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Strictly localizable Borel measures

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I apply ideas from Holický & Spurný 03 to show that the most important examples of non-σ-finite Borel measures are not strictly localizable.

1 Borel classes Let X be a topological space. I define the (additive) **Borel classes** $\mathcal{E}^0_\xi(X)$, for ordinals $\xi \ge 1$, by saying that

$\mathcal{E}^0_0(X)$ is the algebra generated by the open subsets of X,

if $\xi > 0$, then

$$\mathcal{E}^0_\xi(X) = \{\bigcup_{n\in\mathbb{N}} X \setminus E_n : E_n \in \bigcup_{\eta<\xi} \mathcal{E}^0_\eta(X) \text{ for every } n \in \mathbb{N}\}.$$

$\mathcal{E}^0_{\omega_1}(X) = \mathcal{E}^0_{\omega_1+1}(X)$ is the Borel σ-algebra $\mathcal{B}(X)$ of X. I will say that the **Borel order** of X is $\min\{\alpha : \mathcal{B}(X) = \mathcal{E}^0_\alpha(X)\}$.

2 Elementary remarks (a) If $\eta \le \xi$ then $\mathcal{E}^0_\eta(X) \subseteq \mathcal{E}^0_\xi(X)$. For $\xi \ge 1$, $\mathcal{E}^0_\xi(X)$ is closed under countable unions and finite intersections. $\mathcal{E}^0_0(X)$ is the set of finite unions of differences of closed sets; $\mathcal{E}^0_1(X)$ is the set of countable unions of differences of closed sets.

(b) If X and Y are topological spaces and $f : X \to Y$ is such that $f^{-1}[H] \in \mathcal{E}^0_\alpha(X)$ for every open set $H \subseteq Y$, where $\alpha \ge 0$, then $f^{-1}[F] \in \mathcal{E}^0_{\alpha+\xi}(X)$ whenever $\xi \ge 0$ and $F \in \mathcal{E}^0_\xi(Y)$.

In particular, if X is a subspace of Y then $F \cap X \in \mathcal{E}^0_\xi(X)$ whenever $\xi \ge 1$ and $F \in \mathcal{E}^0_\xi(Y)$. If X and Y are separable metrizable spaces and $f : X \to Y$ is Borel measurable, then there is an $\alpha < \omega_1$ such that $f^{-1}[F] \in \mathcal{E}^0_{\alpha+\xi}(X)$ whenever $\xi < \omega_1$ and $F \in \mathcal{E}^0_\xi(Y)$. So if two separable metrizable spaces are Borel isomorphic, one will have countable Borel order iff the other does.

(c) Let X be a topological space. If $Y \in \mathcal{E}^0_\alpha(X)$ and $E \in \mathcal{E}^0_\xi(Y)$, then $E \in \mathcal{E}^0_{\alpha+\xi}(X)$. **P** Induce on ξ. If E is relatively closed in Y, then $E = F \cap Y$ for some closed $F \subseteq X$, so $E \in \mathcal{E}^0_\alpha(X)$, because $\mathcal{E}^0_\alpha(X)$ is closed under $\cap$. Similarly, if E is relatively open in Y, then $E \in \mathcal{E}^0_\alpha(X)$; as $\mathcal{E}^0_\alpha(X)$ is also closed under $\cup$, $\mathcal{E}^0_0(Y) \subseteq \mathcal{E}^0_\alpha(X)$. For the inductive step to $\xi > 0$, take $E \in \mathcal{E}^0_\xi(Y)$, and express E as $\bigcup_{n\in\mathbb{N}} Y \setminus E_n$ where

$$E_n \in \bigcup_{\eta<\xi} \mathcal{E}^0_\eta(Y) \subseteq \bigcup_{\eta<\alpha+\xi} \mathcal{E}^0_\eta(X)$$

for each $n \in \mathbb{N}$. Then $\bigcup_{n\in\mathbb{N}}(X \setminus E_n)$ and $E = Y \cap \bigcup_{n\in\mathbb{N}}(X \setminus E_n)$ belong to $\mathcal{E}^0_{\alpha+\xi}(X)$, so the induction proceeds. **Q**

(d) Let Y be a topological space, $\xi \ge 0$ and $H \in \mathcal{E}^0_\xi(Y)$. Then there is a countable family $\mathcal{V}$ of open subsets of Y such that whenever X is a topological space and $f : X \to Y$ is such that $f^{-1}[V] \in \mathcal{E}^0_0(X)$ for every $V \in \mathcal{V}$, then $f^{-1}[H] \in \mathcal{E}^0_\xi(X)$. **P** Induce on ξ. If $\xi = 0$ then there is a finite family $\mathcal{V}$ of open sets in Y such that H belongs to the algebra of subsets of Y generated by $\mathcal{V}$; now if $f^{-1}[V] \in \mathcal{E}^0_0(X)$ for every $V \in \mathcal{V}$, $f^{-1}[H]$ belongs to the algebra of subsets of X generated by $\{f^{-1}[V] : V \in \mathcal{V}\}$, which is included in $\mathcal{E}^0_0(X)$. For the inductive step to $\xi > 0$, express H as $\bigcup_{n\in\mathbb{N}} X \setminus H_n$ where $H_n \in \bigcup_{\eta<\xi} \mathcal{E}^0_\eta$ for each n. By the inductive hypothesis, there is for each $n \in \mathbb{N}$ a countable family $\mathcal{V}_n$ of open subsets of Y such that $f^{-1}[H_n] \in \bigcup_{\eta<\xi} \mathcal{E}^0_\eta(X)$ whenever $f^{-1}[V] \in \mathcal{E}^0_0(X)$ for every $V \in \mathcal{V}_n$. Set $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$; then if $f^{-1}[V] \in \mathcal{E}^0_0(X)$ for every $V \in \mathcal{V}$,

$$f^{-1}[H] = \bigcup_{n\in\mathbb{N}} X \setminus f^{-1}[H_n] \in \mathcal{E}^0_\xi(X). \textbf{ Q}$$

(e) If X is a topological space and Y is a subspace of X, then $\mathcal{E}^0_\xi(Y) = \{Y \cap E : E \in \mathcal{E}^0_\xi(X)\}$ for every $\xi \le \omega_1$. (Induce on ξ.)

3 Proposition Suppose that X is a topological space and $1 \le \xi < \omega_1$. Let $\langle E_i \rangle_{i \in I}$ be a σ-isolated family in $\mathcal{E}^0_\xi(X)$. Then $\bigcup_{i \in I} E_i$ belongs to $\mathcal{E}^0_\xi(X)$.

proof (a) Consider first the case in which there is a disjoint family $\langle H_i \rangle_{i \in I}$ of open sets such that $E_i \subseteq H_i$ for every i.

