Homogeneous measure spaces

D.H.Fremlin

University of Essex, Colchester, England

1 Introduction I will say that a σ-finite measure space (X,Σ,μ) is **homogeneous** if whenever E, $F\in\Sigma$ have the same non-zero measure the subspace measures μ_E, μ_F are isomorphic. I will say that it is **strongly homogeneous** if whenever E, $F\in\Sigma$ have non-zero finite measure then the subspace measures are isomorphic up to a scalar multiple of the measure.

Lebesgue measure on $\mathbb{R}^r$ is strongly homogeneous; counting measure on $\mathbb{N}$ is homogeneous but not strongly homogeneous. But it seems harder to construct an atomless homogeneous measure which is not strongly homogeneous.

2 Proposition Suppose that the additivity of Lebesgue measure is $\mathfrak{c}$. Then there is a complete atomless homogeneous probability measure (X,T,ν) which is not strongly homogeneous; in fact, for V, $W\in\mathrm{T}$ of different measures, the structures $(V,\mathrm{T}_V,\mathcal{N}(\nu_V))$ and $(W,\mathrm{T}_W,\mathcal{N}(\nu_W))$ are non-isomorphic, where T_V is the subspace σ-algebra, ν_V the subspace measure and $\mathcal{N}(\nu_V)$ is its null ideal.

proof (a) Write μ for Lebesgue measure on $[0,1]$ and Σ for its domain. Let $\langle g_\xi\rangle_{\xi<\mathfrak{c}}$ run over the family of functions g such that $\operatorname{dom} g$ is a Borel subset of $[0,1]$, g is injective and Borel measurable and $\mu(g[\operatorname{dom} g])\ne\mu(\operatorname{dom} g)$. Let $\langle(E_\xi,F_\xi)\rangle_{\xi<\mathfrak{c}}$ run over all pairs (E,F) such that E, $F\subseteq[0,1]$ are Borel sets and $0<\mu E=\mu F<1$. Let $\langle G_\xi\rangle_{\xi<\mathfrak{c}}$ run over all negligible Borel sets.

(b) Choose $\langle f_\xi\rangle_{\xi<\mathfrak{c}}$, $\langle\Phi_\xi\rangle_{\xi<\mathfrak{c}}$, $\langle x_\xi\rangle_{\xi<\mathfrak{c}}$ and $\langle H_\xi\rangle_{\xi<\mathfrak{c}}$ inductively, as follows. Given that $f_\eta:[0,1]\to[0,1]$ is an automorphism for Lebesgue measure and H_η is negligible for every $\eta<\xi$, let Φ_ξ be the subgroup of bijections from $[0,1]$ to itself generated by $\{f_\eta:\eta<\xi\}$. Then there is an $x_\xi\in\operatorname{dom} g_\xi$ such that $g_\xi(x_\xi)\ne f(x_\xi)$ for any $f\in\Phi_\xi$. **P** Set $U_f=\{x:x\in\operatorname{dom} g_\xi,\ g_\xi(x)=f(x)\}$ for each $f\in\Phi_\xi$. Because g_ξ and f are both Σ-measurable, $U_f\in\Sigma$. Because $\#(\Phi_\xi)<\operatorname{add}\mu$, we have a disjoint family $\langle V_f\rangle_{f\in\Phi_\xi}$ of measurable sets such that $V_f\subseteq U_f$ for every f and $\bigcup_{f\in\Phi_\xi}V_f=\bigcup_{f\in\Phi_\xi}U_f=V$ say. Now

$$\mu(g[V])=\textstyle\sum_{f\in\Phi_\xi}\mu(g[V_f])=\sum_{f\in\Phi_\xi}\mu(f[V_f])=\sum_{f\in\Phi_\xi}\mu V_f=\mu V.$$

So $V\ne\operatorname{dom} g$ and we can find an $x_\xi\in\operatorname{dom} g\setminus V$. **Q**

Let H_ξ be a negligible set such that $H_\xi\cap E_\xi$ and $H_\xi\cap F_\xi$ both have cardinal $\mathfrak{c}$, $x_\xi\in H_\xi$, $G_\xi\cup\bigcup_{\eta<\xi}H_\eta\subseteq H_\xi$ and $f[H_\xi]=H_\xi$ for every $f\in\Phi_\xi$. Now $E_\xi\setminus H_\xi$ and $F_\xi\setminus H_\xi$ have the same non-zero finite measure, so their subspace measures are isomorphic; similarly, $([0,1]\setminus E_\xi)\setminus H_\xi$ and $([0,1]\setminus F_\xi)\setminus H_\xi$ have isomorphic subspace measures. There is therefore an automorphism f_ξ of $([0,1],\Sigma,\mu)$ such that $f_\xi(x)=x$ for every $x\in H_\xi$ and $f_\xi[E_\xi\setminus H_\xi]=F_\xi\setminus H_\xi$. Continue.

