Version of 21.9.05

Cardinal functions of density topologies

M.R.Burke & D.H.Fremlin

University of Prince Edward Island; University of Essex, Colchester, England

1 Proposition Let (X, Σ, μ) be a complete locally determined measure space with $\mu X > 0$, $\mathcal{N}(\mu)$ the null ideal of μ and $\mathfrak{A}$ its measure algebra. Let $\underline{\phi} : \Sigma \to \Sigma$ be a lower density for μ such that $\underline{\phi}X = X$, and $\mathfrak{T}$ the associated density topology (Fremlin 03, 414P). Then

(a) the character $\chi(X)$ of X is at most $\max(\mathrm{cf}\,\mathcal{N}(\mu), \#(\mathfrak{A}))$;
(b) the weight $w(X)$ of X is at most $\max(\#(X), \mathrm{cf}\,\mathcal{N}(\mu), \#(\mathfrak{A}))$;
(c) the π-weight $\pi(X)$ of X is $\mathrm{ci}(\Sigma \setminus \mathcal{N}(\mu))$;
(d) (Tall 76) the density $d(X)$ of X is $\min\{\#(A) : A \subseteq X$ has full outer measure$\}$;
(e) (Tall 76) writing $t(X)$ for the tightness of X, $\mathrm{shr}\,\mathcal{N}(\mu) \le t(X) \le \max(\mathrm{shr}\,\mathcal{N}(\mu), c(\mathfrak{A}))$;
(f) The cellularity $c(X)$ of X is $c(\mathfrak{A})$.

proof (a) If both $\mathrm{cf}\,\mathcal{N}(\mu))$ and $\mathfrak{A}$ are finite, then there is a largest negligible set and $\chi(X) = 1 = \mathrm{cf}\,\mathcal{N}(\mu)$. So we may assume that $\max(\mathrm{cf}\,\mathcal{N}(\mu)), \#(\mathfrak{A})) \ge \omega$. For E, $F \in \Sigma$, $\underline{\phi}E = \underline{\phi}F$ iff $E^\bullet = F^\bullet$, so $\#(\underline{\phi}[\Sigma]) = \#(\mathfrak{A})$. Let $\mathcal{E}$ be a cofinal subset of $\mathcal{N}(\mu)$ of cardinal $\mathrm{cf}\,\mathcal{N}(\mu)$. Take $x \in X$. (i) If $\{x\}$ is negligible,

$$\mathcal{V}_x = \{\{x\} \cup (\underline{\phi}F \setminus E) : F \in \Sigma,\ x \in \underline{\phi}F,\ E \in \mathcal{E}\}$$

is a base of neighbourhoods of x. **P** If $F \in \Sigma$, $x \in \underline{\phi}F$, $E \in \mathcal{E}$ and $V = \{x\} \cup (\underline{\phi}F \setminus E)$, then $\underline{\phi}V = \underline{\phi}F$ includes V, so V is open; thus every member of $\mathcal{V}_x$ is a neighbourhood of x. If V is an open neighbourhood of x, then $V \subseteq \underline{\phi}V$ and $E_0 = \underline{\phi}V \setminus V$ is negligible, so is included in some $E \in \mathcal{E}$. Now $\{x\} \cup (\underline{\phi}V \setminus E)$ is a member of $\mathcal{V}_x$ included in V. **Q** So

$$\chi(x, X) \le \#(\mathcal{V}_x) \le \max(\#(\mathfrak{A}), \mathrm{cf}\,\mathcal{N}(\mu)).$$

(ii) If $\{x\}$ is not negligible, set

$$\mathcal{V}_x = \{\underline{\phi}F \setminus E : F \in \Sigma,\ x \in \underline{\phi}F,\ E \in \mathcal{E}\};$$

then the same arguments show that $\mathcal{V}_x$ is a base of neighbourhoods of x so witnesses the required bound on $\chi(x, X)$.

(b) For any topological space X, $w(X) \le \max(\#(X), \chi(X))$.

