A\cap H\setminus F_m(E)$ is disjoint from both E' and $F_m(E)$, and

$$\begin{aligned}\nu_k(X\setminus(E'\cup F(E)))&\ge\nu_k(\hat A\cap H\setminus F_m(E))=\nu_k(\hat A\cap H\setminus B_{l+1}(\tau))\\&\ge\nu_{k-1}(\hat A\cap H\setminus B_{l+1}(\tau))\ge\eta_m,\end{aligned}$$

so that $E'\prec_m E$. **Q**

(iii) It follows that, in the terminology of (a-ii), $\mathrm{rank}'_m(E_l(\tau'))<\mathrm{rank}'_m(E_l(\tau))$.

(c) For each $\tau\in T$, we have a corresponding family $\langle\mathrm{rank}'_{m_j(\tau)}(E_j(\tau))\rangle_{j<N}$ in ω^ω; and if τ' is a one-term extension of τ, then there is an $l<N$ such that $E_j(\tau')=E_j(\tau)$ for $j<l$ and $\mathrm{rank}'_{m_l(\tau')}(E_j(\tau'))<\mathrm{rank}'_{m_l(\tau)}(E_j(\tau))$. We therefore have a function rank from T to the ordinal power $(\omega^\omega)^N=\omega^{\omega\cdot N}$ such that $\mathrm{rank}(\tau')<\mathrm{rank}(\tau)$ whenever τ' is a one-term extension of τ (2Af).

2F Theorem The exhaustivity rank of ν is at least the ordinal power ω^ω and at most ω^{ω^2}.

proof By Fremlin 08?, 539T, the exhaustivity rank of ν is at least ω^ω. In the other direction, given $\epsilon>0$, let $k\ge1$ be such that $2^{-k+2}\le\epsilon$, and let $T\subseteq\bigcup_{n\in\mathbb{N}}\mathfrak{B}^n$ be the corresponding tree as described in 2D. By Lemma 2E, we have a function $\mathrm{rank}:T\to\omega^{\omega\cdot N}$, where $N=N_{k-1}-1$, such that $\mathrm{rank}(\tau)<\mathrm{rank}(\sigma)$ whenever τ properly extends σ. For $E\in\mathfrak{B}$, set

$$r(E)=\min\{\mathrm{rank}(\tau):\tau\in T,\ \tau(i)\cap E=\emptyset\text{ whenever }i<\#(\tau)\}.$$

Then $r(E')<r(E)$ whenever $E'\subseteq E$ and $\nu(E\setminus E')\ge\epsilon$, because then $\tau^\frown{<}E\setminus E'{>}\in T$ whenever $\tau\in T$ and $\bigcup_{i<\#(\tau)}\tau(i)$ is disjoint from E. Consequently, defining r_ϵ as in Fremlin 08?, 539Q, $r_\epsilon(X)\le r(X)<\omega^{\omega\cdot N}$.

As this is true for every $\epsilon>0$, the exhaustivity rank $\sup_{\epsilon>0}r_\epsilon(X)$ of ν is at most ω^{ω^2}.

3 Problems

3A Let $\widehat{\mathfrak{B}}$ be the metric completion of $\mathfrak{B}$ under the metric defined by ν (Remark 1P).

(a) Does $\widehat{\mathfrak{B}}$ have an order-closed subalgebra isomorphic to the measure algebra of Lebesgue measure? In particular, if we take $z\in X$ and $\mathfrak{C}\subseteq\mathfrak{B}$ to be the algebra of sets generated by sets of the form $\{x:x\in X,\ x(n)=z(n)\}$ for $n\in\mathbb{N}$, is $\nu\restriction\mathfrak{C}$ uniformly exhaustive? (Suggested by M.Talagrand.)

(b) Does every $\hat\nu$-preserving automorphism of $\widehat{\mathfrak{B}}$ correspond to the action of an element of the group G of Proposition 1N?

(c) Is the automorphism group $\mathrm{Aut}_{\hat\nu}(\widehat{\mathfrak{B}})$ ergodic, in the sense that for any non-zero elements a, b of $\widehat{\mathfrak{B}}$ there is a $\hat\nu$-preserving automorphism π of $\widehat{\mathfrak{B}}$ such that $a\cap\pi b\ne0$?

(d) Is $\widehat{\mathfrak{B}}$ a homogeneous Boolean algebra?

(e) Does $\widehat{\mathfrak{B}}$ have an atomless rigid order-closed subalgebra?

(vi) Locate $\widehat{\mathfrak{B}}\setminus\{1\}$ in the Tukey classification (Fremlin 08?, chap. 52).

(vii) Find $\mathrm{wdistr}(\widehat{\mathfrak{B}})$, $\pi(\widehat{\mathfrak{B}})$ and $d(\widehat{\mathfrak{B}})$ in terms of the cardinals of Cichoń's diagram (Fremlin n06, §9).

3B In Definition 1H, do we in fact have $\lim_{p\to\infty}\nu_{kp}E$ defined for every $E\in\mathfrak{B}$ and $k\in\mathbb{N}$, so that no kind of selection procedure is needed at this point?

3C Suppose that instead of taking large sets T_n, we simply set $T_n=\{0,1\}$ for every n, but otherwise used the same construction. Should we then find that ν was uniformly exhaustive? (This might be relevant to 3A(a) above.)

3D Let μ be the Radon submeasure extending ν (Fremlin 03, 496F[2], or Fremlin n06, §10). Does μ have a lifting? (Fremlin n06, Problem 11N; Fremlin 08?, 539Zb.)

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[2]Later editions only.