(i) If every E_i is expressible as $F_i \setminus F'_i$ where F_i and F'_i are closed, set $F = \overline{\bigcup_{i \in I} H_i \cap F_i}$ and $F' = \overline{\bigcup_{i \in I} H_i \cap F'_i}$. Then $H_i \cap F = H_i \cap F_i$ and $H_i \cap F' = H_i \cap F'_i$ for each i, so

$$\bigcup_{i \in I} E_i = \bigcup_{i \in I} E_i \cap H_i = (F \setminus F') \cap \bigcup_{i \in I} H_i$$

belongs to $\mathcal{E}^0_0(X)$.

(ii) Now induce on ξ. If $\xi = 1$ then we can express each E_i as $\bigcup_{n \in \mathbb{N}} E_{in}$ where each E_{in} is the difference of two closed sets. So (i) tells us that $\bigcup_{i \in I} E_i = \bigcup_{n \in \mathbb{N}} \bigcup_{i \in I} E_i n$ belongs to $\mathcal{E}^0_1(X)$. For the inductive step to countable $\xi > 1$, we can express each E_i as $\bigcup_{n \in \mathbb{N}} X \setminus E_{in}$ where $E_{in} \in \bigcup_{\eta < \xi} \mathcal{E}^0_\eta(X)$ for every n. Then Set $J_{n\eta} = \{: i \in I,\ E_{in} \in \mathcal{E}^0_\eta(X)\}$ and $W_{n\eta} = \bigcup_{i \in J_{n\eta}} H_i \cap E_{in}$ for $n \in \mathbb{N}$, $\eta < \xi$. Then $H_i \cap E_{in} \in \mathcal{E}^0_\eta(X)$ for every $i \in J_{n\eta}$, so $W_{n\eta} \in \mathcal{E}^0_\eta(X)$, by the inductive hypothesis. Now

$$\bigcup_{i \in J_{n\eta}} H_i \setminus E_{in} = (\bigcup_{i \in J_{n\eta}} H_i) \setminus W_{n\eta}$$

belongs to $\mathcal{E}^0_\xi(X)$, for every $n \in \mathbb{N}$ and $\eta < \xi$. So

$$\bigcup_{i \in I} E_i = \bigcup_{i \in I, n \in \mathbb{N}} H_i \setminus E_{in} = \bigcup_{\eta < \xi, n \in \mathbb{N}} \bigcup_{i \in J_{n\eta}} H_i \setminus E_{in}$$

belongs to $\mathcal{E}^0_\xi(X)$, and the induction proceeds.

(b) Thus the proposition is proved if $\langle E_i \rangle_{i \in I}$ can be separated by open sets. Now suppose that it is isolated, that is, that $E_i \subseteq H_i$ where $H_i = X \setminus \overline{\bigcup_{j \in I, j \ne i} E_j}$ for each $i \in I$. Set $Y = X \setminus \bigcup_{i,j \in I, i \ne j} H_i \cap H_j$, so that Y is closed, $\langle H_i \cap Y \rangle_{i \in I}$ is disjoint and $E_i \subseteq H_i \cap Y$ for each i. Then $E_i \in \mathcal{E}^0_\xi(Y)$ for every i, so $\bigcup_{i \in I} E_i \in \mathcal{E}^0_\xi(Y)$, by (a) applied in the subspace Y. As $Y \in \mathcal{E}^0_0(X)$, $E \in \mathcal{E}^0_\xi(X)$, by 2c.

(c) Finally, if $\langle E_i \rangle_{i \in I}$ is σ-isolated, let $\langle I_n \rangle_{n \in \mathbb{N}}$ be a partition of I such that $\langle E_i \rangle_{i \in I_n}$ is isolated for every n; then (b) tells us that $\bigcup_{i \in I} E_i = \bigcup_{n \in \mathbb{N}} \bigcup_{i \in I_n} E_i$ belongs to $\mathcal{E}^0_\xi(X)$.

4 Weakly θ-refinable spaces (a) I say that a topological space X is **weakly θ-refinable** if for every open cover $\mathcal{G}$ of X there is a σ-isolated family of subsets of X refining $\mathcal{G}$ and covering X.

Every metacompact space (in particular, every metrizable space) is weakly θ-refinable (cf. Fremlin 03, 438Lb).

(b) If X is weakly θ-refinable and $\mathcal{G}$ is an open cover of X, then there is a σ-isolated family $\mathcal{A}$ of subsets of X, refining $\mathcal{G}$ and covering X, such that every member of $\mathcal{A}$ is expressible as the difference of two closed sets. **P** Start with any σ-isolated family $\mathcal{C}$ which refines $\mathcal{G}$ and covers X; let $\langle \mathcal{C}_n \rangle_{n \in \mathbb{N}}$ be a partition of $\mathcal{C}$ into isolated sets. For $n \in \mathbb{N}$ and $C \in \mathcal{C}$, choose $G_C \in \mathcal{G}$ including C, and set $H_C = G_C \setminus \overline{\bigcup \mathcal{C}_n \setminus \{C\}}$; now set $Y_n = X \setminus \bigcup\{H_C \cap H_D : C,\ D \in \mathcal{C}_n \text{ are distinct}\}$. Then $C \subseteq H_C \cap Y_n$ for every $C \in \mathcal{C}$, while $\langle H_C \cap Y_n \rangle_{C \in \mathcal{C}_n}$ is isolated. But this means that $\mathcal{A} = \{H_C \cap Y_n : n \in \mathbb{N},\ C \in \mathcal{C}_n\}$ is a σ-isolated family of differences of closed sets, covering X and refining $\mathcal{G}$. **Q**

5 Proposition Suppose that X is a weakly θ-refinable space and $1 \le \xi < \omega_1$. Let $E \subseteq X$ be such that every point of X belongs to an open set G such that $E \cap G \in \mathcal{E}^0_\xi(X)$. Then $E \in \mathcal{E}^0_\xi(X)$.

proof Let $\mathcal{G}$ be the family of open sets $G \subseteq X$ such that $E \cap G \in \mathcal{E}^0_\xi(X)$; then $\mathcal{G}$ is an open cover. So there is a σ-isolated family $\mathcal{A}$ of subsets of X, refining $\mathcal{G}$ and covering X, such that every member of $\mathcal{A}$ is expressible as the difference of two closed sets. In this case, $\langle E \cap A \rangle_{A \in \mathcal{A}}$ is a σ-isolated family, and $E \cap A \in \mathcal{E}^0_\xi(X)$ for every A; by 4b, its union E belongs to $\mathcal{E}^0_\xi(X)$.