(c)(i) If $\mathcal{H}$ is a coinitial family in $\Sigma \setminus \mathcal{N}(\mu)$, then $\{H \cap \underline{\phi}H : H \in \mathcal{H}\}$ is a π-base for the density topology. (ii) Any π-base for the density topology not containing $\emptyset$ is a coinitial family in $\Sigma \setminus \mathcal{N}(\mu)$.

(d) A subset of X is dense iff it has full outer measure.

(e)(i) If $\mathrm{shr}\,\mathcal{N}(\mu) \le 1$ the result is trivial. Otherwise, if $1 \le \kappa < \mathrm{shr}\,\mathcal{N}(\mu)$, there is a non-negligible set A such that every subset of A of cardinal κ is negligible. Let E be a measurable envelope of A (recall that a complete locally determined measure space with a lower density is strictly localizable); then $\underline{\phi}E \subseteq \overline{A}$. Take any $x \in A \cap \underline{\phi}E$; then $\{x\}$ is negligible so $x \in \overline{A \setminus \{x\}}$. But if $B \in [A \setminus \{x\}]^{\le\kappa}$ then B is negligible therefore closed, and $x \notin \overline{B}$. Thus x and $A \setminus \{x\}$ witness that $t(X) > \kappa$. As κ is arbitrary, $t(X) \ge \mathrm{shr}(X)$.

(ii) If both $\mathrm{shr}\,\mathcal{N}(\mu)$ and $c(\mathfrak{A})$ are finite the result is trivial. Otherwise, take any $A \subseteq X$. Let E be a measurable envelope of A. In $\mathfrak{A}$ consider the set

$$D = \{a : a \cap E^\bullet = 0\} \cup \{F^\bullet : \text{ there is a } B \subseteq A \text{ such that } \#(B) \le \mathrm{shr}\,\mathcal{N}(\mu) \text{ and } F \text{ is a measurable envelope of } B\}.$$

Then D is order-dense so there is a $C \subseteq D$ such that $\#(C) \le c(\mathfrak{A})$ and $\sup C = 1$. There is therefore a family $\langle B_\xi \rangle_{\xi < c(\mathfrak{A})}$ of subsets of A, all of size at most $\mathrm{shr}\,\mathcal{N}(\mu)$, with measurable envelopes F_ξ such that

$E^\bullet = \sup_{\xi<c(\mathfrak{A})} F_\xi^\bullet$. Set $B = \bigcup_{\xi<c(\mathfrak{A})} B_\xi$; then $B \subseteq A$ and E is a measurable envelope of B. But this means that $\overline{A} = A \cup \overline{B}$, while $\#(B) \le \max(c(\mathfrak{A}), \mathrm{shr}(\mathfrak{A}))$. As A is arbitrary, we have the result.

(f) If $\langle E_\xi\rangle_{\xi<\kappa}$ is a family in Σ then

$$\begin{aligned}\{\mathrm{int}\, E_\xi : \xi < \kappa\} &\text{ is a disjoint family of non-empty open sets}\\ &\iff \{\underline{\phi} E_\xi : \xi < \kappa\} \text{ is a disjoint family of non-empty sets}\\ &\iff \{E_\xi^\bullet : \xi < \kappa\} \text{ is a disjoint family in } \mathfrak{A}\setminus\{0\}.\end{aligned}$$

2 Proposition Let (X, Σ, μ) be a countably compact Maharam-type-homogeneous complete σ-finite measure space of Maharam type $\kappa \ge \omega$, and $\underline{\phi}$ a lower density for μ such that $\underline{\phi}X = X$. Give X the associated density topology. Then $\pi\chi(x, X) \ge \max(\mathrm{cf}\,\mathcal{N}, \mathrm{cf}[\kappa]^{\le\omega})$ for every $x \in X$.

proof As there is a probability measure with the same measurable sets and the same null ideal as μ, we can suppose that μ itself is a probability measure. As μ is atomless, $\mu\{x\} = 0$ for every $x \in X$. Let $\mathcal{E}$ be a family of non-empty open subsets of X such that every neighbourhood of x contains some member of $\mathcal{E}$, while $\#(\mathcal{E}) = \pi\chi(x.X)$.

