Version of 29.7.05

On the zeroes of p and p'

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I write out a very pretty result from Pereira 03. This note is based on a talk by M.Laczkovich in March 2004.

Theorem Let p be a complex polynomial of degree $n+1$ where $n \geq 1$, and p' its derivative. Let $z_0, \dots, z_n$ be the zeroes of p and $w_1, \dots, w_n$ the zeroes of p'. Then there is a real matrix $A = \langle \alpha_{ji} \rangle_{0 \leq i \leq n, i \leq j \leq n}$ such that

$$\alpha_{ji} \geq 0 \text{ for all } i \text{ and } j,$$

$$\sum_{i=0}^{n} \alpha_{ji} = 1 \text{ for } 1 \leq j \leq n, \quad \sum_{j=1}^{n} \alpha_{ji} = \frac{n}{n+1} \text{ for } 0 \leq i \leq n,$$

$$w_j = \sum_{i=0}^{n} \alpha_{ji} z_i \text{ for } 1 \leq j \leq n.$$

proof We can assume throughout that p is monic, so that the leading term in p' is $(n+1)z^n$.

(a) We have

$$\begin{aligned} \frac{p'(z)}{p(z)} &= \sum_{i=0}^{n} \frac{1}{z - z_i} \text{ for } z \in \mathbb{C} \setminus \{z_0, \dots, z_n\} \\ &= \frac{1}{z} \sum_{i=0}^{n} \frac{1}{1 - \frac{z_i}{z}} = \frac{1}{z} \sum_{i=0}^{n} \sum_{k=0}^{\infty} (\frac{z_i}{z})^k = \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \sum_{i=0}^{n} z_i^k \end{aligned}$$

when $|z| > \max_{i \leq n} |z_i|$. Let B be the $(n+1) \times (n+1)$ diagonal complex matrix with diagonal entries $z_0, \dots, z_n$, so that B^k is the diagonal matrix with entries $z_0^k, \dots, z_n^k$ for each k. Now we have an orthonormal basis $v_0, \dots, v_n$ of $\mathbb{C}^{n+1}$ such that v_0 is the $(n+1)$-vector with every entry $\frac{1}{\sqrt{n+1}}$, and $(Bv_i|v_j) = 0$ whenever $0 \leq i < j \leq n$. **P** Choose $v_1, \dots, v_n$ inductively so that, for $j \geq 1$, v_j is a normalized eigenvector of the operator $P_j B \restriction W_j$, where W_j is the orthogonal complement of the subspace generated by $v_0, \dots, v_{j-1}$ and P_j is the orthogonal projection onto W_j. If $i < j$ then $v_j \in W_j \subseteq W_i$, so $(Bv_i|v_j) = (P_i B v_i|v_j)$ is a multiple of $(v_i|v_j) = 0$. **Q**

Let E be the orthogonal matrix with columns $v_0, \dots, v_n$.

(b) We have

$$B^k v_0 = \frac{1}{\sqrt{n+1}} \begin{pmatrix} z_0^k \\ \cdot\cdot \\ \cdot\cdot \\ z_n^k \end{pmatrix}$$

for each k. For the moment, fix z such that $|z| > \max_{i \leq n} |z_i|$. Then

$$\begin{aligned} \frac{p'(z)}{p(z)} &= \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} (n+1)(B^k v_0|v_0) = \frac{n+1}{z} \sum_{k=0}^{\infty} ((\tfrac{1}{z}B)^k v_0|v_0) \\ &= \frac{n+1}{z} (\sum_{k=0}^{\infty} (\tfrac{1}{z}B)^k v_0|v_0) = \frac{n+1}{z} ((I_{n+1} - \tfrac{1}{z}B)^{-1} v_0|v_0) \end{aligned}$$

(where I_{n+1} is the $(n+1) \times (n+1)$ identity matrix)

$$= (n+1)((zI_{n+1} - B)^{-1}v_0|v_0).$$

Set

$$C_z = (n+1)E^*(zI_{n+1} - B)^{-1}E = (n+1)(E^*(zI_{n+1} - B)E)^{-1}.$$

Taking $e_0, \dots, e_n$ to be the standard basis of $\mathbb{C}^{n+1}$, $Ee_0 = v_0$ and

$$(C_z e_0|e_0) = (n+1)((zI_{n+1} - B)^{-1}v_0|v_0) = \frac{p'(z)}{p(z)},$$

that is, the top left entry in the matrix C_z is $\frac{p'(z)}{p(z)}$.