6 Lemma (Holický & Spurný 03, Lemma 1) Let X and Y be Hausdorff spaces, and $R \subseteq X \times Y$ an usco-compact relation (see Fremlin 03, §422). If $\langle F_n \rangle_{n \in \mathbb{N}}$ is any sequence of closed subsets of X, there is a selector f for R such that $f^{-1}[F_n] \in \mathcal{E}^0_0(X)$ for every $n \in \mathbb{N}$.

proof For $x \in R^{-1}[Y]$ choose $\langle H_{xn}\rangle_{n\in\mathbb{N}}$ as follows. $H_{x0} = R[\{x\}]$, so that H_{x0} is a non-empty compact subset of Y. Given H_{xn}, set

$$\begin{aligned} H_{x,n+1} &= H_{xn} \cap F_n \text{ if } H_{xn} \cap F_n \neq \emptyset, \\ &= H_{xn} \text{ otherwise.} \end{aligned}$$

Then $\langle H_{xn}\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of non-empty compact sets; choose $f(x) \in \bigcap_{n\in\mathbb{N}} H_{xn}$.

Because $f(x) \in H_{x0}$ for every $x \in R^{-1}[Y]$, f is a selector for R. Now $\{x : H_{xn} \cap F \neq \emptyset\} \in \mathcal{E}^0_0(X)$ for each $n \in \mathbb{N}$ and closed $F \subseteq Y$. **P** Induce on n. For $n = 0$, $\{x : H_{x0} \cap F \neq \emptyset\} = R^{-1}[F]$ is closed because R is usco-compact. For the inductive step to $n + 1$,

$$\begin{aligned} \{x : H_{x,n+1} \cap F \neq \emptyset\} &= \{x : H_{xn} \cap F_n \cap F \neq \emptyset\} \\ &\qquad \cup (\{x : H_{xn} \cap F \neq \emptyset\} \setminus \{x : H_{xn} \cap F_n \neq \emptyset\}) \end{aligned}$$

belongs to $\mathcal{E}^0_0(X)$ by the inductive hypothesis. **Q**

Now $f^{-1}[F_n] = \{x : H_{xn} \cap F_n \neq \emptyset\}$ belongs to $\mathcal{E}^0_0(X)$ for each n.

7 Theorem (Holický & Spurný 03, Theorem 3) Let X and Y be Hausdorff spaces and $R \subseteq X \times Y$ an usco-compact relation. If $\xi \geq 1$ and $H \in \mathcal{E}^0_\xi(Y)$ is such that $R^{-1}[H] \cap R^{-1}[Y \setminus H] = \emptyset$, then $R^{-1}[H] \in \mathcal{E}^0_\xi(X)$.

proof By 2d, there is a countable family $\mathcal{V}$ of open sets in Y such that $f^{-1}[H] \in \mathcal{E}^0_\xi(X)$ whenever $f : X \to Y$ is such that $f^{-1}[V] \in \mathcal{E}^0_0(X)$ for every $V \in \mathcal{V}$. By Lemma 6, there is a selector f for R with this property. Because $R^{-1}[H]$ and $R^{-1}[Y \setminus H]$ are disjoint, $R^{-1}[H] = f^{-1}[H]$ belongs to $\mathcal{E}^0_\xi(X)$.

8 Corollary Let X be a compact non-scattered Hausdorff space. Then $\mathcal{E}^0_\xi(X) \neq \mathcal{E}^0_{\xi+1}(X)$ for all $\xi < \omega_1$.

proof (a) When $X = [0, 1]$ this is the standard theorem that all Borel classes are different; see Jayne & Rogers 80, Theorem 4.3.2, or Kechris 95, .

(b) In general, there is a continuous surjection $g : X \to [0, 1]$ (Fremlin 03, 4A2Gi). Now $R = \{(g(x), x) : x \in X\}$ is an usco-compact relation in $[0, 1] \times X$. Take $E \in \mathcal{E}^0_{\xi+1}([0, 1]) \setminus \mathcal{E}^0_\xi([0, 1])$. Then $H = g^{-1}[E] \in \mathcal{E}^0_{\xi+1}(X)$, because g is continuous. Also $R^{-1}[H] = g[H] = E$ and $R^{-1}[X \setminus H] = [0, 1] \setminus E$ are disjoint and $R^{-1}[H] \notin \mathcal{E}^0_\xi([0, 1])$, so Theorem 4 tells us that $H \notin \mathcal{E}^0_\xi(X)$.

9 Proposition Let X be a Hausdorff space and μ an atomless tight strictly localizable Borel measure on X. Then μ is σ-finite.

proof ? Otherwise, let $\langle X_i\rangle_{i\in I}$ be a decomposition for μ; then there is a family $\langle i_\xi\rangle_{\xi<\omega_1}$ of distinct elements of I such that $\mu X_{i_\xi} > 0$ for each ξ. Let $K_\xi \subseteq X_{i_\xi}$ be a compact set of non-zero measure for each $\xi < \omega_1$. Then K_ξ is not scattered. **P** The subspace measure on K_ξ is τ-additive (Fremlin 03, 411E), so has a support (Fremlin 03, 411Nd) which is a non-empty set without an isolated point. **Q** So there is an $E_\xi \in \mathcal{B}(K_\xi) \setminus \mathcal{E}^0_\xi(K_\xi)$, by Corollary 8. Set $E = \bigcup_{\xi<\omega_1} E_\xi$. Because $\langle X_i\rangle_{i\in I}$ is a decomposition for the measure μ with domain $\mathcal{B}(X)$, $E \in \mathcal{B}(X)$. So there is a $\xi < \omega_1$ such that $E \in \mathcal{E}^0_\xi(X)$. But in this case $E_\xi = E \cap K_\xi$ ought to belong to $\mathcal{E}^0_\xi(K_\xi)$. **X**

10 Corollary Let X be a locally compact Hausdorff topological group, and μ be a left Haar measure on X. Then the Borel measure $\mu\restriction\mathcal{B}(X)$ is strictly localizable iff X is either discrete or σ-compact.

proof: case 1 If X is discrete then $\mu = \mu\restriction\mathcal{B}(X)$ is a multiple of counting measure on X, so is strictly localizable.

case 2 Suppose that X is not discrete. Then μ is atomless. **P?** Otherwise, there is an x such that $\mu\{x\} > 0$ (Fremlin 03, 414G); but now $\{e\} = \{x\}^{-1}\{x\}$ has non-empty interior (Fremlin 03, 443Dc). **X** **Q** If X is σ-compact, then μ and $\mu\restriction\mathcal{B}(X)$ are σ-finite, therefore strictly localizable. If $\mu\restriction\mathcal{B}(X)$ is strictly localizable, then it is σ-finite, by Proposition 9, so there is a dense σ-compact subset of X and X is itself σ-compact.

11 Corollary Let X be a locally compact Hausdorff topological group which is neither discrete nor σ-compact, and μ a left Haar measure on X. Then $\mu\restriction\mathcal{B}(X)$ is not strictly localizable, but its completion is strictly localizable.

proof By Corollary 10, $\mu\restriction\mathcal{B}(X)$ is not strictly localizable. But the completion of $\mu\restriction\mathcal{B}(X)$ is the Radon measure μ (Fremlin 03, 443Jb), which is strictly localizable (Fremlin 03, 416B).