(a) Let $f : X \to [0, 1]$ be inverse-measure-preserving for μ and Lebesgue measure on $[0, 1]$. μ is perfect (Fremlin 03, 451C), so for each $E \in \mathcal{E}$ there is a compact set $K_E \subseteq f[E \setminus \{x\}]$ such that $E \cap f^{-1}[K_E]$ is non-negligible. In particular, K_E is not Lebesgue negligible; set $F_E = \mathbb{R} \setminus (K_E + \mathbb{Q})$, so that F_E is Lebesgue negligible.

If $F \subseteq \mathbb{R}$ is Lebesgue negligible, set $A_F = f^{-1}[F + \mathbb{Q}]) \setminus \{x\}$; then A_F is negligible so is closed and there must be an $E \in \mathcal{E}$ disjoint from A_F. Now $E \setminus \{x\}$ is disjoint from $f^{-1}[F + \mathbb{Q}]$ so $f[E \setminus \{x\}]$ is disjoint from $F + \mathbb{Q}$ and F is disjoint from $K_E + \mathbb{Q}$, that is, $F \subseteq F_E$.

So $\{F_E : E \in \mathcal{E}\}$ is cofinal with $\mathcal{N}$ and $\mathrm{cf}\,\mathcal{N} \le \pi\chi(x, X)$.

(b) By Fremlin 07?, 524Ga, there is a family $\langle H_\xi\rangle_{\xi<\kappa}$ in $\mathcal{N}(\mu)$ such that $\bigcup_{\xi\in J} H_\xi$ has full outer measure for every uncountable $J \subseteq \kappa$. For $E \in \mathcal{E}$, set $I_E = \{\xi : \xi < \kappa,\ H_\xi \cap E \subseteq \{x\}\}$. Then I_E is countable. Also, for any countable $I \subseteq \kappa$, $\bigcup_{\xi\in I} H_\xi \setminus \{x\}$ is closed, so is disjoint from some $E \in \mathcal{E}$, and $I \subseteq I_E$. Thus $\{I_E : E \in \mathcal{E}\}$ witnesses that $\mathrm{cf}[\kappa]^{\le\omega} \le \pi\chi(x, X)$.

3 Proposition Let (X, Σ, μ) be a measure space with Maharam type $\kappa \ge \omega$, and $\langle V_n\rangle_{n\in\mathbb{N}}$ a sequence of measurable sets of non-zero finite measure. Let $\mathcal{F}$ be the filter generated by

$$\{E : E \in \Sigma,\ \lim_{n\to\infty} \frac{\mu(E\cap V_n)}{\mu V_n} = 1\}.$$

Then

$$\mathrm{ci}\,\mathcal{F} \le \max(\mathrm{cf}\,\mathcal{N}(\mu), \mathrm{cf}\,\mathcal{N}, \mathrm{cf}[\kappa]^{\le\omega}),$$

where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$.

proof (a) Let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of μ, and $c_n = V_n^\bullet$ the equivalence class of V_n for each n. Let $D \subseteq \mathbb{N}^\mathbb{N}$ be a cofinal set of cardinal $\mathfrak{d}$ consisting of strictly positive strictly increasing functions. Let $\mathfrak{B}$ be a τ-generating subalgebra of $\mathfrak{A}$ of cardinal κ containing every c_n; then $\mathfrak{B} \cap \mathfrak{A}_{c_n}$ is dense (for the measure-algebra topology) in the principal ideal $\mathfrak{A}_{c_n}$ for every n.

