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On a question of W.Ricker

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1 The question (Ricker & Väth 04; see Fokkink & Neerven 03) Let U be the smallest linear subspace of $\mathbb{C}^{\mathbb{C}}$ such that the identity function belongs to U and $|u| \in U$ for every $u \in U$. Does U contain the function $z \mapsto \bar{z}$?

2 The answer No (as expected).

proof (a) Let $\mathcal{G}$ be the family of sets $G \subseteq \mathbb{C}$ such that $\mathbb{C} \setminus G$ is a finite union of sets of the forms

$$\{x_0 + iy : y \ge \beta\}, \quad \{x_0 + iy : y \le \gamma\}$$

for $x_0 \in \mathbb{R}$, $\beta > 0$ and $\gamma \le 0$. Note that $\mathcal{G}$ is closed under intersection and that every member of $\mathcal{G}$ is open and simply connected. Let Φ be the set of analytic functions ϕ such that $\operatorname{dom}\phi \in \mathcal{G}$ and there is some polynomial p of two variables, with complex coefficients, such that $p(\phi(z), z) = 0$ for every $z \in \operatorname{dom}\phi$. Then $\phi + \psi$ and $\phi \times \psi$ (defined on $\operatorname{dom}\phi \cap \operatorname{dom}\psi$) belong to Φ for all ϕ, $\psi \in \Phi$; also $c\phi \in \Phi$ for $\phi \in \Phi$ and $c \in \mathbb{C}$. If $\phi \in \Phi$ is not identically zero, then it has only finitely many zeroes. **P?** Otherwise, let p be a polynomial of minimal degree such that $p(\phi(z), z) = 0$ for every $z \in \operatorname{dom}\phi$. Then $p(0, z) = 0$ for infinitely many z so $p(0, z) = 0$ for every z, that is, p is of the form $(w, z) \mapsto wq(w, z)$ for some polynomial q. Now $\phi(z)q(\phi(z), z) = 0$ for every $z \in \operatorname{dom}\phi$. But as $\operatorname{dom}\phi$ is connected and ϕ is not identically zero, $q(\phi(z), z) = 0$ for every $z \in \operatorname{dom}\phi$, and p did not have minimal degree. **XQ**

It follows that for every $\phi \in \Phi$ there is a $\psi \in \Phi$ such that $\operatorname{dom}\psi \subseteq \operatorname{dom}\phi$ and $\psi(z)^2 = \phi(z)$ for every $z \in \operatorname{dom}\psi$. **P** For each zero $z_0 = x_0 + iy_0$ of ϕ, delete either $\{x_0 + iy : y \ge y_0\}$ (if $y_0 > 0$) or $\{x_0 + iy : y \le y_0\}$ (if $y_0 \le 0$) from the domain of ϕ. This cuts ϕ down to a member of Φ with no zeroes which will have just two square roots both belonging to Φ. **Q**

Of course all polynomials belong to Φ. In particular, we have a function $\chi \in \Phi$ representing a square root of the function $z \mapsto 1 - z^2$; for definiteness, let us take it that $\operatorname{dom}\chi = \mathbb{C} \setminus \{x + iy : |x| = 1, y \le 0\}$ and that

$$\begin{aligned} \chi(x) &= \sqrt{1 - x^2} \text{ for } |x| < 1, \\ &= -i\sqrt{x^2 - 1} \text{ for } x > 1, \\ &= i\sqrt{x^2 - 1} \text{ for } x < -1. \end{aligned}$$

(b) For $\alpha \ge 0$ let Φ_α be the set of those $\phi \in \Phi$ such that $\limsup_{z \in \operatorname{dom}\phi, |z| \to \infty} \frac{\phi(z)}{|z|^\alpha}$ is finite. Then $\phi + \psi$ and $c\phi$ belong to Φ_α for all ϕ, $\psi \in \Phi_\alpha$, while $\phi \times \psi \in \Phi_{\alpha+\beta}$ whenever $\phi \in \Phi_\alpha$ and $\psi \in \Phi_\beta$. Also if $\psi \times \psi \subseteq \phi$, as in (a), and $\phi \in \Phi_\alpha$, then $\psi \in \Phi_{\alpha/2}$.

Note that our function χ, and the identity function $z \mapsto z$, belong to Φ_1.

(c) For $\alpha \ge 0$ let F_α be the set of continuous functions $f :]-\pi, \pi] \to \mathbb{R}$ such that

for every non-trivial interval $I \subseteq]-\pi, \pi]$ there are a non-trivial interval $J \subseteq I$ and a $\phi \in \Phi_\alpha$ such that $\phi(\cos t)$ is defined and equal to $f(t)$ for every $t \in J$.