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(c) At the end of the induction, let $\Phi=\bigcup_{\xi<\mathfrak{c}}\Phi_\xi$ be the group generated by $\{f_\xi:\xi<\mathfrak{c}\}$ and $\sim$ the corresponding orbit equivalence relation on $[0,1]$.

If $\xi<\mathfrak{c}$ then $E_\xi\cap H_\xi$ and $F_\xi\cap H_\xi$ meet $\mathfrak{c}$ different $\sim$-equivalence classes. **P** Since $f_\eta(x)=x$ for every $x\in H_\xi$ and $\eta\ge\xi$, while $f_\eta[H_\xi]=H_\xi$ for every $\eta<\xi$, H_ξ is a union of $\sim$-equivalence classes, all of cardinal at most $\max(\omega,\#(\xi))$. Since $E_\xi\cap H_\xi$ and $F_\xi\cap H_\xi$ both have cardinal $\mathfrak{c}$, they must both meet $\mathfrak{c}$ equivalence classes. **Q**

Note that if $\xi<\mathfrak{c}$, then $x_\xi\in H_\xi$ and $g_\xi(x_\xi)\ne f(x_\xi)$ for any $f\in\Phi_\xi$; it follows that $g_\xi(x_\xi)\not\sim x_\xi$.

(d) Let $\langle\kappa_x\rangle_{x\in[0,1]}$ be a family of non-zero cardinals less than $\omega_{\mathfrak{c}}$ such that $\kappa_x=\kappa_y$ iff $x\sim y$. Set $X=\{(x,\alpha):x\in[0,1],\,\alpha<\kappa_x\}$, and set $\pi(x,\alpha)=x$ for $(x,\alpha)\in X$. Set

$$\mathrm{T}=\{W:W\subseteq X,\,\pi[W]\in\Sigma,\,\mu(\pi[W]\cap\pi[X\setminus W])=0\},$$

$$\nu W=\mu(\pi[W])\text{ for }W\in\mathrm{T};$$

then ν is a complete probability measure on X and π is inverse-measure-preserving, so ν is atomless.

(e) ν is homogeneous. **P** Let V, $W\in\mathrm{T}$ be sets of the same non-zero measure. Then there are Borel sets E, $F\subseteq[0,1]$ such that $\pi^{-1}[E]\subseteq V$, $\pi^{-1}[F]\subseteq W$ and $\mu E=\nu V=\nu W=\mu F$.

Suppose for the moment that this common measure is not 1. Then there is a $\xi<\mathfrak{c}$ such that $E=E_\xi$ and $F=F_\xi$. Now $E\cap H_\xi$ and $F\cap H_\xi$ both meet $\mathfrak{c}$ equivalence classes, so

$$\#(\pi^{-1}[E\cap H_\xi])=\max(\mathfrak{c},\sup\nolimits_{x\in E\cap H_\xi}\kappa_x)=\omega_{\mathfrak{c}}$$

and

$$\#(V\setminus\pi^{-1}[E\setminus H_\xi])=\omega_{\mathfrak{c}}=\#(W\setminus\pi^{-1}[F\setminus H_\xi]).$$

On the other hand, if $x\in E\setminus H_\xi$, then $f_\xi(x)\in F\setminus H_\xi$ and $x\sim f_\xi(x)$, so $\kappa_{f_\xi(x)}=\kappa_x$. So we can find a bijection $h:V\to W$ such that $h(x,\alpha)=(f_\xi(x),\alpha)$ whenever $x\in E\setminus H_\xi$ and $\alpha<\kappa_x$. Now h is an isomorphism of the subspace measures on V and W.