(b) For $h \in D$ define $p_h \in \mathbb{N}^\mathbb{N}$ and $\langle \epsilon_{hn}\rangle_{n\in\mathbb{N}}$ by setting

$$\begin{aligned}p_h(n) &= 1 \text{ if } n < h(0)\\ &= k \text{ if } h(k) \le n < h(k+1),\\ \epsilon_{hn} &= \frac{1}{p_h(n)^2} \text{ for every } n.\end{aligned}$$

Note that $\lim_{n\to\infty} p_h(n) = \infty$ and $\lim_{n\to\infty} p_h(n)\epsilon_{hn} = 0$. For each $n \in \mathbb{N}$, let $\delta_{hn} \in]0,1]$ be such that $p_h(n)\delta_{hn} \le 2^{j-n}\epsilon_{hj}\mu V_j$ for every $j < n$.

For each $n \in \mathbb{N}$, let $\langle (a_{hn\xi}, b_{hn\xi})\rangle_{\xi<\kappa}$ run over the set of pairs (a,b) such that

$$a,\, b \in \mathfrak{B}, \quad \bar\mu(c_n \setminus a) \le 2\epsilon_{hn}\bar\mu c_n, \quad \bar\mu b \le \delta_{hn}.$$

Let $\mathcal{S}_h$ be the family of sets $S \subseteq \mathbb{N}\times\kappa$ such that $\#(S[\{n\}]) \le p_h(n)$ for every $n \in \mathbb{N}$. Then there is a family $\mathcal{A}'_h \subseteq \mathcal{S}_h$ such that $\#(\mathcal{A}'_h) \le \max(\mathrm{cf}\mathcal{N}, \mathrm{cf}[\kappa]^{\le\omega})$ and for every $f \in \kappa^{\mathbb{N}}$ there is an $S \in \mathcal{A}'_h$ such that $f \subseteq^* S$ (Fremlin 07?, 522L and 522M); set

$$\mathcal{A}_h = \{S : S \in \mathcal{S}_h,\ S[\{n\}] \ne \emptyset \text{ for every } n,\ S\triangle S' \text{ is finite for some } S' \in \mathcal{A}'_h\},$$

so that $\#(\mathcal{A}_h) \le \max(\mathrm{cf}\mathcal{N}, \mathrm{cf}[\kappa]^{\le\omega})$ and for every $f \in \kappa^{\mathbb{N}}$ there is an $S \in \mathcal{A}_h$ such that $f \subseteq S$.

(c) For $h \in D$ and $S \in \mathcal{A}_h$ set

$$e_{hSn} = \inf_{\xi\in S[\{n\}]} a_{hn\xi} \setminus \sup_{(k,\xi)\in S, k>n} b_{hk\xi}.$$

Then

$$\mu(c_n \setminus e_{hSn}) \le 2p_h(n)\epsilon_{hn}\bar\mu c_n + \textstyle\sum_{k>n} p_h(k)\delta_{hk} \le (2p_h(n)+1)\epsilon_{hn}\bar\mu c_n$$

for every n. Set $e_{hS} = \sup_{n\in\mathbb{N}} e_{hSn}$; then $\lim_{n\to\infty} \frac{\bar\mu(e_{hS}\cap c_n)}{\bar\mu c_n} = 1$.

(d) If $E \in \mathcal{F}$ and $e = E^\bullet$ then there are $h \in D$ and $S \in \mathcal{A}_h$ such that $e_{hS} \subseteq e$. **P** Set $\epsilon_n = \max(\frac{1}{n+1}, \frac{\bar\mu(c_n\setminus e)}{\bar\mu c_n})$ for each n; then $\langle \epsilon_n\rangle_{n\in\mathbb{N}} \to 0$. Let $h \in D$ be such that $\epsilon_n \le \frac{1}{k^2}$ whenever $k \ge 1$ and $n \ge h(k)$. Then $\epsilon_n \le \epsilon_{hn}$ for every n.