12 For Borel measures which are not tight, there is still something to be said.

Lemma Let X be a separable metrizable space and μ an atomless σ-finite Borel measure on X. Then there is a Borel isomorphism between X and a subset Y of $\mathbb{R}$ which identifies μ with the subspace measure on Y induced by Lebesgue measure.

proof (a) If $\mu X = 0$ this is elementary. The completion $\hat{X}$ of X is a Polish space, so is either countable or Borel isomorphic to $\mathbb{N}^{\mathbb{N}}$ and to the Cantor set (Fremlin 03, 424Cb); in either case $\hat{X}$ is Borel isomorphic to a Lebesgue negligible subset of $[0,1]$, so X also is.

(b) So henceforth let us suppose that $\mu X > 0$. In this case we can find a family $\langle E_\sigma\rangle_{\sigma\in S^*}$ of Borel sets, where $S^* = \bigcup_{n\in\mathbb{N}}\mathbb{N}^n$, such that $E_\emptyset = X$ and, for every $\sigma\in S^*$, $\langle E_{\sigma^\frown i}\rangle_{i\in\mathbb{N}}$ is a partition of E_σ into Borel sets of measure and diameter at most 2^{-n}; moreover, we can do this in such a way that if $\mu E_\sigma > 0$ then $\mu E_{\sigma^\frown i} > 0$ for infinitely many i. Set $X_0 = \bigcup\{E_\sigma : \sigma\in S^*,\ \mu E_\sigma = 0\}$, so that X_0 is a negligible Borel set. Set $X_1 = X\setminus X_0$. Now define s_σ, for $\sigma\in S^*$, by setting $s_\emptyset = 0$ and

$$s_{\sigma^\frown i} = s_\sigma + \textstyle\sum_{j<i}\mu E_{\sigma^\frown j}$$

for each σ; set $t_\sigma = s_\sigma + \mu E_\sigma$ for each σ (allowing $t_\emptyset = \infty$).

(c) For $x\in X$, set

$$f(x) = \sup\{s_\sigma : \sigma\in S^*,\ x\in E_\sigma\} = \inf\{t_\sigma : \sigma\in S^*,\ x\in E_\sigma\}.$$

Then $f : X\to[0,\infty[$ is Borel measurable. Now if $x\in E_\sigma$ and $\mu E_\sigma > 0$, we have $f(x) < t_\sigma$. **P** There are infinitely many i such that $\mu E_{\sigma^\frown i} > 0$, so if $x\in E_{\sigma^\frown j}$ then $f(x)\le t_{\sigma^\frown j} < t_\sigma$. **Q** Thus $f[X_1\cap E_\sigma] = f[X_1]\cap[s_\sigma,t_\sigma[$ for every σ. It follows that $f\restriction X_1$ is injective, and is a Borel isomorphism between X_1 and $f[X_1]$, because $\mu E_\sigma\le 2^{-n}$ and $\operatorname{diam}E_\sigma\le 2^{-n}$ for every $\sigma\in\mathbb{N}^{n+1}$, so that $\mathcal{B}(X)$ is the σ-algebra generated by $\{E_\sigma : \sigma\in S^*\}$ and $\mathcal{B}(f[X])$ is the σ-algebra generated by $\{f[X]\cap[s_\sigma,t_\sigma[,\ \sigma\in S^*\}$.

(d) Now $f\restriction X_1$ is an isomorphism for the subspace measures μ_{X_1} and $\lambda_{f[X_1]}$, where λ is the restriction of Lebesgue measure to $\mathcal{B}(\mathbb{R})$. **P** Let ν be the Borel measure $(\mu f^{-1})\restriction\mathcal{B}(\mathbb{R})$, where μf^{-1} is the image measure on $\mathbb{R}$. Then

$$\begin{aligned}\nu\,[s_\sigma,t_\sigma[&= \mu f^{-1}[\,[s_\sigma,t_\sigma[\,] = \mu(X_1\cap f^{-1}[\,[s_\sigma,t_\sigma[\,])\\ &= \mu(X_1\cap E_\sigma) = \mu E_\sigma = t_\sigma - s_\sigma\end{aligned}$$

for every $\sigma\in S^*$. Because $t_\sigma - s_\sigma\le 2^{-n}$ whenever $\sigma\in\mathbb{N}^{n+1}$, $\nu\,[s,t[= t-s$ whenever $0\le s\le t\le t_\emptyset$ and ν agrees with λ on $\mathcal{B}([0,t_\emptyset[)$. If now $E\subseteq X_1$ is a Borel set, then $f[E]$ is of the form $F\cap f[X_1]$ for some Borel set $F\subseteq\mathbb{R}$; we can suppose that $F\subseteq[0,t_\emptyset[$ and $\lambda F = \lambda^*(f[E]) = \lambda_{f[X_1]}(f[E])$. Now

$$\begin{aligned}\mu E &= \mu(X_1\cap f^{-1}[f[E]]) = \mu(X_1\cap f^{-1}[F])\\ &= \mu f^{-1}[F] = \nu F = \lambda F = \lambda_{f[X_1]}(f[E]).\ \mathbf{Q}\end{aligned}$$

(e) By (a), we have a Borel isomorphism g between X_0 and a negligible subset of $[-2,-1]$. Set

$$\begin{aligned}h(x) &= f(x) \text{ if } x\in X_1,\\ &= g(x) \text{ if } x\in X_0;\end{aligned}$$

then h is a Borel isomorphism between X and $Y = f[X_1]\cup g[X_0]$, and also is an isomorphism between μ and the subspace Borel measure λ_Y.

13 Proposition The following are equiveridical:

(i) every atomless strictly localizable Borel measure on a metrizable space is σ-finite;

(ii) if X is metrizable and there is an atomless Borel probability measure on X then the Borel order of X is ω_1;

(iii)(α) there is no atomlessly-measurable cardinal,

(β) if $A \subseteq \mathbb{R}$ is not Lebesgue negligible then the Borel order of A is ω_1.

proof $\neg$**(ii)**$\Rightarrow \neg$**(i)** Suppose that (ii) is false; let X be a metrizable space such that $\mathcal{E}^0_\alpha(X) = \mathcal{B}(X)$, where $1 \leq \alpha < \omega_1$, and there is an atomless Borel probability measure μ on X. Let I be an uncountable set and λ the direct-sum measure on $X \times I$, so that $\lambda W = \sum_{i \in I} \mu W^{-1}[\{i\}]$ whenever this is defined. Then λ is atomless and strictly localizable but not σ-finite. If we give I its discrete topology and $X \times I$ the product topology, then of course λ is a topological measure; the point is that its domain is exactly $\mathcal{B}(X \times I)$. **P** If $W \subseteq X \times I$ and $W^{-1}[\{i\}] \in \mathcal{B}(X)$ for every $i \in I$, then W belongs to $\mathcal{E}^0_\alpha \mathcal{B}(X \times I)$, by Proposition 3, so is a Borel set. **Q**

Thus λ is a Borel measure and witnesses that (i) is false.