Let D be the $n \times n$ matrix obtained by deleting the top row and left-hand column of the $(n+1)\times(n+1)$-matrix E^*BE. If we do the same to $E^*(zI_{n+1} - B)E = zI_{n+1} - E^*BE$ we get $zI_n - D$, where I_n is the $n \times n$ identity matrix. By Cramer's rule the top left entry of C_z is

$$\begin{aligned}(n+1)\frac{\det(zI_n - D)}{\det(E^*(zI_{n+1}-B)E)} &= (n+1)\frac{\det(zI_n - D)}{\det(zI_{n+1}-B)} \\ &= (n+1)\frac{\det(zI_n - D)}{\prod_{i=0}^n (z - z_i)} = (n+1)\frac{\det(zI_n - D)}{p(z)}\end{aligned}$$

and $p'(z) = (n+1)\det(zI_n - D)$. Since p' has leading coefficient $n+1$, $\det(zI_n - D) = \prod_{j=1}^n (z - w_j)$.

(c) At the same time, for $0 \le i < j \le n$,

$$(E^*BEe_i|e_j) = (BEe_i|Ee_j) = (Ev_i|v_j) = 0.$$

So E^*BE and $zI_n - D$ are triangular matrices, and $\det(zI_n - D)$ is the product of the diagonal entries of $zI_n - D$. And this is true whenever $|z|$ is large enough. But this means that the diagonal entries of $zI_n - D$ are $z - w_1, \dots, z - w_n$, and the diagonal entries of D are $w_1, \dots, w_n$ (in some order); that is, there is a permutation π of $1, \dots, n$ such that

$$(Bv_j|v_j) = (E^*BEe_j|e_j) = w_{\pi(j)}$$

for $1 \le j \le n$.

(d) For $0 \le j \le n$, express v_j as $\sum_{i=0}^n \gamma_{ji} e_i$. Then $Bv_j = \sum_{i=0}^n \gamma_{ji} z_i e_i$ and if $j \ge 1$ then $w_{\pi(j)} = \sum_{i=0}^n |\gamma_{ji}|^2 z_i$. Set $\alpha_{ji} = |\gamma_{\pi^{-1}(j),i}|^2$ for $0 \le i \le n$, $1 \le j \le n$. Then $\alpha_{ji} \ge 0$ for each i, j. We have

$$\sum_{i=0}^n \alpha_{ji} = \sum_{i=0}^n |\gamma_{\pi^{-1}(j),i}|^2 = (v_{\pi^{-1}(j)}|v_{\pi^{-1}(j)}) = 1$$

if $j \ge 1$. For the column sums, we have, for any i,

$$\begin{aligned}\sum_{j=1}^n \alpha_{ji} &= \sum_{j=1}^n |\gamma_{\pi^{-1}(j),i}|^2 = \sum_{j=1}^n |\gamma_{ji}|^2 \\ &= \sum_{j=0}^n |\gamma_{ji}|^2 - |\gamma_{0i}|^2 = \sum_{j=0}^n |(v_j|e_i)|^2 - |\gamma_{0i}|^2 = \sum_{j=0}^n |(e_i|v_j)|^2 - \frac{1}{n+1}\end{aligned}$$

(by the definition of v_0)

$$= (e_i|e_i) - \frac{1}{n+1}$$

(because $v_0, \dots, v_n$ form an orthonormal basis)

$$= \frac{n}{n+1}.$$

So we have a suitable matrix $A = \langle \alpha_{ji} \rangle_{1 \le j \le n, 0 \le i \le n}$.

Corollary (a) w_j belongs to the convex hull of $\{z_0, \dots, z_n\}$ for every j.
(b) If $\phi : \mathbb{C} \to \mathbb{R}$ is any convex function then

$$\frac{1}{n}\sum_{j=1}^{n} \phi(w_j) \leq \frac{1}{n+1}\sum_{i=0}^{n} \phi(z_i).$$

Remark For non-polynomial entire functions nothing like this works; consider $z \mapsto ze^z$.

References

Pereira R. [03] 'Differentiators and the geometry of polynomials', J. Math. Anal. Appl. 285 (2003) 336-348.