Then F_α is a real linear space for each α; $f \times g \in F_{\alpha+\beta}$ whenever $f \in F_\alpha$ and $g \in F_\beta$; and if $f \in F_\alpha$ is non-negative then $\sqrt{f} \in F_{\alpha/2}$. **P** Let $I \subseteq]-\pi, \pi]$ be a non-trivial interval. Then there are a non-trivial interval $J_0 \subseteq I$ and a $\phi \in \Phi_\alpha$ such that $\phi(\cos t) = f(t)$ for every $t \in J_0$. Let $\psi \in \Phi_{\alpha/2}$ be such that $\psi(z)^2 = \phi(z)$ for every $z \in \operatorname{dom}\psi$. Then there is a non-trivial interval $J_1 \subseteq J_0$ such that $\psi(\cos t)$ is defined for every $t \in J_1$. Let $J \subseteq J_1$ be a non-trivial interval such that $\psi\cos$ does not change sign on J; then one of $\psi\cos$, $-\psi\cos$ agrees with $\sqrt{f}$ on J, while J is a non-trivial subinterval of I. **Q**

(d) Let F be the set of all functions from $]-\pi, \pi]$ to $\mathbb{C}$ of the form $t \mapsto f(t) + ig(t) + ce^{it}$ where $f, g \in \bigcup_{0 \le \alpha < 1} F_\alpha$ and $c \in \mathbb{C}$. Then F is a complex linear space. Now $|h| \in F$ for every $h \in F$. **P** Let f,

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$g \in \bigcup_{0\le\alpha<1} F_\alpha$ and $c \in \mathbb{C}$ be such that $h(t) = f(t) + ig(t) + ce^{it}$ for every $t \in]-\pi, \pi]$; then there are real numbers $\gamma_0, \dots, \gamma_4$ such that

$$|h(t)|^2 = f(t)^2 + g(t)^2 + \gamma_0 + \gamma_1 f(t)\cos t + \gamma_2 f(t)\sin t + \gamma_3 g(t)\cos t + \gamma_4 g(t)\sin t$$

for every $t \in]-\pi, \pi]$. Let $\alpha < 1$ be such that both f and g belong to F_α. Let $I \subseteq]-\pi, \pi]$ be a non-trivial interval. Then there are a non-trivial interval $J_0 \subseteq I$ and ϕ, $\psi \in \Phi_\alpha$ such that $f(t) = \phi(\cos t)$ and $g(t) = \psi(\cos t)$ for every $t \in J_0$. Let $J \subseteq J_0$ be a non-trivial interval on which sin is either strictly positive or strictly negative; then there is an $\epsilon \in \{-1, 1\}$ such that $\sin t = \epsilon\chi(\cos t)$ for every $t \in J$. So if we set

$$\theta(z) = \phi(z)^2 + \psi(z)^2 + \gamma_0 + \gamma_1\phi(z)z + \epsilon\gamma_2\phi(z)\chi(z) + \gamma_3\psi(z)z + \epsilon\gamma_4\psi(z)\chi(z)$$

whenever this is defined, we shall have $\theta \in \Phi_{\alpha+1}$ and $|h(t)|^2 = \theta(\cos t)$ for every $t \in J$. As I is arbitrary, $|h|^2 \in F_{\alpha+1}$ and $|h| \in F_{(\alpha+1)/2} \subseteq F$. **Q**

(e) It follows that if $u \in U$, as defined in Question 1, then $t \mapsto u(e^{it}) :]-\pi, \pi] \to \mathbb{C}$ belongs to F.

(f) The function $\cos \restriction]-\pi, \pi]$ does not belong to F. **P?** Otherwise, there are f, $g \in \bigcup_{0\le\alpha<1} F_\alpha$ and β, $\gamma \in \mathbb{R}$ such that $\cos t = f(t) + ig(t) + (\beta + i\gamma)(\cos t + i\sin t)$ for every $t \in]-\pi, \pi]$. There must be an $\alpha < 1$, functions ϕ, $\psi \in \Phi_\alpha$ and a non-trivial interval $J \subseteq]-\pi, \pi]$ such that $f(t) = \phi(\cos t)$ and $g(t) = \psi(\cos t)$ for every $t \in J$. There are therefore a non-trivial interval $K \subseteq]-1, 1[$ and an $\epsilon \in \{-1, 1\}$ such that

$$x = \phi(x) + i\psi(x) + (\beta + i\gamma)(x + i\epsilon\chi(x))$$

for every $x \in K$. Taking real and imaginary parts,

$$x = \phi(x) + \beta x - \gamma\epsilon\chi(x), \quad 0 = \psi(x) + \gamma x + \beta\epsilon\chi(x)$$

for $x \in K$. But this means that

$$0 = \phi(z) + (\beta - 1)z - \gamma\epsilon\chi(z) = \psi(z) + \gamma z + \beta\epsilon\chi(z)$$

for every z in the connected open set $G = \operatorname{dom}\phi \cap \operatorname{dom}\psi \cap \operatorname{dom}\chi$, and

$$0 = \frac{\phi(x)}{x} + \beta - 1 - \gamma\epsilon\frac{\chi(x)}{x} = \frac{\psi(x)}{x} + \gamma + \beta\epsilon\frac{\chi(x)}{x}$$

for all sufficiently large real x. Letting $x \to \infty$ we get

$$0 = \beta - 1 + i\gamma\epsilon = \gamma - i\beta\epsilon,$$

which is impossible since β, γ and ϵ are supposed all to be real and $\epsilon \ne 0$. **X**

(g) Consequently the function $\mathcal{R}\mathrm{e}$ cannot belong to U; as $\mathcal{R}\mathrm{e}\, z = \frac{1}{2}(z + \bar{z})$, the function $z \mapsto \bar{z}$ does not belong to U, as expected.

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Reference

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