Version of 7.3.05

A strictly positive Borel measure

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1 Lemma Let I be a countably infinite set disjoint from the ordinal ω_1+1. Then there is a Hausdorff topology $\mathfrak{S}$ on $Z=(\omega_1+1)\cup I$ such that I is dense in Z, each point of I is isolated in Z and the induced topology on ω_1+1 is the order topology of ω_1+1.

proof (Compare Fremlin 03, 419A.) Choose $\langle I_\xi\rangle_{\xi<\omega_1}$ inductively so that every I_ξ is an infinite and co-infinite subset of I, $I_\eta\setminus I_\xi$ is finite whenever $\eta<\xi$, and $I_{\xi+1}\setminus I_\xi$ is infinite for every ξ; set $I_{\omega_1}=I$. Now let $\mathfrak{S}$ be the family of those sets $U\subseteq Z$ such that

— if $0\in U$ then $I_0\setminus U$ is finite;

— if $0<\xi\le\omega_1$ and $\xi\in U$ then there is a $\zeta<\xi$ such that $\{\eta:\zeta<\eta\le\xi\}\subseteq U$ and $(I_\xi\setminus I_\eta)\setminus U$ is finite.

2 Example There is a σ-compact Hausdorff space X with a strictly positive Borel probability measure μ on X for which there is a non-empty open set G^* such that $\mu K=0$ for every compact subset K of G^*.

Construction (a) Take I, Z and $\mathfrak{S}$ from Lemma 1. Set $S^*=\bigcup_{n\in\mathbb{N}}I^n$ and $X=S^*\times(\omega_1+1)$. For $V\subseteq Z$, $n\in\mathbb{N}$ and $\sigma\in I^n$ set

$$H(V,\sigma)=\{(\sigma,\xi):\xi\in V\cap(\omega_1+1)\}\cup\{(\tau,\eta):\sigma\subset\tau\in S^*,\ \tau(n)\in V\cap I,\ \eta<\omega_1\}.$$

Give X the topology generated by $\{H(V,\sigma):V\in\mathfrak{S},\ \sigma\in S^*\}$.

(b) X is Hausdorff. **P** Let $x=(\sigma,\xi)$ and $y=(\tau,\eta)$ be distinct members of X.

(i) If $\sigma\not\subseteq\tau$ and $\tau\not\subseteq\sigma$ then $H(Z,\sigma)$ and $H(Z,\tau)$ are disjoint open sets containing x, y respectively.

(ii) If $\sigma\subset\tau$, set $n=\#(\sigma)$ (so that $\sigma\in I^n$). Then $H(Z\setminus\{\tau(n)\},\sigma)$, $H(Z,\tau)$ are disjoint open sets containing x, y respectively.

Similarly, x and y can be separated if $\tau\subset\sigma$.

(iii) If $\sigma=\tau$ then there are disjoint open sets U, $V\subseteq Z$ such that $\xi\in U$ and $\eta\in V$; now $H(U,\sigma)$ and $H(V,\sigma)$ are disjoint open sets in X containing x, y respectively. **Q**

(c) For $\sigma\in S^*$, the map $\xi\mapsto(\sigma,\xi):\omega_1+1\to X$ is continuous. So $X=\bigcup_{\sigma\in S^*}\{\sigma\}\times(\omega_1+1)$ is σ-compact.

(d) Let ν be Dieudonné's measure on ω_1 (Fremlin 03, 411Q); then ν is a complete topological probability measure. Let $\langle\epsilon_\sigma\rangle_{\sigma\in S^*}$ be a family of strictly positive real numbers summing to 1. For Borel sets $E\subseteq X$ set

$$\mu E=\textstyle\sum_{\sigma\in S^*}\epsilon_\sigma\nu\{\xi:\xi<\omega_1,\ (\sigma,\xi)\in E\}.$$

Then μ is a Borel probability measure on X.

(e) If $G\subseteq X$ is a non-empty open set then it includes a set of the form $H(Z,\tau)$ for some $\tau\in S^*$, so $\mu G\ge\epsilon_\tau>0$.

(f) Set

$$G^*=H(Z\setminus\{\omega_1\},\emptyset)=S^*\times\omega_1,$$

so that G^* is open and not empty. If $K\subseteq G$ is compact then $\mu K=\sum_{\sigma\in S^*}\nu K[\{\sigma\}]=0$. So we have the required example.

3 Remark Note that if X is regular, Hausdorff and σ-compact and μ is a strictly positive topological measure on X and $G\subseteq X$ is open and not empty, then there is a non-empty open set H such that $\overline{H}\subseteq G$. Now $\overline{H}$ is σ-compact and non-negligible so includes a non-negligible compact set.

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Reference

Fremlin D.H. [03] *Measure Theory, vol. 4: Topological Measure Spaces.* Torres Fremlin, 2003.