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Baire σ-algebras in product spaces

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1 Notation (a) If X is a topological space, $\mathcal{B}\mathfrak{a}(X)$ will be its Baire σ-algebra, that is, the σ-algebra generated by the zero sets.

(b)(i) If X and Y are sets, Σ is a σ-algebra of subsets of X and T is a σ-algebra of subsets of Y, then $\Sigma\widehat{\otimes}\mathrm{T}$ will be the σ-algebra of subsets of $X\times Y$ generated by $\{E\times F: E\in\Sigma,\ F\in\mathrm{T}$.

(ii) If $\langle X_i\rangle_{i\in I}$ is a family of sets, with product X, and Σ_i is a σ-algebra of subsets of X_i for each i, then $\widehat{\bigotimes}_{i\in I}\Sigma_i$ will be the σ-algebra of subsets of X generated by sets of the form $\{x: x\in X,\ x(i)\in E\}$ where $i\in I$ and $E\in\Sigma_i$.

2 Lemma Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces, with product X. Then $\mathcal{B}\mathfrak{a}(X)\supseteq\widehat{\bigotimes}_{i\in I}\mathcal{B}\mathfrak{a}(X_i)$.

3 Lemma Let $\langle X_i\rangle_{i\in I}$ be a family of compact Hausdorff spaces, with product X. Then $\mathcal{B}\mathfrak{a}(X)=\widehat{\bigotimes}_{i\in I}\mathcal{B}\mathfrak{a}(X_i)$.

proof If F, H are disjoint closed subsets of X, there is an open set G including F and disjoint from H which is a finite union of open cylinder sets whose coordinates are all cozero sets. So $G\in\widehat{\bigotimes}_{i\in I}\mathcal{B}\mathfrak{a}(X_i)$.

If $F\subseteq X$ is a zero set, there is a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ of closed sets with union $X\setminus F$. For each n, let $G_n\in\widehat{\bigotimes}_{i\in I}\mathcal{B}\mathfrak{a}(X_i)$ be such that $F\subseteq G_n\subseteq X\setminus H_n$; then $F=\bigcap_{n\in\mathbb{N}}G_n$ belongs to $\widehat{\bigotimes}_{i\in I}\mathcal{B}\mathfrak{a}(X_i)$.

4 Lemma Let X be any topological space and Y a separable metrizable space. Then $\mathcal{B}\mathfrak{a}(X\times Y)=\mathcal{B}\mathfrak{a}(X)\widehat{\otimes}\mathcal{B}\mathfrak{a}(Y)$.

proof Let $f:X\times Y\to[0,1]$ be a continuous function. Let D be a countable dense subset of Y and $\mathcal{U}$ a countable base for the topology of y. For $q\in\mathbb{Q}$ and $U\in\mathcal{U}$ set $G_q=\{(x,y):f(x,y)>q\}$, $H_q=\{(x,y):f(x,y)\ge q\}$,

$$E_{qU}=\{x:\{x\}\times U\subseteq H_q\}=\{x:f(x,d)\le q\text{ for every }d\in D\cap U\}\in\mathcal{B}\mathfrak{a}(X);$$

then

$$W_q=\bigcup_{U\in\mathcal{U}}E_{qU}\times U\in\mathcal{B}\mathfrak{a}(X)\widehat{\otimes}\mathcal{B}\mathfrak{a}(Y)$$

and $G_q\subseteq W_q\subseteq F_q$. But this means that

$$G_0=\bigcup_{q>0}G_q=\bigcup_{q>0}F_q=\bigcup_{q>0}W_q$$

belongs to $\mathcal{B}\mathfrak{a}(X)\widehat{\otimes}\mathcal{B}\mathfrak{a}(Y)$. As f is arbitrary, every cozero set belongs to $\mathcal{B}\mathfrak{a}(X)\widehat{\otimes}\mathcal{B}\mathfrak{a}(Y)$ and $\mathcal{B}\mathfrak{a}(X\times Y)\subseteq\mathcal{B}\mathfrak{a}(X)\widehat{\otimes}\mathcal{B}\mathfrak{a}(Y)$.

5 Lemma Let X be a compact Hausdorff space and Y a separable space. Then $\mathcal{B}\mathfrak{a}(X\times Y)=\mathcal{B}\mathfrak{a}(X)\widehat{\otimes}\mathcal{B}\mathfrak{a}(Y)$.

proof (a) Let $f:X\times Y\to[0,1]$ be a continuous function. Let $D\subseteq Y$ be a dense set. Define $\phi:X\to[0,1]^D$ by setting $\phi(x)(d)=f(x,d)$ for $x\in X$ and $d\in D$; then ϕ is continuous, so $Z=\phi[X]$ is compact; because D is countable, Z is metrizable.