(ii)$\Rightarrow$**(i)** Repeat the argument of Proposition 9, but using hypothesis (ii) in place of Corollary 8.

(ii)$\Rightarrow$**(iii)(α)** **?** Suppose that (ii) is true and that κ is an atomlessly-measurable cardinal. Let ν be a witnessing probability on κ. If κ is given its discrete topology, then ν is an atomless Borel measure; but $\mathcal{B}(\kappa) = \mathcal{E}^0_0(\kappa)$, so this is impossible. **X**

(β) If $A \subseteq \mathbb{R}$ is not Lebesgue measurable, then we can find a Borel probability measure on A, so $\mathcal{B}(A) \neq \mathcal{E}^0_\xi(A)$ for every $\xi < \omega_1$.

(iii)$\Rightarrow$**(ii)** Assume (iii). Note first that (iii)(β) implies that whenever X is a separable metrizable space with a non-zero atomless totally finite Borel measure, the Borel order of X is ω_1. **P** By Lemma 12, there is a measure-preserving Borel isomorphism between X and a subset Y of $\mathbb{R}$ with the subspace measure induced by Lebesgue measure; now Y has Borel order ω_1 so X also does, by the last remark in 2b. **Q**

Now let X be any metrizable space, and μ an atomless Borel probability measure on X. Let $\mathcal{U}$ be a σ-disjoint base for the topology of X; express $\mathcal{U}$ as $\bigcup_{n \in \mathbb{N}} \mathcal{U}_n$ where each $\mathcal{U}_n$ is a disjoint family of open sets; set $\mathcal{V}_n = \mathcal{U}_n \cup \{X \setminus \bigcup \mathcal{U}_n\}$ so that $\mathcal{V}_n$ is a partition of X and $\bigcup \mathcal{V}$ is a Borel set for every $\mathcal{V} \subseteq \mathcal{V}_n$. Set $\nu_n \mathcal{V} = \mu(\bigcup \mathcal{V})$ for $\mathcal{V} \subseteq \mathcal{V}_n$; then ν_n is a probability measure with domain $\mathcal{P}\mathcal{V}_n$. Because there is no atomlessly-measurable cardinal, ν_n is purely atomic and $\mathfrak{c}^+$-additive (Fremlin 07?, 543Bf). Let $\langle \mathcal{V}_{ni} \rangle_{i \in I_n}$ be a partition of $\mathcal{V}_n$ into ν_n-atoms.

Let $\mathcal{I}$ be the σ-ideal of subsets of X generated by $\bigcup_{n \in \mathbb{N}} \{\bigcup \mathcal{V} : \mathcal{V} \subseteq \mathcal{V}_n,\ \nu_n \mathcal{V} = 0\}$. Then $\mathcal{I}$ is $\mathfrak{c}^+$-additive and every member of $\mathcal{I}$ is μ-negligible.

Set $Z = \prod_{n \in \mathbb{N}} I_n$, where each I_n is given its discrete topology, so that Z is separable and metrizable. Define $f : X \to Z$ by saying that $f(x)(n) = i$ if $x \in \bigcup \mathcal{V}_{ni}$. Then f is Borel measurable. Now for every open $G \subseteq X$ there is an open $H \subseteq Z$ such that $G \triangle f^{-1}[H] \in \mathcal{I}$. **P** For each $n \in \mathbb{N}$ set $\mathcal{W}_n = \{U : U \in \mathcal{U}_n,\ U \subseteq G\}$, so that $\mathcal{W}_n \subseteq \mathcal{V}_n$ and $G = \bigcup_{n \in \mathbb{N}} \bigcup \mathcal{W}_n$. For each n, set $J_n = \{i : i \in I_n,\ \nu_n(\mathcal{V}_{ni} \setminus \mathcal{W}_n) = 0\}$ and $\mathcal{W}'_n = \bigcup_{i \in J_n} \mathcal{V}_{ni}$, so that $\nu_n(\mathcal{W}_n \triangle \mathcal{W}'_n) = 0$ and $\bigcup \mathcal{W}_n \triangle \bigcup \mathcal{W}'_n \in \mathcal{I}$. Set $H = \bigcup_{n \in \mathbb{N}} \{z : z \in Z,\ z(n) \in J_n\}$; then $f^{-1}[H] = \bigcup_{n \in \mathbb{N}} \mathcal{W}'_n$ so $G \triangle f^{-1}[H] \in \mathcal{I}$. **Q**

It follows that if $\xi < \omega_1$ and $E \in \mathcal{E}^0_\xi(X)$ there is an $F \in \mathcal{E}^0_\xi(Z)$ such that $E \triangle f^{-1}[F] \in \mathcal{I}$. **P** Induce on ξ. **Q**

Set $Z_1 = \{z : z \in Z,\ f^{-1}[\{z\}] \notin \mathcal{I}\}$. Then $f^{-1}[Z \setminus Z_1] \in \mathcal{I}$ so $f^{-1}[Z_1]$ is conegligible. We therefore have a Borel probability measure λ on Z_1 defined by saying that $\lambda F = \mu f^{-1}[F]$ for every $F \in \mathcal{B}(Z_1)$. Also λ is atomless. **P** For any $n \in \mathbb{N}$ we have a partition of X into Borel sets $E_0, \dots, E_n$ of measure $\frac{1}{n+1}$. Now for each $i \leq n$ we can find a Borel set $F_i \subseteq Z$ such that $E_i \triangle f^{-1}[F_i]$ is negligible. But this means that $\lambda(F_i \cap Z_1) = \frac{1}{n+1}$ for each i and $\lambda(Z_1 \setminus \bigcup_{i \leq n} F_i) = 0$. So λ can have no atom of measure greater than $\frac{1}{n+1}$. **Q**

For any $\xi < \omega_1$, there is a Borel set $F \subseteq Z$ such that $F \cap Z_1 \notin \mathcal{E}^0_\xi(Z_1)$. Set $E = f^{-1}[F] \in \mathcal{B}(X)$. **?** If $E \in \mathcal{E}^0_\xi(X)$, let $F' \in \mathcal{E}^0_\xi(Z)$ be such that $E \triangle f^{-1}[F'] \in \mathcal{I}$. Then $f^{-1}[F \triangle F'] \in \mathcal{I}$ so $F \triangle F'$ does not meet Z_1 and $F \cap Z_1 = F' \cap Z_1 \in \mathcal{E}^0_\xi(Z_1)$. **X**

Thus $\mathcal{B}(X) \neq \mathcal{E}^0_\xi(X)$ for any $\xi < \omega_1$; as X and μ are arbitrary, (iii) is true.

Remark Compare Theorem 1.1 of Džamonja & Kunen 95.

14 Examples (a) Suppose there is a Sierpiński set; that is, an uncountable subset A of $\mathbb{R}$ such that every Lebesgue negligible subset of A is countable. Then every Borel subset of A is the union of a relatively F_σ subset and a countable set, so is in fact F_σ and G_δ; in the language of §1, $\mathcal{B}(A) = \mathcal{E}_1^0(A)$. So 13(iii)($\beta$) is false.