If $\mu E=\mu F=1$ let $\xi<\omega_1$ be such that $[0,1]\setminus(E\cup F)\subseteq G_\xi$ and $G_\xi\cap E$, $G_\xi\cap F$ both have cardinal $\mathfrak{c}$. Proceed as above, but using the identity function in place of f_ξ, so that h is the identity on $X\setminus\pi^{-1}[G_\xi]$. **Q**

(e) **?** Suppose, if possible, that $0<\nu V<\nu W$ and $(V,\mathrm{T}_V,\mathcal{N}(\nu_V))\cong(W,\mathrm{T}_W,\mathcal{N}(\nu_W))$. Let $\phi:V\to W$ be an isomorphism of these structures. Let T_0 be a countable subalgebra of T such that

V, $W\in\mathrm{T}_0$,
$C_q=\pi^{-1}[\,[0,q]\,]\in\mathrm{T}_0$ for every $q\in\mathbb{Q}\cap[0,1]$,
$\pi^{-1}[\pi[U]]\in\mathrm{T}_0$ for every $U\in\mathrm{T}_0$,
$\phi[U]\in\mathrm{T}_0$, $\phi^{-1}[U]\in\mathrm{T}_0$ for every $U\in\mathrm{T}_0$.

Set $G=[0,1]\setminus\bigcup_{U\in\mathrm{T}_0}\pi[U]\cap\pi[X\setminus U]$, so that G is μ-conegligible. If $x\in G$, $(x,\alpha)\in V$ and $\beta<\kappa_x$, then $(x,\beta)\in V$, because $x\notin\pi[X\setminus V]$. Also, for every $q\in\mathbb{Q}\cap[0,1]$,

$$\pi\phi(x,\alpha)\le q\Rightarrow\phi(x,\alpha)\in W\cap C_q\Rightarrow(x,\alpha)\in\phi^{-1}[C_q]\Rightarrow(x,\beta)\in\phi^{-1}[C_q]$$

(because $x\notin\pi[X\setminus\phi^{-1}[C_q]]$)

$$\Rightarrow \pi\phi(x,\beta) \le q;$$

so $\pi\phi(x,\beta) \le \pi\phi(x,\alpha)$. It follows that in fact $\pi\phi(x,\beta) = \pi\phi(x,\alpha)$. So we have a function $h_0 : G \cap \pi[V] \to [0,1]$ such that $\pi\phi(x,\alpha) = h_0(x)$ whenever $x \in G \cap \pi[V]$ and $\alpha < \kappa_x$. For each $q \in \mathbb{Q} \cap [0,1]$, $\{x : x \in G \cap \pi[V],\ h_0(x) \le q\} = G \cap \pi[\phi^{-1}[C_q]] \in \Sigma$, so h_0 is Σ-measurable. If $H \subseteq [0,1]$ is μ-negligible, then

$$h_0^{-1}[H] = G \cap \pi[\phi^{-1}[\pi^{-1}[H]]]$$

is μ-negligible because $W \cap \pi^{-1}[H]$ and $\phi^{-1}[\pi^{-1}[H]]$ are ν-negligible.

Similarly, we have a Σ-measurable function $h_1 : G \cap \pi[W] \to [0,1]$ such that $\pi\phi^{-1}(x,\alpha) = h_1(x)$ whenever $x \in G \cap \pi[W]$ and $\alpha < \kappa_x$, and $\mu h_1^{-1}[H] = 0$ whenever $\mu H = 0$.

There is therefore a conegligible Borel set $G' \subseteq G$ such that $E = G' \cap \pi[V]$ and $F = G' \cap \pi[W]$ are Borel sets, $h_0[G'] \subseteq G'$, $h_1[G'] \subseteq G'$ and $h_0{\restriction}G'$, $h_1{\restriction}G'$ are Borel functions. If $x \in E$ and $\alpha < \kappa_x$, then $\phi(x,\alpha) \in W$ so $h_0(x) \in \pi[W]$; since also $h_0(x) \in G'$, $h_0(x) \in F$. Similarly, $h_1(x) \in E$ for every $x \in F$. Next, if $x \in E$ and $\alpha < \kappa_x$, there is a $\beta < \kappa_{h_0(x)}$ such that $\phi(x,\alpha) = (h_0(x),\beta)$, in which case $\phi^{-1}(h_0(x),\beta) = (x,\alpha)$, so $h_1h_0(x) = x$. Similarly, $h_0h_1(x) = x$ for every $x \in F$. Thus $h = h_0{\restriction}E$ and $h^{-1} = h_1{\restriction}F$ are two halves of a Borel isomorphism between E and F.

As $\mu E = \nu V \ne \nu W = \mu F$, there is some $\xi < \mathfrak{c}$ such that $h = g_\xi$. In this case, $x_\xi \in E$ and $x_\xi \not\sim h(x_\xi)$. But this means that $\kappa_{x_\xi} \ne \kappa_{h(x_\xi)}$ and ϕ cannot act as a bijection between the fibers $\pi^{-1}[\{x_\xi]$ and $\pi^{-1}[\{h(x_\xi)]$. $\maltese$

3 Problem Can we do this, or something similar, in ZFC?