We can define $f : \mathbb{N} \to \kappa$ inductively so that

$$\bar\mu((a_{h,n,f(n)} \setminus \sup_{n<k\le m} b_{h,k,f(k)}) \setminus e) < \tfrac{1}{m+1}\delta_{h,m+1}$$

whenever $n \le m$ in $\mathbb{N}$. To see this, start by choosing $a'_0 \in \mathfrak{B}$ such that $\bar\mu(c_0 \setminus a'_0) \le 2\epsilon_{h0}\bar\mu c_0$ and $\bar\mu(a'_0 \setminus e) < \delta_{h1}$; this is possible because $\bar\mu(c_0 \setminus e) < 2\epsilon_0\bar\mu c_0 \le 2\epsilon_{h0}\bar\mu c_0$, so we have only to ensure that $\bar\mu(a'_0\triangle(e\cap c_0))$ is small enough. Take $f(0)$ such that $a'_0 = a_{h,0,f(0)}$. When choosing $f(m)$ for $m \ge 1$, take $a'_m \in \mathfrak{B}$ such that $\bar\mu(c_m \setminus a'_m) \le 2\epsilon_{hm}\bar\mu c_m$ and $\bar\mu(a'_m \setminus e) < \frac{1}{m+1}\delta_{h,m+1}$; now for $n < m$ take $b'_{mn} \in \mathfrak{B}$ such that $\bar\mu b'_{mn} \le \frac{1}{m}\delta_{hm}$ and

$$\bar\mu(((a_{h,n,f(n)} \setminus \sup_{n<k<m} b_{h,k,f(k)}) \setminus e) \setminus b'_{mn}) < \tfrac{1}{m+1}\delta_{h,m+1}.$$

Take $f(m)$ such that $a_{h,m,f(m)} = a'_m$ and $b_{h,m,f(m)} = \sup_{n<m} b'_{mn}$, and continue.

Let $S \in \mathcal{A}_h$ be such that $f \subseteq S$. Then

$$e_{hSn} \subseteq a_{h,n,f(n)} \setminus \sup_{k>n} b_{h,k,f(k)}$$

so

$$\bar\mu(e_{hSn} \setminus e) \le \tfrac{1}{m+1}\delta_{h,m+1}$$

whenever $m \ge n$, and $\bar\mu(e_{hS} \setminus e) = 0$. **Q**

(e) For $h \in D$, $S \in \mathcal{A}_h$ choose E_{hS} such that $E^\bullet_{hS} = e_{hS}$. Then $E_{hS} \in \mathcal{F}$. Let $\mathcal{W}$ be a cofinal family in $\mathcal{N}(\mu)$ of cardinal $\mathrm{cf}\mathcal{N}(\mu)$. If $F \in \mathcal{F}$ there are $h \in D$, $S \in \mathcal{S}_h$ such that $e_{hS} \subseteq F^\bullet$, and now there is a $W \in \mathcal{W}$ such that $E_{hS} \setminus W \subseteq F$. So $\{E_{hS} \setminus W : h \in D,\ S \in \mathcal{A}_h,\ W \in \mathcal{W}\}$ witnesses that $\mathrm{ci}\,\mathcal{F} \le \max(\mathrm{cf}\mathcal{N}, \mathrm{cf}[\kappa]^{\le\omega}, \mathrm{cf}\mathcal{N}(\mu))$.

4 Theorem Give $\mathbb{R}$ the density topology defined by Lebesgue lower density. Then, writing $\mathcal{N}$ for the null ideal of Lebesgue measure on $\mathbb{R}$,

(a) (Tall 76) $w(\mathbb{R}) = \mathfrak{c}$;

(b) (Tall 76) $d(\mathbb{R}) = \mathrm{non}\,\mathcal{N}$;

(c) $\chi(x,\mathbb{R}) = \pi\chi(x,\mathbb{R}) = \mathrm{cf}\mathcal{N}$ for every $x \in \mathbb{R}$;

(d) $\pi(\mathbb{R}) = \operatorname{cf}\mathcal{N}$.

proof (a) By 1b, $w(\mathbb{R}) \le \mathfrak{c}$. In the other direction, there is a negligible set of cardinal $\mathfrak{c}$, which is a closed discrete set, so has weight $\mathfrak{c}$, and $w(\mathbb{R}) \ge \mathfrak{c}$.