(b) Martin's Axiom, without CH, does not give us a Sierpiński set in this sense, but if $\mathfrak{p}$ is equal to $\mathrm{cf}\,\mathcal{N}$, the cofinality of the Lebesgue null ideal, then we do have a set $A \subseteq \mathbb{R}$ of non-zero measure such that every negligible subset of A has cardinal less than $\mathfrak{p}$; it follows that every negligible subset of A is a G_δ set in A (use Fremlin 84, 23B), so that every Borel set is the difference of two G_δ sets and and $\mathcal{B}(A) = \mathcal{E}_2^0(A)$.

(c) On the other side, Miller 79 describes a model in which $\mathcal{B}(A) \ne \mathcal{E}_\xi^0(A)$ for any uncountable $A \subseteq \mathbb{R}$ and any $\xi < \omega_1$. His model can be built by forcing from a model of V=L so will have no atomlessly-measurable cardinals and will satisfy 13(iii).

15 Scattered spaces In certain types of scattered space we can be sure that the Borel order is small. In particular, we have the following.

(a) If X is any topological space, we can define its **Cantor-Bendixson derivatives** $\langle X_\xi\rangle_{\xi<\mathrm{On}}$ as follows. $X_0 = X$; $X_{\xi+1}$ is the set of non-isolated points of X_ξ; $X_\xi = \bigcap_{\eta<\xi} X_\eta$ for non-zero limit ordinals ξ. Now $\langle X_\xi\rangle_{\xi<\mathrm{On}}$ is a non-increasing family of closed subsets of X. X is scattered iff there is some $\zeta \in \mathrm{On}$ such that $X_\zeta = \emptyset$; the first such ζ is the **Cantor-Bendixson rank** $\mathrm{CB}(X)$ of X. Note that if $Y \subseteq X$ has Cantor-Bendixson derivatives Y_ξ, then $Y_\xi \subseteq X_\xi$ for every ξ, so that $\mathrm{CB}(Y) \le \mathrm{CB}(X)$, with $\mathrm{CB}(Y) \le \xi$ if $Y \subseteq X \setminus X_\xi$. If Y is open, then $Y_\xi = Y \cap X_\xi$ for every ξ.

(b) If X is scattered and hereditarily weakly θ-refinable (in particular, if X is any scattered metrizable space) then $\mathcal{B}(X) = \mathcal{E}_1^0(X)$. **P** Induce on $\xi = \mathrm{CB}(X)$. Let $\langle X_\eta\rangle_{\eta<\mathrm{On}}$ be the Cantor-Bendixson derivatives of X. If $\xi = 0$ the result is trivial. For the inductive step to $\xi = \eta+1$, any Borel subset of X is the union of a subset of X_η with a Borel subset of $X \setminus X_\eta$; the former is relatively open in X_η so belongs to $\mathcal{E}_0^0(X)$; the latter is in $\mathcal{E}_1^0(X \setminus X_\eta) \subseteq \mathcal{E}_1^0(X)$ by the inductive hypothesis (noting that $X \setminus X_\eta$ is still hereditarily weakly θ-refinable and $\mathrm{CB}(X \setminus X_\eta) \le \eta$). For the inductive step to a non-zero limit ordinal ξ, then X is weakly θ-refinable and $\{X \setminus X_\eta : \eta < \xi\}$ is an open cover of X. By the inductive hypothesis,

$$E \setminus X_\eta \in \mathcal{E}_1^0(X \setminus X_\eta) \subseteq E_1^0(X)$$

for every $\eta < \xi$; by Proposition 8, $E \in \mathcal{E}_1^0(X)$. Thus the induction continues. **Q**

(c) If X is a totally ordered set which is scattered in its order topology (in particular, if X is well-ordered), then $\mathcal{B}(X) = \mathcal{E}_1^0(X)$. **P** Again induce on $\xi = \mathrm{CB}(X)$. As in (b), the case $\xi = 0$ and the inductive step to a successor ordinal ξ are elementary. For the inductive step to a non-zero limit ordinal ξ, let $\hat{X}$ be the Dedekind completion of X and $\langle X_\eta\rangle_{\eta<\mathrm{On}}$ the family of Cantor-Bendixson derivatives of X; set $Z = \bigcap_{\eta<\xi} \overline{X}_\xi$, the closures being taken in $\hat{X}$, so that $X \cap Z = \emptyset$. For the rest of this argument, all intervals $[a, b]$, $[x, v[$ etc. are to be interpreted as intervals in $\hat{X}$ rather than in X. Note that because every member of $\hat{X} \setminus X$ is approximated both from above and from below by a member of X, the order topology of X is just the subspace topology inducted by the order topology of $\hat{X}$.

Express $\hat{X} \setminus Z$ as $\bigcup \mathcal{G}$ where $\mathcal{G}$ is a disjoint family of open intervals in $\hat{X}$; by Proposition 3 as usual, it will be enough to show that $\mathcal{B}(X \cap G) = \mathcal{E}_1^0(X \cap G)$ for every $G \in \mathcal{G}$. Take $G \in \mathcal{G}$ such that $X \cap G \ne \emptyset$, and express G as $]u, v[$ where u, $v \in \hat{X} \cup \{-\infty, \infty\}$. Fix $x_0 \in X \cap G$; it will be enough if I can show that $\mathcal{B}(X \cap [x_0, v[)$ and $\mathcal{B}(X \cap]u, x_0])$ are both included in $\mathcal{E}_1^0(X)$; and of course it will be enough to consider just one of these, say the first. Note that if $a \le b$ in $\hat{X}$ then $[a, b]$ is compact, so if $x_0 \le a \le b < v$ there is some $\eta < \xi$ such that $[a, b] \cap \overline{X}_\eta = \emptyset$ and $\mathrm{CB}(X \cap [a, b]) < \xi$ and $\mathcal{B}(X \cap [a, b]) = \mathcal{E}_1^0(X \cap [a, b])$.

case 1 If the cofinality $\mathrm{cf}\,[x_0, v[$ of the half-open interval $[x_0, v[$ is countable, let $\langle y_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $[x_0, v[$ such that $[x_0, v[= \bigcup_{n\in\mathbb{N}}[x_0, y_n]$. Now $\mathcal{B}(X \cap [x_0, y_n]) = \mathcal{E}_1^0(X \cap [x_0, y_n]) \subseteq \mathcal{E}_1^0(X)$ for every n, so $\mathcal{B}(X \cap [x_0, v[) \subseteq \mathcal{E}_1^0(X)$.

case 2 If $\mathrm{cf}\,[x_0, v[$ is uncountable, let $\mathcal{F}$ be the family of closed cofinal subsets of $[x_0, v[$. Then every closed subset of $[x_0, v[$ either belongs to $\mathcal{F}$ or is disjoint from a member of $\mathcal{F}$, while $\mathcal{F}$ is closed under countable intersections, because every non-decreasing sequence in $[x_0, v[$ has an upper bound in $[x_0, v[$. So every Borel subset of $[x_0, v[$ either includes a member of $\mathcal{F}$ or is disjoint from a member of $\mathcal{F}$. If $E \in \mathcal{B}(X)$

and $E \subseteq [x_0, v[\setminus F$ for some $F \in \mathcal{F}$, then $]x_0, v[\setminus F$ is a disjoint union of open intervals with closures disjoint from Z, so $E \in \mathcal{E}_1^0(X)$ by the inductive hypothesis and Proposition 3. For any other $E \in \mathcal{B}(X \cap [x_0, v[)$, there is an $F \in \mathcal{F}$ such that $F \cap X \subseteq E$, so that $E = (F \cap X) \cup (E \setminus F) \in \mathcal{E}_1^0(X)$.