If x, $x'\in X$ and $\phi(x)=\phi(x')$, then $f(x,d)=f(x',d)$ for every $d\in D$, so $f(x,y)=f(x',y)$ for every $y\in Y$. We therefore have a function $g:Z\times Y\to[0,1]$ defined by setting $g(z,y)=f(x,y)$ whenever $\phi(x)=z$.

(b) g is continuous. $\mathbf{P}$ Let $\mathcal{F}$ be a filter on $Z\times Y$ converging to (z_0,y_0). Set $\psi(x,y)=(\phi(x),y)$ for $x\in X$, $y\in Y$. Then there is an ultrafilter $\mathcal{G}$ on $X\times Y$ containing $\psi^{-1}[A]$ for every $A\in\mathcal{F}$. If V is any neighbourhood of y_0 then $Z\times V\in\mathcal{F}$ so $X\times V\in\mathcal{G}$; accordingly $\pi_2[[\mathcal{G}]]\to y_0$, where $\pi_2(x,y)=y$ for $x\in X$, $y\in Y$. Because X is compact, the ultrafilter $\pi_1[[\mathcal{G}]]$ has a limit $x_0\in X$, so $\mathcal{G}\to(x_0,y_0)$ in $X\times Y$. Now $\psi[[\mathcal{G}]]\supseteq\mathcal{F}$ so $\psi[[\mathcal{G}]]\to(z_0,y_0)$ and $\phi[[\pi_1[[\mathcal{G}]]\,]]\to z_0$; as ϕ is continuous and Z is Hausdorff, $\phi(x_0)=z_0$ and $g(z_0,y_0)=f(x_0,y_0)$.

Let U be a neighbourhood of $g(z_0,y_0)$. Then $f^{-1}[U]\in\mathcal{G}$, so $\psi[f^{-1}[U]]\in\psi[[\mathcal{G}]]$. But if $(z,y)\in\psi[f^{-1}[U]]$ then $g(z,y)\in U$. So $g^{-1}[U]\in\psi[[\mathcal{G}]]$. As U is arbitrary, $\lim_{(z,y)\to\psi[[\mathcal{G}]]}g(z,y)=g(z_0,y_0)$. But $\psi[[\mathcal{G}]]\supseteq\mathcal{F}$, so $g(z_0,y_0)$ is a cluster point of $g[[\mathcal{F}]]$.

As $\mathcal{F}$ and (z_0,y_0) are arbitrary, g is continuous. **Q**

(c) Because Z is second-countable, Lemma 4 tells us that $\mathcal{B}\mathfrak{a}(Z\times Y)=\mathcal{B}\mathfrak{a}(Z)\widehat{\otimes}\mathcal{B}\mathfrak{a}(Y)$. So g is $\mathcal{B}\mathfrak{a}(Z)\widehat{\otimes}\mathcal{B}\mathfrak{a}(Y)$-measurable; as ϕ is continuous, it is $(\mathcal{B}\mathfrak{a}(X),\mathcal{B}\mathfrak{a}(Z))$-measurable, so ψ is $(\mathcal{B}\mathfrak{a}(X)\widehat{\otimes}\mathcal{B}\mathfrak{a}(Y),\mathcal{B}\mathfrak{a}(Z)\widehat{\otimes}\mathcal{B}\mathfrak{a}(Y))$-measurable and $f=g\psi$ is $\mathcal{B}\mathfrak{a}(X)\widehat{\otimes}\mathcal{B}\mathfrak{a}(Y)$-measurable. As f is arbitrary, $\mathcal{B}\mathfrak{a}(X\times Y)\subseteq\mathcal{B}\mathfrak{a}(X)\widehat{\otimes}\mathcal{B}\mathfrak{a}(Y)$.