(b) By 1d,

$$d(\mathbb{R}) = \min\{\#(A) : A \subseteq \mathbb{R} \text{ has full outer measure}\}.$$

Now $(\mathbb{R}, \mathcal{N})$ is isomorphic to $(\{0,1\}^{\mathbb{N}}, \mathcal{N}(\nu_{\mathbb{N}}))$ where $\nu_{\mathbb{N}}$ is the usual measure on $\{0,1\}^{\mathbb{N}}$ (Fremlin 07?, 522Va), so

$$d(\mathbb{R}) = \min\{\#(A) : A \subseteq \{0,1\}^{\mathbb{N}} \text{ has full outer measure}\} = \operatorname{non}\mathcal{N}(\nu_{\mathbb{N}}) = \operatorname{non}\mathcal{N}$$

by Fremlin 07?, 523Dd.

(c) Take any $x \in \mathbb{R}$. By Proposition 2, $\operatorname{cf}\mathcal{N} \le \pi\chi(x, \mathbb{R})$. Next, if we set $V_n = [x - \frac{1}{n+1}, x + \frac{1}{n+1}]$ for each n, the filter $\mathcal{F}$ defined in Proposition 3 is just the neighbourhood filter of x. So $\chi(x, \mathbb{R}) \le \operatorname{cf}\mathcal{N}$. Since we surely have $\pi\chi(x, \mathbb{R}) \le \chi(x, \mathbb{R})$, both are equal to $\operatorname{cf}\mathcal{N}$.

(d) This follows immediately on putting (b) and (c) together, or from 1c.

5 Lemma If $\mathfrak{A}$ is an infinite Dedekind complete Boolean algebra then its Stone space has a point of character $\#(\mathfrak{A})$.

proof (cf. Koppelberg 89, 10.16 and Douwen 89, 9.3) By the Balcar-Franěk theorem (Fremlin 07?, 515H), $\mathfrak{A}$ has a Boolean-independent subset A of cardinal $\kappa = \#(\mathfrak{A})$. So if Z is the Stone space of $\mathfrak{A}$, there is a continuous surjection $f : Z \to \{0,1\}^\kappa$. Let $F \subseteq Z$ be a closed subset such that $f{\restriction}F$ is an irreducible surjection onto $\{0,1\}^\kappa$. Take any $x \in F$. Then $\pi\chi(x, F) \ge \kappa$. **P** If $\mathcal{U}$ is any family of non-empty relatively open subsets of F with $\#(\mathcal{U}) < \kappa$, then for each $U \in \mathcal{U}$ there is a non-empty open cylinder set $V_U \subseteq \{0,1\}^\kappa$ such that $F \cap f^{-1}[V_U] \subseteq U$. Each V_U is determined by coordinates in a finite set $I_U \subseteq \kappa$; as $\#(U) < \kappa$, there is a $\xi < \kappa$ not belonging to any I_U. Set $H = \{w : w \in \{0,1\}^\kappa,\ w(\xi) = f(x)(\xi)\}$; then $F \cap f^{-1}[H]$ is a neighbourhood of x in F. But if $U \in \mathcal{U}$ then $V_U \not\subseteq H$ so $U \setminus f^{-1}[H] \supseteq F \cap f^{-1}[V_U \setminus H]$ is not empty. Thus $\mathcal{U}$ cannot be a local π-base at x; as $\mathcal{U}$ is arbitrary, $\pi\chi(x, F) \ge \kappa$. **Q**

Now

$$\kappa \le \pi\chi(x, F) \le \chi(x, F) \le \chi(x, Z) \le w(Z) = \kappa,$$

so $\chi(x, Z) = \kappa$, as required.