So the induction proceeds. **Q**

16 We cannot extend the results of §15 even to all scattered compact spaces. The following example is based on the construction of Fremlin 03, 439K, itself derived from Juhász Kunen & Rudin 76; I am grateful to R.Pol for the hint.

Example Let $\mathfrak{S}$ be the usual topology on $Z = \{0,1\}^{\mathbb{N}}$.

(a) There is a locally compact, locally countable topology $\mathfrak{T}$ on Z, finer than $\mathfrak{S}$, such that whenever $E \in \mathcal{B}(Z, \mathfrak{S})$ is uncountable and $\langle A_n \rangle_{n \in \mathbb{N}}$ is a sequence of subsets of Z, then there is a point x of E such that $\{n : x \in \overline{A}_n^{(\mathfrak{S})}\} = \{n : x \in \overline{A}_n^{(\mathfrak{T})}\}$.

(b) If $E \in \mathcal{B}(Z, \mathfrak{S}) \cap \mathcal{E}_\xi^0(Z, \mathfrak{T})$, where $\xi < \omega_1$, then $E \in \mathcal{E}^0_{\max(\xi,2)}(Z, \mathfrak{S})$.

(c) If X is the one-point compactification of $(Z, \mathfrak{T})$ then X is a compact scattered space with uncountable Borel order.

proof (a)(i) $\mathfrak{T}$ will be the last in a family $\langle \mathfrak{T}_\xi \rangle_{\xi \le \mathfrak{c}}$ of topologies. We must begin by choosing a family $\langle (E_\xi, \langle I_{\xi n} \rangle_{n \in \mathbb{N}}) \rangle_{\xi < \mathfrak{c}}$ running over $\mathcal{B}(Z, \mathfrak{S}) \times ([Z]^{\le \omega})^{\mathbb{N}}$ with cofinal repetitions. Let $\langle z_\xi \rangle_{\xi < \mathfrak{c}}$ enumerate Z. Together with $\langle \mathfrak{T}_\xi \rangle_{\xi \le \mathfrak{c}}$ we choose simultaneously families $\langle x_\xi \rangle_{\xi < \mathfrak{c}}$ and $\langle y_\xi \rangle_{\xi < \mathfrak{c}}$ in Z and the inductive hypothesis will be

$\mathfrak{T}_\xi$ is a topology on $X_\xi = \{x_\eta : \eta < \xi\} \cup \{y_\eta : \eta < \xi\}$ finer than the topology on X_ξ induced by $\mathfrak{S}$;

if $\eta < \xi \le \mathfrak{c}$, then $X_\eta \in \mathfrak{T}_\xi$ and $\mathfrak{T}_\eta$ is the subspace topology on X_η induced by $\mathfrak{T}_\xi$;

every point of X_ξ belongs to a countable set which is compact and open for $\mathfrak{T}_\xi$.

The induction starts with $X_0 = \emptyset$, $\mathfrak{T}_0 = \{\emptyset\}$.

(ii) *Inductive step to a successor ordinal* Suppose that we have found X_ξ and $\mathfrak{T}_\xi$ where $\xi < \mathfrak{c}$. Take $y_\xi = z_\xi$ if $z_\xi \notin X_\xi$, and otherwise y_ξ any point of $Z \setminus X_\xi$.

(α) Examine the $\mathfrak{S}$-Borel set E_ξ and the sequence $\langle I_{\xi n} \rangle_{n \in \mathbb{N}}$. If *either* $\bigcup_{n \in \mathbb{N}} I_{\xi n} \not\subseteq X_\xi$ *or* E_ξ is countable, take x_ξ to be any point of $Z \setminus (X_\xi \cup \{y_\xi\})$ and set $K_m = \emptyset$ for every m before proceeding to (γ) below.

(β) If $I_{\xi n} \subseteq X_\xi$ for every n and E_ξ is uncountable, then $\#(E_\xi) = \mathfrak{c}$; take any $x_\xi \in E_\xi \setminus (X_\xi \cup \{y_\xi\})$. Set $J = \{n : x_\xi \in \overline{I}_{\xi n}^{(\mathfrak{S})}\}$. If $J = \emptyset$, set $K_m = \emptyset$ for every m and proceed to (γ). Otherwise, let $\langle n_m \rangle_{m \in \mathbb{N}}$ run over J with cofinal repetitions. Let $\langle t_m \rangle_{m \in \mathbb{N}}$ be a sequence in Z such that $t_m \restriction m = x_\xi \restriction m$ for every $m \in \mathbb{N}$ and $t_m \in I_{\xi n_m}$ whenever $m \in \mathbb{N}$. (Thus, for each $n \in J$, $t_m \in I_{\xi n}$ for infinitely many m, while $\langle t_m \rangle_{m \in \mathbb{N}} \to x_\xi$ for $\mathfrak{S}$.) By the inductive hypothesis, we can find countable sets $K_m \subseteq X_\xi$, compact and open for $\mathfrak{T}_\xi$, such that $t_m \in K_m$ for each m. Because $\{t : t \in X_\xi,\ t\restriction m = x_\xi \restriction m\}$ is open-and-closed for $\mathfrak{T}_\xi$ and contains t_m, we may suppose that $t \restriction m = x_\xi \restriction m$ for every $t \in K_m$.

(γ) Let $\mathfrak{T}_{\xi+1}$ be the topology on $X_{\xi+1} = X_\xi \cup \{x_\xi, y_\xi\}$ generated by

$$\mathfrak{T}_\xi \cup \{\{y_\xi\}\} \cup \{L_n : n \in \mathbb{N}\},$$

where $L_n = \{x_\xi\} \cup \bigcup_{m \ge n} K_m$ for each n.

