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Extensions of Talagrand's measure

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1 Lemma Let ν be the usual measure on $\mathcal{P}\mathbb{N}$ (Fremlin 01, 254Jb) and μ_T Talagrand's measure on $\mathcal{P}\mathbb{N}$ (Fremlin 03, 464D); write $\mathcal{N}(\mu_T)$ for the null ideal of μ_T. Suppose that there is an uncountable set $A \subseteq \mathcal{P}\mathbb{N}$ such that no uncountable subset of A is ν-negligible. Then $\operatorname{cov}\mathcal{N}(\mu_T)$ (definition: Fremlin 07?, 511Fc) is ω_1.

proof Choose $\langle a_\xi\rangle_{\xi<\omega_1}$ in A as follows. Given that $b_J = \bigcap_{\eta\in J} a_\eta$ is infinite for every finite $J \subseteq \xi$, where $\xi < \omega_1$, then $\{a : a \cap b_J \text{ is finite}\}$ is ν-negligible for every $J \in [\xi]^{<\omega}$, so there is an $a_\xi \in A \setminus \{a_\eta : \eta < \xi\}$ such that $a_\xi \cap b_J$ is infinite for every $J \in [\xi]^{<\omega}$; continue.

Let $\mathcal{F}$ be the filter generated by $\{a_\xi : \xi < \omega_1\} \cup \{\mathbb{N} \setminus \{n\} : n \in \mathbb{N}\}$. As $A \cap \mathcal{F}$ is uncountable, $\nu^*\mathcal{F} > 0$; by the zero-one law (Fremlin 03, 464Ac), $\nu^*\mathcal{F} = 1$ and $\mu_T\mathcal{F} = 1$. On the other hand, $\mathcal{F} = \bigcup\{a : a \subseteq \mathbb{N}, b_J \setminus a \text{ is finite}\}$ is the union of ω_1 ν-negligible sets. So $\mathcal{P}\mathbb{N}$ is covered by ω_1 μ_T-negligible sets.

2 Theorem There is no extension of Talagrand's measure μ_T on $\mathcal{P}\mathbb{N}$ to a measure defined on every subset of $\mathcal{P}\mathbb{N}$.

proof ? Suppose, if possible, otherwise. Let μ be such an extension. Then $\mu\{a\} = 0$ for every $a \in \mathcal{P}\mathbb{N}$, so $\kappa = \operatorname{add}\mu$ is a real-valued-measurable cardinal (Fremlin 07?, 543Ba); as $\kappa \le \mathfrak{c}$, κ is atomlessly-measurable (Fremlin 07?, 543Bc). By Fremlin 07?, 544G there is an uncountable set $A \subseteq \mathcal{P}\mathbb{N}$ such that no uncountable subset of A is ν-negligible, where ν is the usual measure on $\mathcal{P}\mathbb{N}$. By Lemma 1, $\mathcal{P}\mathbb{N}$ can be covered by ω_1 μ_T-negligible sets, which are all μ-negligible, and κ is at most ω_1. But ω_1 is not real-valued-measurable, by Ulams theorem. **X**

3 Corollary Interpreting Talagrand's measure on $\mathcal{P}\mathbb{N}$ as a measure on ℓ^∞ (Fremlin 03, 464R), it has no extension to a measure on ℓ^∞ measuring every $\| \|_\infty$-open set.

proof The norm $\| \|_\infty$ induces the discrete topology on the natural copy of $\mathcal{P}\mathbb{N}$ in ℓ^∞.

Reference

Fremlin D.H. [01] *Measure Theory, Vol. 2: Broad Foundations.* Torres Fremlin, 2001.

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Fremlin D.H. [07?] *Measure Theory, Vol. 5: Set-theoretic measure theory,* in preparation. *Partial drafts available via* `http://www.essex.ac.uk/maths/staff/fremlin/mt.htm`. *Note that paragraph references are subject to fluctuation.*