6 Example There is a lower density $\underline{\phi}$ for Lebesgue measure on $\mathbb{R}$, taking $\mathbb{R}$ to itself, such that the associated density topology on $\mathbb{R}$ has a point of character $\mathfrak{c}$.

proof (a) Let μ_1 be Lebesgue measure on $[0,1]$, Σ_1 its domain and $\mathfrak{A}$ its measure algebra; let Z be the Stone space of $\mathfrak{A}$ and $f : Z \to [0,1]$ the canonical inverse-measure-preserving map (Fremlin 03, 416V). For $E \in \Sigma$, let $E^* \subseteq Z$ be the open-and-closed set corresponding to $E^\bullet \in \mathfrak{A}$, By Lemma 5, Z has a point z_0 of character $\mathfrak{c}$; set $x_0 = f(z_0) \in [0,1]$. Let $\underline{\phi}_0 : \Sigma \to \Sigma$ be any lower density such that $\underline{\phi}_0[0,1] = [0,1]$. Define $\underline{\phi} : \Sigma \to \Sigma$ by setting

$$\begin{aligned} \underline{\phi}E &= \underline{\phi}_0 E \cup \{x_0\} \text{ if } z_0 \in E^\bullet \\ &= \underline{\phi}_0 E \setminus \{x_0\} \text{ otherwise.} \end{aligned}$$

Then $\underline{\phi}$ is a lower density and $\underline{\phi}[0,1] = [0,1]$. Now $\chi(x_0, [0,1]) \ge \mathfrak{c}$. **P?** Otherwise, let $\mathcal{U}$ be a base of neighbourhoods of x_0 with $\#(\mathcal{U}) < \mathfrak{c}$. If $E \in \mathcal{U}$, then $x_0 \in E \subseteq \underline{\phi}E$ so $z_0 \in E^*$. Because $\chi(z_0, Z) = \mathfrak{c}$, there is a neighbourhood V of z_0 not including E^* for any $E \in \mathcal{U}$; as Z is zero-dimensional, we can suppose that V is open-and-closed so is of the form F^* for some $F \in \Sigma$. In this case, $x_0 \in \underline{\phi}F$ so $\operatorname{int}(F \cup \{x_0\})$ is a neighbourhood of x_0; but it does not include any member of $\mathcal{U}$. **XQ**

(b) Writing μ for Lebesgue measure on $\mathbb{R}$ and Σ for its domain, $(\mathbb{R}, \Sigma, \mathcal{N}(\mu))$ is isomorphic to $([0,1], \Sigma_1, \mathcal{N}(\mu_1))$. So we can translate $\underline{\phi}$ into a lower density for μ for which the density topology has a point of character at least $\mathfrak{c}$. But 1a tells us that the character is at most $\mathfrak{c}$, so we have equality, as required.

References

Douwen E.K.van [89] 'Cardinal functions on Boolean spaces', pp. 417-467 in Monk 89.

Fremlin D.H. [02] *Measure Theory, Vol. 3: Measure Algebras.* Torres Fremlin, 2002.

Fremlin D.H. [03] *Measure Theory, Vol. 4: Topological Measure Spaces.* Torres Fremlin, 2003.

Fremlin D.H. [07?] *Measure Theory, Vol. 5: Set-theoretic measure theory,* in preparation. *Partial drafts available via* `http://www.essex.ac.uk/maths/staff/fremlin/mt.htm`. *Note that paragraph references are subject to fluctuation.*

Goffman C. & Waterman D. [61] 'Approximately continuous transformations', Proc. Amer. Math. Soc. 12 (1961) 116-121.

Koppelberg S. [89] *General Theory of Boolean Algebras*, vol. 1 of Monk 89.

Monk J.D. [89] (ed.) *Handbook of Boolean Algebra.* North-Holland, 1989.

Tall F.D. [76] 'The density topology', Pacific J. Math. 62 (1976) 275-284.

Tall F.D. [78] 'Normal subspaces of the density topology', Pacific J. Math. 75 (1978) 579-588.