(δ) Because $\mathfrak{T}_\xi \subseteq \mathfrak{T}_{\xi+1}$, X_ξ will be open in $X_{\xi+1}$. Because the K_m are always $\mathfrak{T}_\xi$-open, and x_ξ, y_ξ are distinct points of $Z \setminus X_\xi$, the topology on X_ξ induced by $\mathfrak{T}_{\xi+1}$ is just $\mathfrak{T}_\xi$. Consequently (by the inductive hypothesis) the topology on X_η induced by $\mathfrak{T}_{\xi+1}$ is $\mathfrak{T}_\eta$ for every $\eta \le \xi$. We have $t \restriction n = x_\xi \restriction n$ for every $t \in L_n$, so $\mathfrak{T}_{\xi+1}$ is finer than $\mathfrak{S}$ on $X_{\xi+1}$.

If $x \in X_\xi$, then there is a countable $\mathfrak{T}_\xi$-open $\mathfrak{T}_\xi$-compact set containing x, which is still $\mathfrak{T}_{\xi+1}$-open and $\mathfrak{T}_{\xi+1}$-compact. Of course $\{y_\xi\}$ is a countable $\mathfrak{T}_{\xi+1}$-open $\mathfrak{T}_{\xi+1}$-compact set containing y_ξ. As for x_ξ, L_0 is surely countable and $\mathfrak{T}_{\xi+1}$-open. To see that it is $\mathfrak{T}_{\xi+1}$-compact, observe that any ultrafilter containing L_0 either contains every L_n, and converges to x_ξ, or contains some K_m and converges to a point of K_m.

Thus the induction proceeds at successor stages.

(iii) *Inductive step to a limit ordinal* If $\xi \le \mathfrak{c}$ is a non-zero limit ordinal, then we have $X_\xi = \bigcup_{\eta<\xi} X_\eta$, and can take $\mathfrak{T}_\xi$ to be the topology generated by $\bigcup_{\eta<\xi} \mathfrak{T}_\eta$. It is easy to check that this works (because the topologies $\mathfrak{T}_\eta$ are consistent with each other).

(iv) At the end of the induction, we have $X_\mathfrak{c} = Z$ because $z_\xi \in X_{\xi+1} \subseteq X_\mathfrak{c}$ for every ξ. The final topology $\mathfrak{T} = \mathfrak{T}_\mathfrak{c}$ on Z is finer than $\mathfrak{S}$, locally countable and locally compact. Now let $E \in \mathcal{B}(Z,\mathfrak{S})$ be uncountable and $\langle A_n\rangle_{n\in\mathbb{N}}$ a sequence of subsets of Z. For each n, let I_n be a countable dense subset of A_n. Then there is a $\zeta < \mathfrak{c}$ such that $I_n \subseteq X_\zeta$ for every $n \in \mathbb{N}$, and a $\xi \ge \zeta$ such that $E_\xi = E$, $I_{\xi n} = I_n$ for every n. In this case, we shall have constructed $X_{\xi+1}$ and $\mathfrak{T}_{\xi+1}$ such that $x_\xi \in E$ and there is a sequence $\langle t_m\rangle_{m\in\mathbb{N}}$ in X_ξ, converging to x_ξ for $\mathfrak{T}_{\xi+1}$, such that $\{m : t_m \in I_n\}$ is infinite whenever $x_\xi \in \overline{I}_n^{(\mathfrak{S})}$; so that, for $n \in \mathbb{N}$,

$$\begin{aligned} x_\xi \in \overline{A}_n^{(\mathfrak{S})} &\Longrightarrow x_\xi \in \overline{I}_n^{(\mathfrak{S})} \\ &\Longrightarrow x_\xi \in \overline{I}_n^{(\mathfrak{T}_{\xi+1})} \\ &\Longrightarrow x_\xi \in \overline{I}_n^{(\mathfrak{T})} \\ &\Longrightarrow x_\xi \in \overline{A}_n^{(\mathfrak{T})} \\ &\Longrightarrow x_\xi \in \overline{A}_n^{(\mathfrak{S})}. \end{aligned}$$

Thus $\mathfrak{T}$ has the required properties.

(b) Take $E \in \mathcal{B}(Z,\mathfrak{S}) \cap \mathcal{E}_\xi^0(Z,\mathfrak{T})$. There is a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{T}$ such that whenever W is a topological space and $f : W \to Z$ is such that $f^{-1}[H_n] \in \mathcal{E}_0^0(W)$ for every $n \in \mathbb{N}$, then $f^{-1}[E] \in \mathcal{E}_\xi^0(W)$, by 2d. Let W be the set of points $x \in Z$ such that $\{n : x \in \overline{Z \setminus H_n}^{(\mathfrak{S})}\} = \{n : x \in \overline{Z \setminus H_n}^{(\mathfrak{T})}\}$. Then every $W \cap H_n$ is open for the subspace topology on W induced by $\mathfrak{S}$, so $E \cap W \in \mathcal{E}_\xi^0(W,\mathfrak{S})$, and there is an $F \in \mathcal{E}_\xi^0(Z,\mathfrak{S})$ such that $E \cap W = F \cap W$, by 2e. But this means that $E \triangle F$ is an $\mathfrak{S}$-Borel subset of Z not meeting W. By (a), $E \triangle F$ is countable. So E is of the form $(F \cup F_0) \cap F_1$ where F_0 is countable and F_1 is cocountable; in this case, $F_0 \in \mathcal{E}_1^0(Z,\mathfrak{S})$ and $F_1 \in \mathcal{E}_2^0(Z,\mathfrak{S})$; so $E \in \mathcal{E}_{\max(2,\xi)}^0(Z,\mathfrak{S})$.

(c) Of course $\mathcal{B}(Z,\mathfrak{S}) \subseteq \mathcal{B}(Z,\mathfrak{T})$, while $\mathcal{B}(Z,\mathfrak{S}) \ne \mathcal{E}_\xi^0(Z,\mathfrak{S})$ for any $\xi < \omega_1$, so $\mathcal{B}(Z,\mathfrak{T}) \ne \mathcal{E}_\xi^0(Z,\mathfrak{T})$ for any $\xi < \omega_1$. It follows at once that X also has uncountable Borel order. Finally, Z is scattered because it is locally compact and locally countable, so every point of Z has a scattered neighbourhood; so $X = Z \cup \{\infty\}$ is the union of two scattered subspaces and is scattered.

Remark Note that if we combine the method of (a) above with the ideas of Fremlin 03, 439K, then we can arrange that X has an atomless Borel probability measure.

17 Problem Is there a ZFC example of a topological space X with countable Borel order which has an atomless Borel probability measure? I shall be surprised if there isn't one, but don't know where to look. The arguments of (i)⇔(iii) in Proposition 13 apply in general, so this is the same as asking for a ZFC example of an atomless Borel measure which is strictly localizable but not σ-finite.

Note that ω_1 has Borel order 1, carries no atomless Borel probability measure, but is not 'universally negligible' on the definition of Fremlin 03, 439B, because singletons are negligible for Dieudonné's measure. The method of Fremlin 03, 439K, always gives us a scattered compact Hausdorff space with an atomless Borel probability measure, but I suspect that even without the refinement in §16 here it has uncountable Borel order.

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