6 Lemma Let X be a compact Hausdorff space and $\langle Y_i\rangle_{i\in I}$ a family of Polish spaces with product Y. If $f:X\times Y\to[0,1]$ is continuous, there is a countable $J\subseteq I$ such that $f(x,y)=f(x,y')$ whenever $x\in X$, y, $y'\in Y$ and $y{\restriction}J=y'{\restriction}J$.

proof For each $n\in\mathbb{N}$ and $y\in Y$, there is an open neighbourhood V_{ny} of y such that $|f(x,y')-f(x,y)|\le 2^{-n}$ whenever $x\in X$ and $y\in V_{ny}$; and we can suppose that V_{ny} is determined by coordinates in a finite set $J_{ny}\subseteq I$. Because Y is ccc, there is a countable set $D\subseteq Y$ such that $\bigcup_{y\in D}V_{ny}$ is dense for every n; set $J=\bigcup_{n\in\mathbb{N},y\in D}J_{ny}$. For each n, $Y_n=\bigcup_{d\in D}V_{nd}$ is a dense open subset of Y determined by coordinates in J, so (because $\prod_{i\in J}Y_i$ is a Baire space) $Y'=\bigcap_{n\in\mathbb{N}}Y_n$ is a dense G_δ subset of Y determined by coordinates in J.

If y, $y'\in Y'$ and $y{\restriction}J=y'{\restriction}J$, then $f(x,y)=f(x,y')$ for every $x\in X$. **P** For each n there is a $d\in D$ such that $y\in V_{nd}$. In this case $y'\in V_{nd}$ so

$$|f(x,y)-f(x,y')|\le|f(x,y)-f(x,d)|+|f(x,d)-f(x,y')|\le 2^{-n+1}. \textbf{ Q}$$

If y, $y'\in Y$ and $y{\restriction}J=y'{\restriction}J$, then $f(x,y)=f(x,y')$ for every $x\in X$. **P** $(y,y')\in\overline{\{(v,v'):v,\,v'\in Y',\,v{\restriction}J=v'{\restriction}J\}}$. **Q**

7 Lemma Let X be a compact Hausdorff space and $\langle Y_i\rangle_{i\in I}$ a family of Polish spaces with product Y. Then $\mathcal{B}\mathfrak{a}(X\times Y)=\mathcal{B}\mathfrak{a}(X)\widehat{\otimes}\mathcal{B}\mathfrak{a}(Y)$.

proof Any continuous $f:X\times Y\to[0,1]$ is $\mathcal{B}\mathfrak{a}(X)\widehat{\otimes}\mathcal{B}\mathfrak{a}(Y)$-measurable. **P** Let J be as in Lemma 6. Set $Y_J=\prod_{i\in J}Y_i$ so that Y_J is Polish. Fix any $w\in\prod_{i\in I\setminus J}Y_i$ (I am passing over the trivial case $Y=\emptyset$), and define $g:X\times Y_J\to[0,1]$ by setting $g(x,u)=f(x,u\cup w)$ for every $x\in X$ and $u\in Y_J$. Then g is continuous, so is $\mathcal{B}\mathfrak{a}(X\times Y_J)=\mathcal{B}\mathfrak{a}(X)\widehat{\otimes}\mathcal{B}\mathfrak{a}(Y_J)$-measurable (using Lemma 5). Now, writing π_J for the canonical map from Y to Y_J, $f(x,y)=g(x,\pi_J(y))$ for all $x\in X$, $y\in Y$; as in the proof of Lemma 5, π_J is $(\mathcal{B}\mathfrak{a}(Y),\mathcal{B}\mathfrak{a}(Y_J))$-measurable, so $(x,y)\mapsto(x,\pi_J(y))$ is $(\mathcal{B}\mathfrak{a}(X)\widehat{\otimes}\mathcal{B}\mathfrak{a}(Y),\mathcal{B}\mathfrak{a}(X)\widehat{\otimes}\mathcal{B}\mathfrak{a}(Y_J))$-measurable and f is $\mathcal{B}\mathfrak{a}(X)\widehat{\otimes}\mathcal{B}\mathfrak{a}(Y)$-measurable. **Q**

8 Lemma Let $\langle X_i\rangle_{i\in I}$ be a family of separable metrizable spaces with product X. Then $\mathcal{B}\mathfrak{a}(X)=\widehat{\bigotimes}_{i\in I}\mathcal{B}\mathfrak{a}(X_i)$.

proof Fremlin 03, 4A3Na.

9 Theorem Let $\langle X_i\rangle_{i\in I}$ be a family of Hausdorff spaces, each either compact or Polish, with product X. Then $\mathcal{B}\mathfrak{a}(X)=\widehat{\bigotimes}_{i\in I}\mathcal{B}\mathfrak{a}(X_i)$.

proof Put Lemmas 3, 7 and 8 together.

References

Fremlin D.H. [03] *Measure Theory, Vol. 4: Topological Measure Theory.* Torres Fremlin, 2003.